Title: Lecture - Classical Physics, PHYS 612

Speakers: Aldo Riello

Collection/Series: Classical Physics (Core), PHYS 612, September 2 - October 7, 2025

Subject: Mathematical physics, Other

Date: September 16, 2025 - 1:00 PM

URL: https://pirsa.org/25090017

RECAP

- V = vect sp, dim $V = n$
  $\underline{v}, \underline{w}, \ldots \in V$ basis $\{\underline{e}_i\}_{i=1}^n$

- $V^* = \mathrm{Lin}(V, \mathbb{R})$ also a vector sp of dim n -
  $\underline{\alpha}, \underline{\beta}, \ldots \in V^*$ basis $\{\underline{f}^i\}_{i=1}^n$ w/ $\underline{f}^i(\underline{e}_j) = \delta^i_j$

<u>Notation</u>: $\underline{\alpha}(\underline{v}) \equiv \langle \underline{\alpha}, \underline{v} \rangle \in \mathbb{R}$

<u>Thm</u> $V^{**} \simeq V$ $(n < \infty)$

$\{\ \}_{i=1}^{n}$

vector sp

n -

$\{f^i\}_{i=1}^{n}$ w/ $\underline{f}^i(\underline{e}_j) = \delta^i_j$

$\langle\underline{\alpha}, \underline{v}\rangle \in \mathbb{R}$

$< \infty)$

- $v^i := \langle \underline{f}^i, \underline{v} \rangle \in \mathbb{R}$

$\underline{v} = \sum_{i=1}^{n} v^i \underline{e}_i \in V$

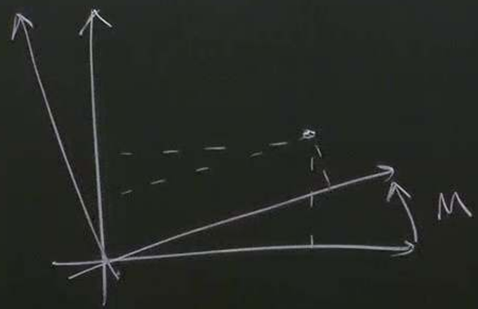
- $\alpha_i = \langle \underline{\alpha}, \underline{e}_i \rangle \in \mathbb{R}$

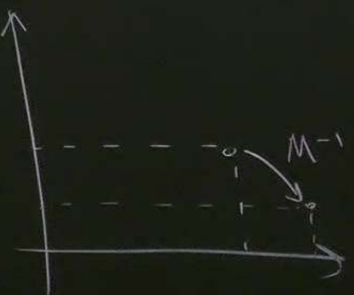
$\underline{\alpha} = \sum_{i=1}^{n} \alpha_i \underline{f}^i \in V^*$

$M \in Lin(V, V)$, $M : V \to V$

$M^\dagger : V^* \longrightarrow V^*$ such that

for all $\underline{\alpha}, \underline{v}$ $\quad \langle M^\dagger \underline{\alpha}, \underline{v} \rangle = \langle \underline{\alpha}, M \underline{v} \rangle$





$$\left[\begin{array}{l} \underline{w} = M\underline{v} \\ w^i = \sum_j M^i{}_j v^j \end{array}\right.$$

$$\left[\begin{array}{l} \underline{e}'_i = M\underline{e}_i \\ v'^i = \sum_j (M^{-1})^i{}_j v^j \end{array}\right.$$

$$M^i{}_j = \langle \underline{f}^i, M\underline{e}_j \rangle$$

$$\left[\ \alpha'_i = \sum_j \alpha_j M^j{}_i \right.$$

$$A : V \longrightarrow V$$

$$A^i{}_j = \langle \underline{f}^i, A\underline{e}_j \rangle$$

$$A'^i{}_j = \langle \underline{f}'^i, A\underline{e}'_j \rangle$$

$$= \sum_{k,l} (M^{-1})^j{}_k A^k{}_l M^l{}_j$$

Note: $A : V \longrightarrow V$ can also be thought as an element of $V \otimes V^*$

$$A = \sum_{k,l} A^k{}_l\, \underline{e}_k \otimes \underline{f}^l$$

Tensors

$$T \in \underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell}$$

tensor of rank $m = k + \ell$, of type (k, ℓ)

$$T \underset{\sum_{\{i,j\}}}{=} T^{i_1 \cdots i_k}{}_{j_1 \cdots j_\ell} \, \underline{e}_{i_1} \otimes \cdots \otimes \underline{e}_{i_k} \otimes \underline{f}^{j_1} \otimes \cdots \otimes \underline{f}^{j_\ell}$$

$$T_{ij} \quad \text{vs} \quad S^{ij} \quad \text{vs} \quad U^{i}{}_{j}$$

Einstein's convention

$$U^{j}{}_{kp} = T^{ij}{}_{k\ell}\, S_{i}{}^{\ell}{}_{p}$$

$$= \sum_{i,\ell} \left(\; \cdots\cdots \; \right)$$

$$A^{i}{}_{i} = \sum_{i} A^{i}{}_{i} \qquad (\text{trace})$$

$$= \sum_{i,j} A^{i}{}_{j} \langle \underline{f}^{j}, \underline{e}_{i} \rangle$$

Gradients

$f \in C^\infty(V)$

rethink of $v \in V$ as a directional derivative i.e. $v : C^\infty(V) \longrightarrow C^\infty(V)$

$$f \longmapsto v(f) \quad \text{s.t.} \quad \underline{v}(f)(\underline{w}) = \lim_{\epsilon \to 0} \frac{f(\underline{w} + \epsilon \underline{v}) - f(\underline{w})}{\epsilon}$$

Basis : $\boxed{\nabla_i f := \underline{e}_i(f)}$ gradient of f

$$\hookrightarrow \underline{v}(f) = \sum_i v^i \nabla_i f$$

Rmk f & $\underline{v}(f)$ are basis indep $\Rightarrow$ gradient of f is a covector

4-vectors in relativity

$$i \rightsquigarrow \mu = 0, \underbrace{1, 2, 3}_{i}$$

We now have a metric

$$\eta : V \times V \to \mathbb{R}$$

$$\eta \in V^* \otimes V^*$$

Orthonormal basis $\{\underline{e}_\mu\}_{\mu=0}^{3}$

s.t. $\eta(\underline{e}_0, \underline{e}_0) = -1 \quad \eta(\underline{e}_i, \underline{e}_j) = \delta_{ij}$

c

$\mathbb{R}$

$\{\underline{e}_\mu\}_{\mu=0}^{3}$

$\eta(\underline{e}_i, \underline{e}_j) = \delta_{ij}$

$$\eta_{\mu\nu} X^\mu Y^\nu = \eta_{\mu\nu} \underbrace{\Lambda^\mu{}_{\mu'} X^{\mu'}} \underbrace{\Lambda^\nu{}_{\nu'} Y^{\nu'}}$$

$$[\Lambda^T \eta \Lambda = \eta]$$

$$\eta_{\mu\nu} \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} = \eta_{\mu'\nu'}$$

$$(X^\nu \eta_{\mu\nu}) \Lambda^\mu{}_{\mu'} = \eta_{\mu'\nu'} \left[(\Lambda^{-1})^{\nu'}{}_{\nu} X^\nu \right]$$

$$\eta : V \longrightarrow V^*$$

$$X^\mu \mapsto \alpha_\nu = \eta_{\nu\mu} X^\mu$$

$$\underbrace{\Lambda^{\mu}{}_{\mu'} X^{\mu'}}\ \underbrace{\Lambda^{\nu}{}_{\nu'} Y^{\nu'}}$$

$$\begin{array}{ccc} V & \xrightarrow{\Lambda} & V \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\ V^* & \xrightarrow{\Lambda} & V^* \end{array}$$

$$\nu' = \eta_{\mu'\nu'}$$

$$= \eta_{\mu'\nu'} \left[(\Lambda^{-1})^{\nu'}{}_{\nu} X^{\nu} \right]$$

Therefore, as long as we restrict changes of frames to Lorentz transf., it makes sense to denote

$$\eta_{\mu\nu} X^{\nu} \text{ by } X_{\mu}$$

(here η is fixed)

V^*

$\alpha_{\nu} = \eta_{\nu\mu} X^{\mu}$

$\longrightarrow V$

$\downarrow \eta$

$\longrightarrow V^*$

as long as

changes of

Lorentz transf.,

sense to denote

by X_μ

is fixed)

Lorentz algebra revisited

$so(1,3)$ basis $\{\mathbb{J}_i, \mathbb{K}_i\}$

$$\mathbb{J}_{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha = \beta \\ -\mathbb{K}_i & \text{if } \alpha = 0, \beta = i \\ \mathbb{K}_i & \text{if } \alpha = i, \beta = 0 \\ \epsilon_{ij}{}^{k}\,\mathbb{J}_k & \text{if } \alpha, \beta = i, j \end{cases}$$

$$[\mathbb{J}_{\alpha\beta}, \mathbb{J}_{\gamma\delta}] = -\eta_{\alpha\gamma}\mathbb{J}_{\beta\delta} + \eta_{\beta\gamma}\mathbb{J}_{\alpha\delta} + \eta_{\alpha\delta}\mathbb{J}_{\beta\gamma} - \eta_{\beta\delta}\mathbb{J}_{\alpha\gamma}$$

INDICES HAVE DIFFERENT STATUS!

$$(\mathbb{J}_{\underline{\alpha\beta}})^{\mu}{}_{\nu} = \delta^{\mu}_{\alpha}\,\eta_{\beta\nu} - \delta^{\mu}_{\beta}\,\eta_{\alpha\nu}$$

$$\Lambda^T \eta \Lambda = \eta \qquad \Lambda = \mathbb{1} + \epsilon \lambda + \mathcal{O}(\epsilon^2)$$

$$\lambda^T \eta = -\eta \lambda \qquad (*)$$

with index notation:

$$\lambda^\mu{}_\nu = \frac{1}{2} \lambda^{\alpha\beta} \left(J_{\alpha\beta}\right)^\mu{}_\nu, \quad \lambda^{\alpha\beta} = -\lambda^{\beta\alpha} \qquad (1)$$

$$\cap$$

$$\mathfrak{so}(1,3)$$

$$(*) \text{ becomes } \lambda_{\mu\nu} = -\lambda_{\nu\mu} \qquad (2)$$

These (1-2) are two ways of seeing $\mathfrak{so}(1,3) = \underbrace{\mathbb{R}^4 \wedge \mathbb{R}^4}_{\text{antisym tensor product}}$

$\Lambda^T \eta \Lambda = \eta$ $\qquad \Lambda = \mathbb{1} + \epsilon \lambda + \mathcal{O}(\epsilon^2)$

$\lambda^T \eta = -\eta \lambda$ ⊛

$\frac{4\times 3}{2} = 6$

with index notation:

$\lambda^\mu{}_\nu = \frac{1}{2} \lambda^{\alpha\beta} (J_{\alpha\beta})^\mu{}_\nu, \quad \lambda^{\alpha\beta} = -\lambda^{\beta\alpha}$ ①

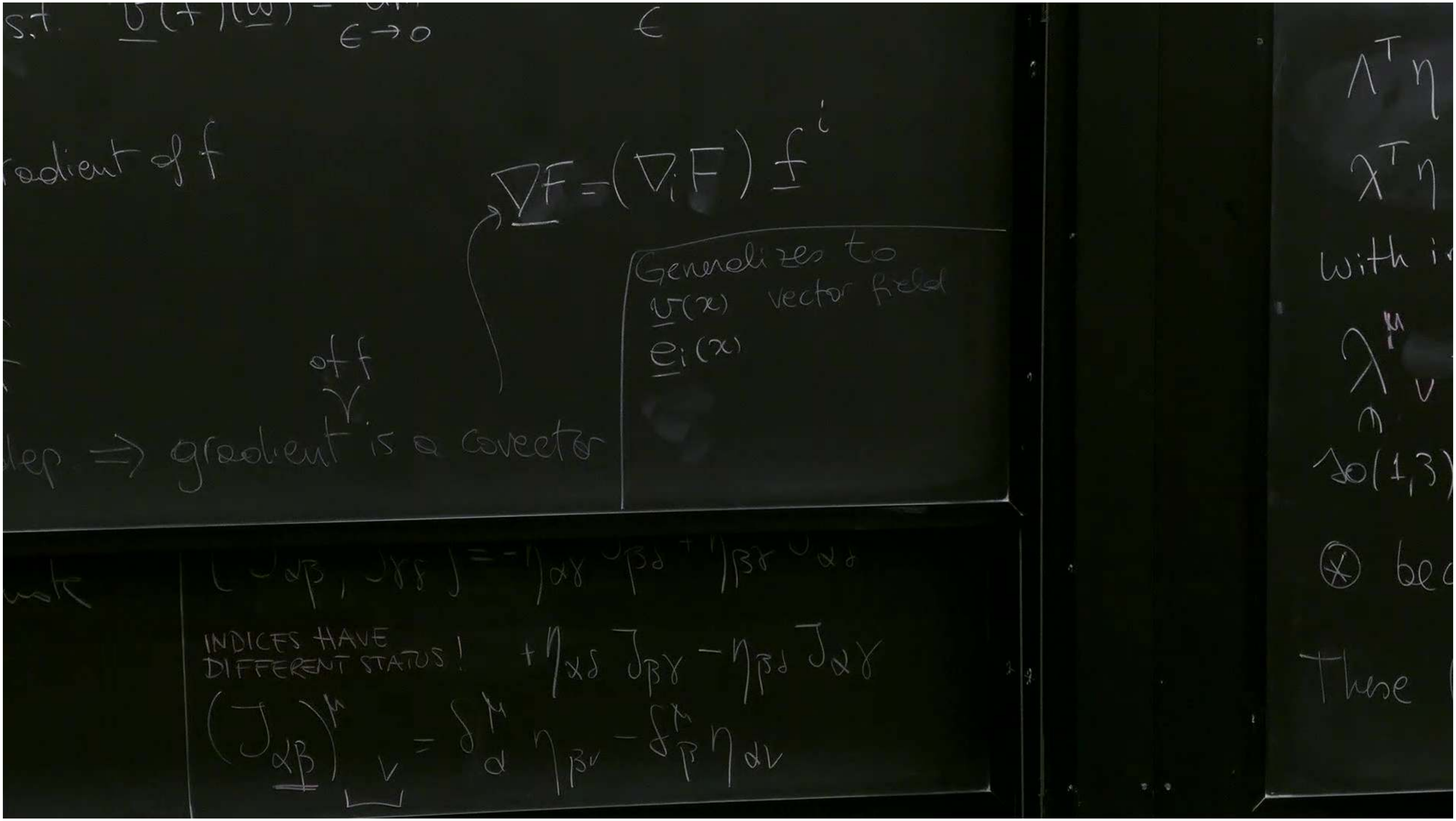
$\in$

$so(1,3)$

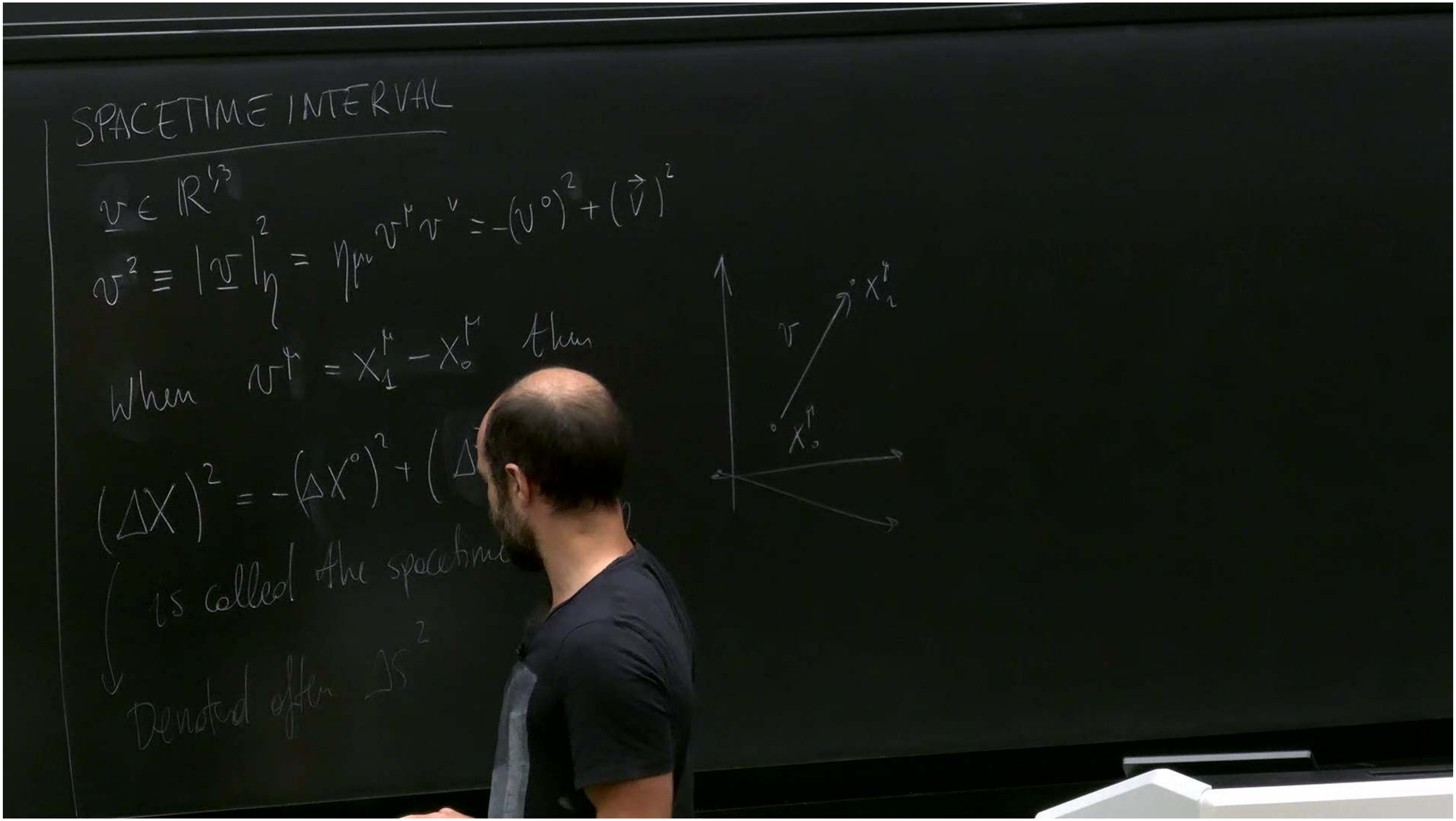
② antisym tensor product

⊛ becomes $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$

These (1-2) are two ways of seeing $so(1,3) = \underbrace{\mathbb{R}^4 \wedge \mathbb{R}^4}_{\text{antisym tensor product}}$



$\epsilon \to 0$
radient of f
$\underline{\nabla F} = (\nabla_i F)\, \underline{f}^i$
Generalizes to
$\underline{v}(x)$ vector field
$\underline{e}_i(x)$
of f
dep ⇒ gradient is a covector
INDICES HAVE DIFFERENT STATUS!
$+\eta_{\alpha\delta} J_{\beta\gamma} - \eta_{\beta\delta} J_{\alpha\gamma}$
$(J_{\alpha\beta})^{\mu}{}_{\nu} = \delta^{\mu}_{\alpha}\, \eta_{\beta\nu} - \delta^{\mu}_{\beta}\, \eta_{\alpha\nu}$
$\Lambda^T \eta$
$\lambda^T \eta$
with i
$\lambda^{\mu}{}_{\nu}$
$so(1,3)$
These


SPACETIME INTERVAL
$\underline{v} \in \mathbb{R}^{1,3}$
$v^2 \equiv |\underline{v}|^2_\eta = \eta_{\mu\nu} v^\mu v^\nu = -(v^0)^2 + (\vec{v})^2$
When $v^\mu = X_1^\mu - X_0^\mu$ then
$(\Delta X)^2 = -(\Delta X^0)^2 + ($
is called the spacetime
Denoted often Δs^2
v
X_1^μ
X_0^μ

Relativistic Dynamics

- Newton's second law $\dfrac{d\vec{p}}{dt} = \vec{F}$

if $\vec{p} = m\vec{v} \rightarrow m\vec{a} = \vec{F}$

const

absolute wrt

Galilean relativity

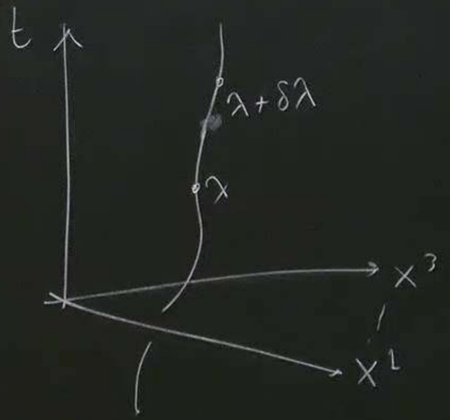
$\vec{v} \mapsto \vec{v} + \vec{u}$

$\vec{a} \mapsto \vec{a}$

does not hold in Sp Rel!

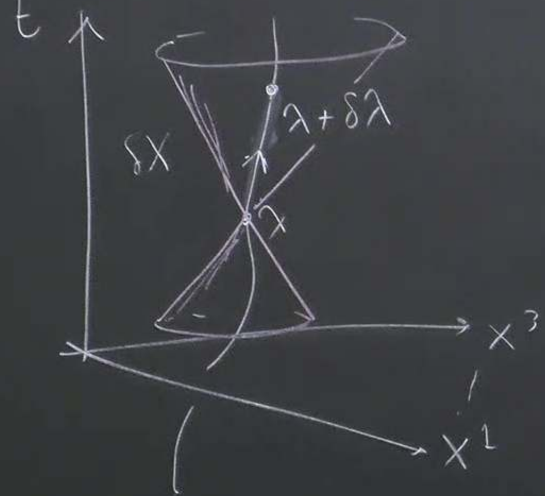
particle history $\vec{x} = \vec{x}(t)$ (Newton)

- Spacetime trajectory



$\lambda \longmapsto x^\mu(\lambda) = \begin{pmatrix} t(\\ \vec{x}(\end{pmatrix}$

- spacetime trajectory



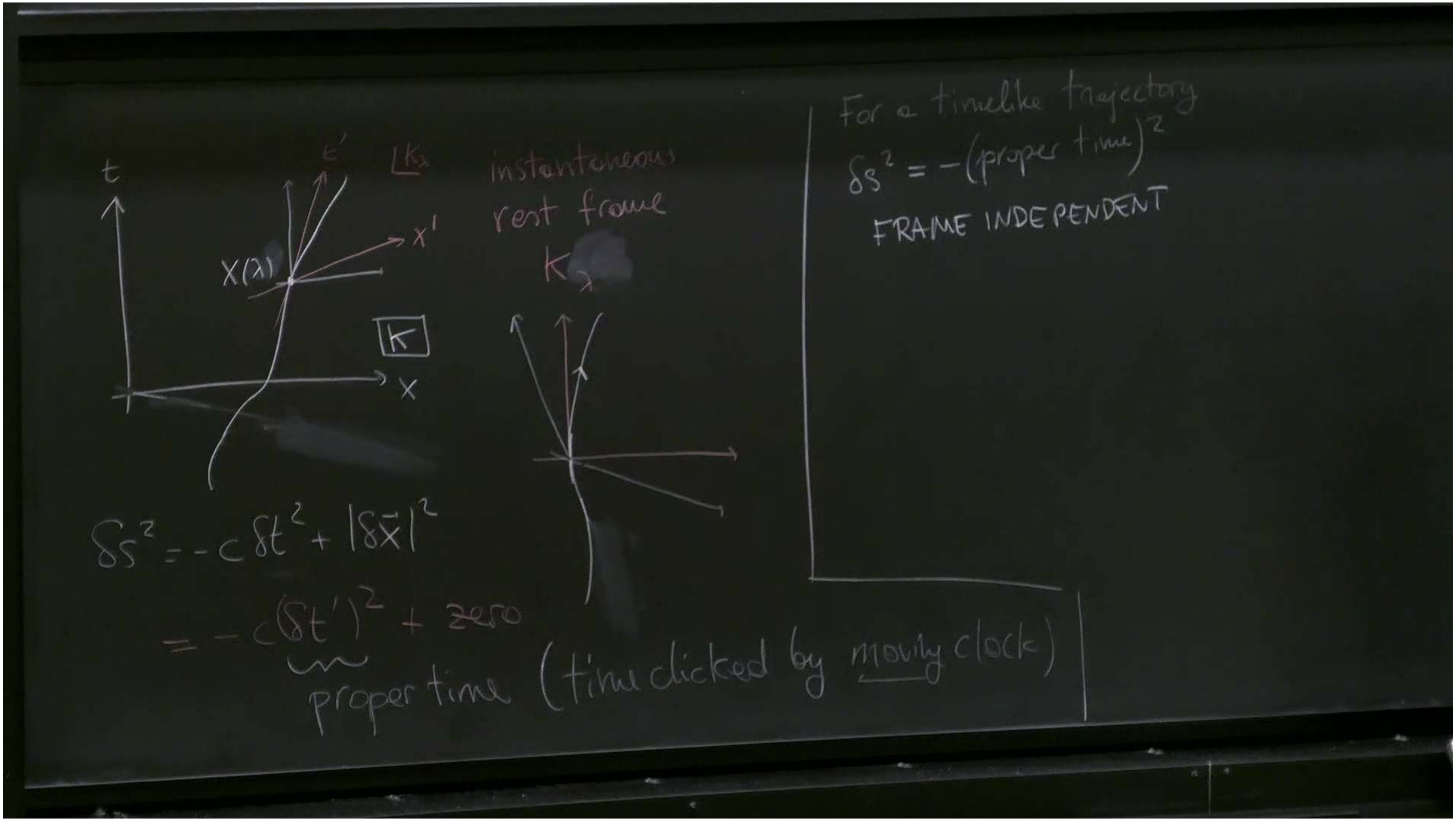
$$\lambda \longmapsto X^\mu(\lambda) = \begin{vmatrix} t(\lambda) \\ \vec{x}(\lambda) \end{vmatrix}$$

parameter, frame indep, a priori "unphysical" (it can be anything)

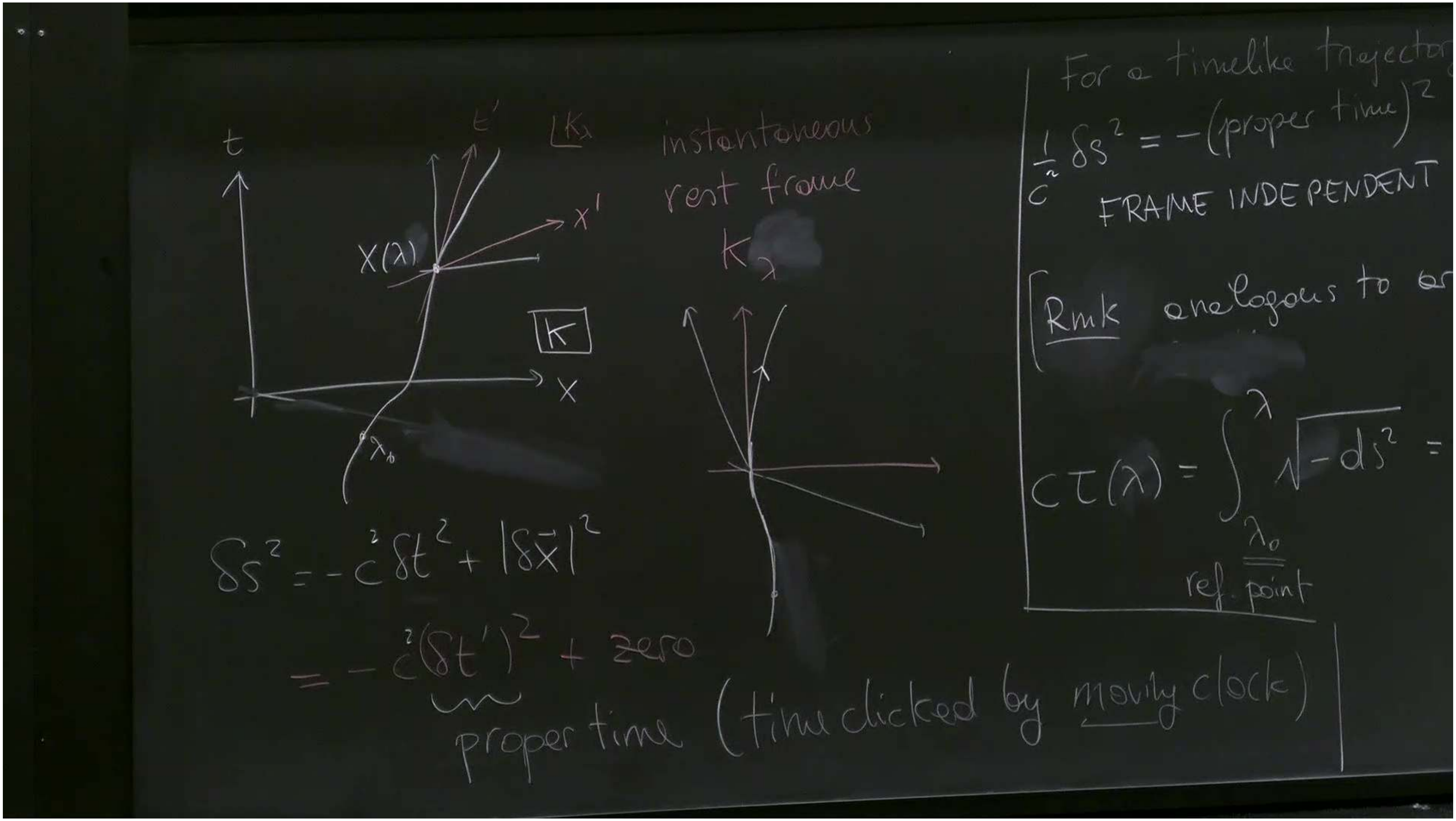
$$\delta\lambda \mapsto \delta X^\mu = (c\delta t, \delta\vec{x}) \leadsto \delta s^2 = -c\delta t^2 + |\delta\vec{x}|^2 \quad \text{frame independent.}$$

<u>Rmk</u> if trajectory is subluminal (causal)

$$\Rightarrow \delta s^2 < 0$$



t
t'
K_λ
instantaneous
rest frame
K_λ
x'
X(λ)
K
x
$\delta s^2 = -c\,\delta t^2 + |\delta\vec{x}|^2$
$= -c(\delta t')^2 + \text{zero}$
proper time (time clicked by moving clock)
For a timelike trajectory
$\delta s^2 = -(\text{proper time})^2$
FRAME INDEPENDENT



For a timelike trajector
$\frac{1}{c^2}\delta s^2 = -(\text{proper time})^2$
FRAME INDEPENDENT
Rmk analogous to ar
$c\tau(\lambda) = \int_{\lambda_0}^{\lambda} \sqrt{-ds^2} =$
ref. point
instantaneous
rest frame
K_λ
$\boxed{K}$
$X(\lambda)$
λ_0
$\delta s^2 = -c^2\delta t^2 + |\delta\vec{x}|^2$
$= -c^2(\delta t')^2 + \text{zero}$
proper time (time clicked by moving clock)

For a timelike trajectory

$$\frac{1}{c^2}\delta s^2 = -(\text{proper time})^2$$

FRAME INDEPENDENT

[Rmk analogous to arc length]

$$c\tau(\lambda) = \int_{\lambda_0}^{\lambda} \sqrt{-ds^2} = \int_{\lambda_0}^{\lambda} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda'}\frac{dx^\nu}{d\lambda'}}\, d\lambda'$$

λ_0 ref. point

licked by moving clock)