Title: Lecture - Classical Physics, PHYS 612

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Subject: Mathematical physics, Other

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Lorentz g/p $SO^+(1,3) \ni \Lambda$ s.t. $\Lambda^T \eta \Lambda = \eta$

↳ Infinitesimally generated by L. alg. $so(1,3)$

$\Lambda = \mathbb{1} + \epsilon \underbrace{\lambda}_{\text{matrix}} + \mathcal{O}(\epsilon^2)$ $\quad \lambda^T = -\eta \lambda \eta$ $\quad \eta = \left(\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

$\left[\Lambda = \exp(\lambda)\right]$ $\quad \lambda = \vec{\theta} \cdot \vec{J} + \vec{\psi} \cdot \vec{K}$

$$\Lambda = \exp(\vec{\theta}\cdot\vec{J} + \vec{\psi}\cdot\vec{K}) = \left(\begin{array}{c|ccc} 0 & \psi_1 & \psi_2 & \psi_3 \\ \hline \psi_1 & 0 & -\theta^3 & \theta^2 \\ \psi_2 & \theta^3 & 0 & -\theta^1 \\ \psi_3 & -\theta^2 & \theta^1 & 0 \end{array}\right)$$

$$\neq \exp(\vec{\theta}\vec{J})\exp(\vec{\psi}\vec{K})$$

$$[J^i, J^j] = -\epsilon^{ij}{}_k J^k$$

$$[J^i, K^j] = -\epsilon^{ij}{}_k K^k$$

$$[K^i, K^j] = \epsilon^{ij}{}_k J^k$$

$$(J^i)^l{}_m = -\epsilon^{il}{}_m$$

$$(J^3)^1{}_2 = -\epsilon^{31}{}_2$$

Exercice

$$R_{\vec{\theta}} = \exp(\vec{\theta}\vec{J})$$

$$R_{\vec{\theta}} \exp(\vec{\phi}\vec{J} + \vec{\psi}\vec{K}) R_{\vec{\theta}}^{-1}$$

$$= R_{\vec{\theta}} \sum_i \frac{(\vec{\phi}\vec{J} + \vec{\psi}\vec{K})^i}{i!} R_{\vec{\theta}}^{-1}$$

$$= \sum_i \frac{1}{i!} \left(\vec{\phi}.(R_{\vec{\theta}}\vec{J}R_{\vec{\theta}}^{-1}) + \vec{\psi}.(R_{\vec{\theta}}\vec{K}R_{\vec{\theta}}^{-1})\right)^i$$

Exercise

$$R_{\vec{\theta}} = \exp(\vec{\theta}\vec{J})$$

$$R_{\vec{\theta}} \exp(\vec{\phi}\vec{J} + \vec{\psi}\vec{K}) R_{\vec{\theta}}^{-1}$$

$$= R_{\vec{\theta}} \sum_i \frac{(\vec{\phi}\vec{J} + \vec{\psi}\vec{K})^i}{i!} R_{\vec{\theta}}^{-1}$$

MATRIX MULT

$$= \sum_i \frac{1}{i!} \left(\vec{\phi}\cdot(R_{\vec{\theta}}\vec{J}R_{\vec{\theta}}^{-1}) + \vec{\psi}\cdot(R_{\vec{\theta}}\vec{K}R_{\vec{\theta}}^{-1})\right)^i$$

$$R J^2 R^{-1} = R J J R^{-1}$$
$$= R J R^{-1} R J R^{-1}$$
$$= (R J R^{-1})^2$$

$$* \quad = \sum_i \frac{1}{i!} \left((R_{\vec{\theta}}\vec{\phi})\cdot\vec{J} + (R_{\vec{\theta}}\vec{\psi})\cdot\vec{K} \right)^i = \exp\left((R_{\vec{\theta}}\vec{\phi})\vec{J} + (R_{\vec{\theta}}\vec{\psi})\vec{K} \right)$$

MATRIX ACTING ON VECTOR

Rmk

rotations $so(3)$ form a subalgebra

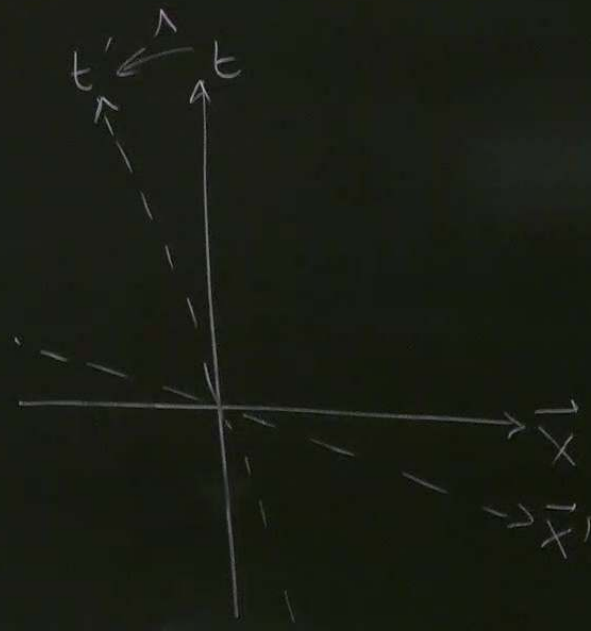
but boosts do not $\left([\mathbb{k}, \mathbb{k}] \neq \mathbb{k}\right)$

Rmk

$so(3) \hookrightarrow so(1,3)$

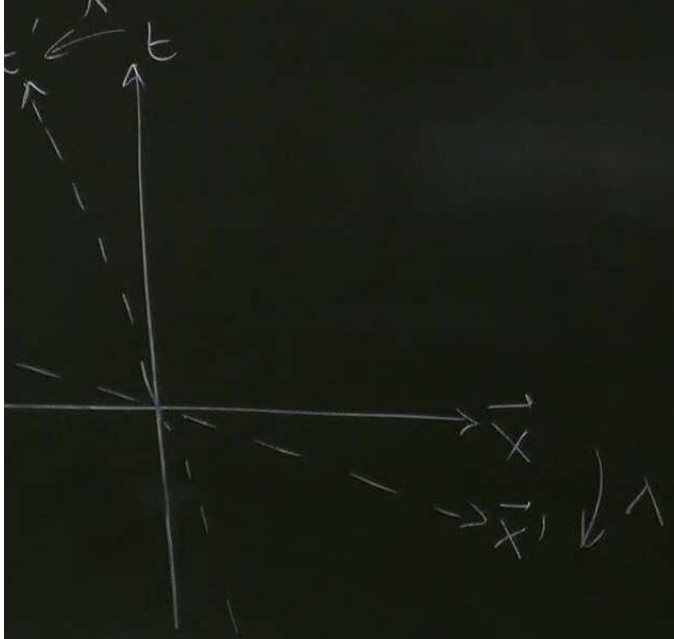
one per choice of frame!

they are related by a Lorentz transf



$$R' = \Lambda R \Lambda^{-1} \rightsquigarrow \Lambda R v = R' \Lambda v \quad \forall \text{ vecto } v$$

$\neq ik$)



$R v = R' \Lambda v$ $\vec{v}$ vecto v

$J'^i = \Lambda J^i \Lambda^{-1}$

$[J'^i, J'^j] = -\epsilon^{ij}{}_k J'^k$

Rmk $\tilde{J}^i = -J^i$

$\rightsquigarrow [\tilde{J}^i, \tilde{J}^j] = \epsilon^{ij}{}_k \tilde{J}^k$

$[\tilde{J}(\pm ik)] = \epsilon(\pm ik)$

$[(\pm ik), (\pm iK)] = -\epsilon \tilde{J}$

Rmk

$SO(3$

$\downarrow e$

$SO(3$

$= SU$

$\mathbb{Z}_2 =$

$J'^i = \Lambda J^i \Lambda^{-1}$

$[J'^i, J'^j] = -\epsilon^{ij}{}_k J'^k$

3×3 ℝ

Rmk

$\mathfrak{so}(3) \simeq \mathfrak{su}(2)$ — 2×2 ℂ

exp

$SO(3)$ $\quad$ $SU(2)$ — simply connected

$= SU(2)/\mathbb{Z}_2$ — this is not

$\mathbb{Z}_2 = \{\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\}$

4×4 ℝ

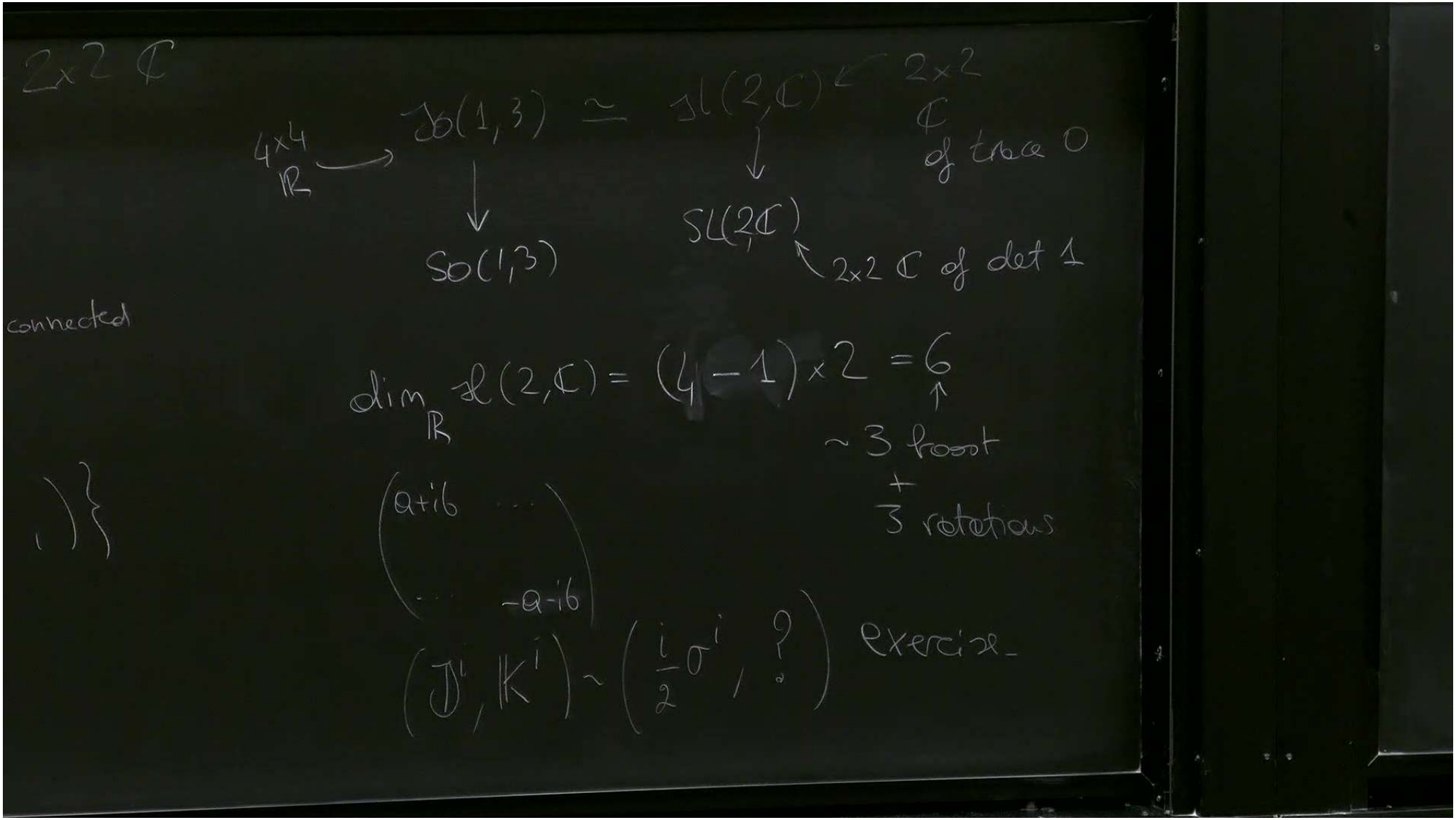
Rmk $\quad \tilde{J}^i = -J^i$

$\rightsquigarrow [\tilde{J}^i, \tilde{J}^j] = \epsilon^{ij}{}_k \tilde{J}^k$

$[\tilde{J}(\pm k)] = \epsilon(\pm k)$

$[(\pm k), (\pm k)] = -\epsilon \tilde{J}$

vecto
s


2×2 ℂ
4×4 ℝ
so(1,3) ≃ sl(2,ℂ)
2×2 ℂ of trace 0
SO(1,3)
SL(2,ℂ)
2×2 ℂ of det 1
connected
dim_ℝ sl(2,ℂ) = (4−1)×2 = 6
~3 boost + 3 rotations
(a+ib ...
... −a−ib)
(J^i, K^i) ~ (i/2 σ^i, ?) exercise

$$\vec{J} \longleftrightarrow \vec{\tau} = -\frac{i}{2}\vec{\sigma}$$

$$U = \exp\left(\vec{\theta}\cdot\vec{\tau}\right) \in \mathbb{C}_{2,1}$$

$$R = \exp\left(\vec{\theta}\cdot\vec{J}\right) \in \mathbb{R}_{3,3} \qquad SO(3) = SU(2)/\mathbb{Z}_2$$

$$R \subset \mathbb{R}^3 \simeq su(2)$$

$$e_1 \leftrightarrow \tau_1$$

$$Re_1 \leftrightarrow U\tau_1 U^{-1} = (-U)\tau_1(-U)^{-1}$$

fundam rep of $SO(3)$ — adj rep of $SO(2)$ [on its Lie algebra] (spin 1)

INDEX NOTATION

- V = (real) vector space, $\dim V = n < \infty$

  $\Downarrow$

  $\underline{v}, \underline{w}, \ldots$

- $V^* = \mathrm{Lin}(V, \mathbb{R})$ also a ($\mathbb{R}$) vector sp of $\dim n$

  $\Downarrow$

  $\underline{\alpha}, \underline{\beta}, \ldots$

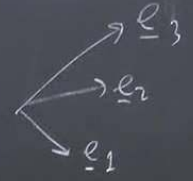
  - linear $\underline{\alpha}(a\underline{v} + \underline{w}) = a\,\underline{\alpha}(\underline{v}) + \underline{\alpha}(\underline{w}) \in \mathbb{R}$
  - vect. sp $(\underline{\alpha} + a\underline{\beta})(\underline{v}) := \underline{\alpha}(\underline{v}) + a\,\underline{\beta}(\underline{v})$ also linear

"Covectors"

- A basis of V : $\{\underline{e}_i\}_{i=1}^{n}$

$$\underline{v} = \sum_{i=1}^{n} v^i \underline{e}_i$$

$v^i \in \mathbb{R}$, $\underline{e}_i \in V$



Dual basis of V^* wrt $\{\underline{e}_i\}$ call it $\{\underline{f}^i\}$

$$\underline{f}^i(\underline{e}_j) = \delta^i{}_j = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{\alpha} = \sum_{i=1}^{n} \alpha_i \underline{f}^i$$

$\alpha_i \in \mathbb{R}$, $\underline{f}^i \in V^*$

$$\underline{\alpha}(\underline{v}) = \sum_{i,j=1}^{n} v^i \alpha_j \underbrace{\underline{f}^j(\underline{e}_i)}_{\delta_i^j} = \sum_{i=1}^{n} v^i \alpha_i$$

↑ linearity

(2)

$M \in Lin(V, V)$ $\qquad M : V \longrightarrow V$

$\underline{v} \longmapsto \underline{w} = M\underline{v}$

① Components of $\underline{w}' = M\underline{v}$ in original basis $\underline{e}_i$

$$w^i = \underline{f}^i(\underline{w}) = \sum_j \underbrace{\underline{f}^i(M\underline{e}_j)}_{M^i{}_j} v^j \equiv \sum_i M^i{}_j v^j$$

(2) Components of $\underline{v}$ in new basis $\underline{e}'_i = M\underline{e}_i$

First I need to figure out how $\underline{f}^i$ transforms

For this I need the following:

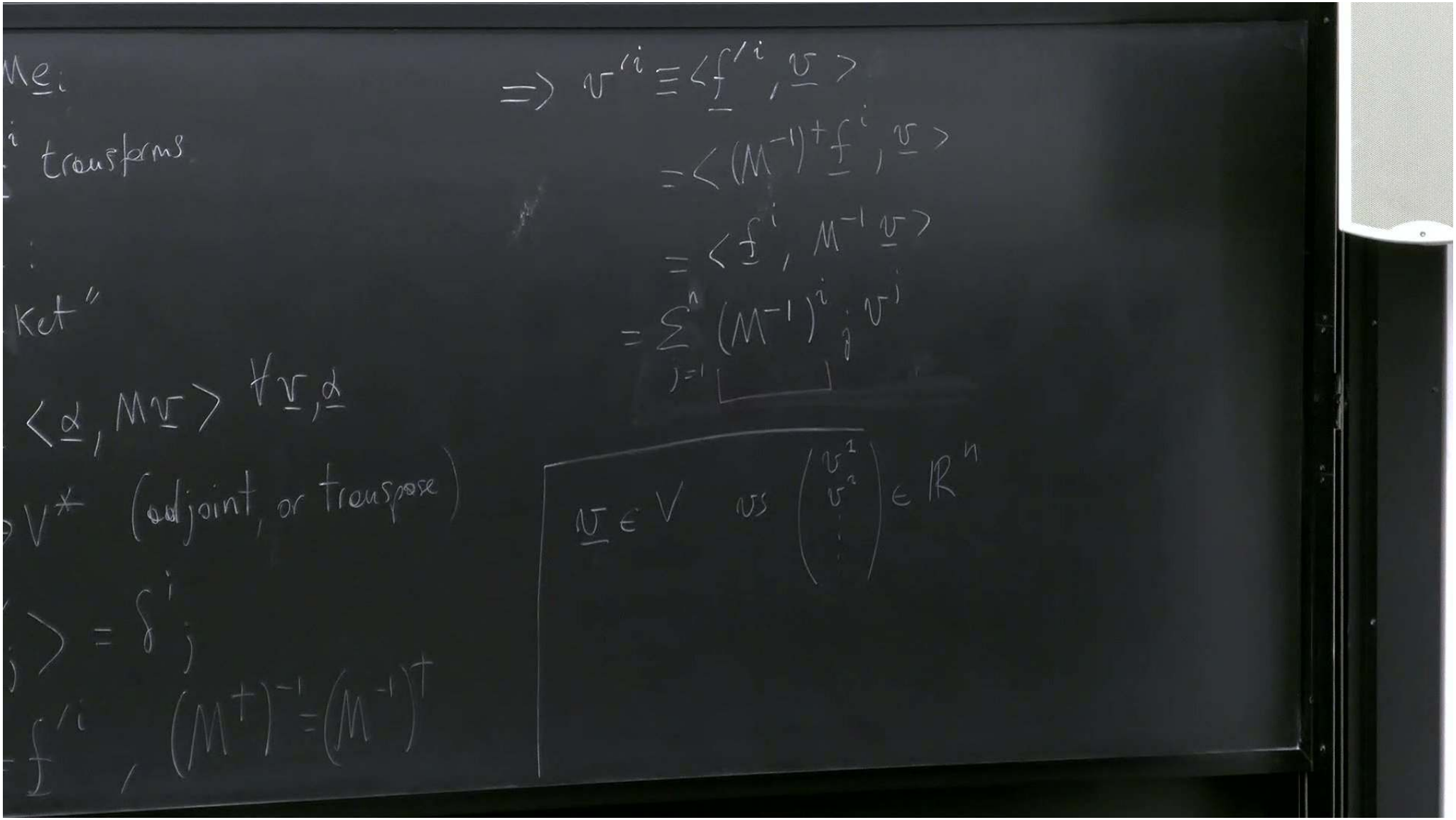
$\underline{\alpha}(\underline{v}) \equiv \langle \underline{\alpha}, \underline{v} \rangle$ "bra-ket"

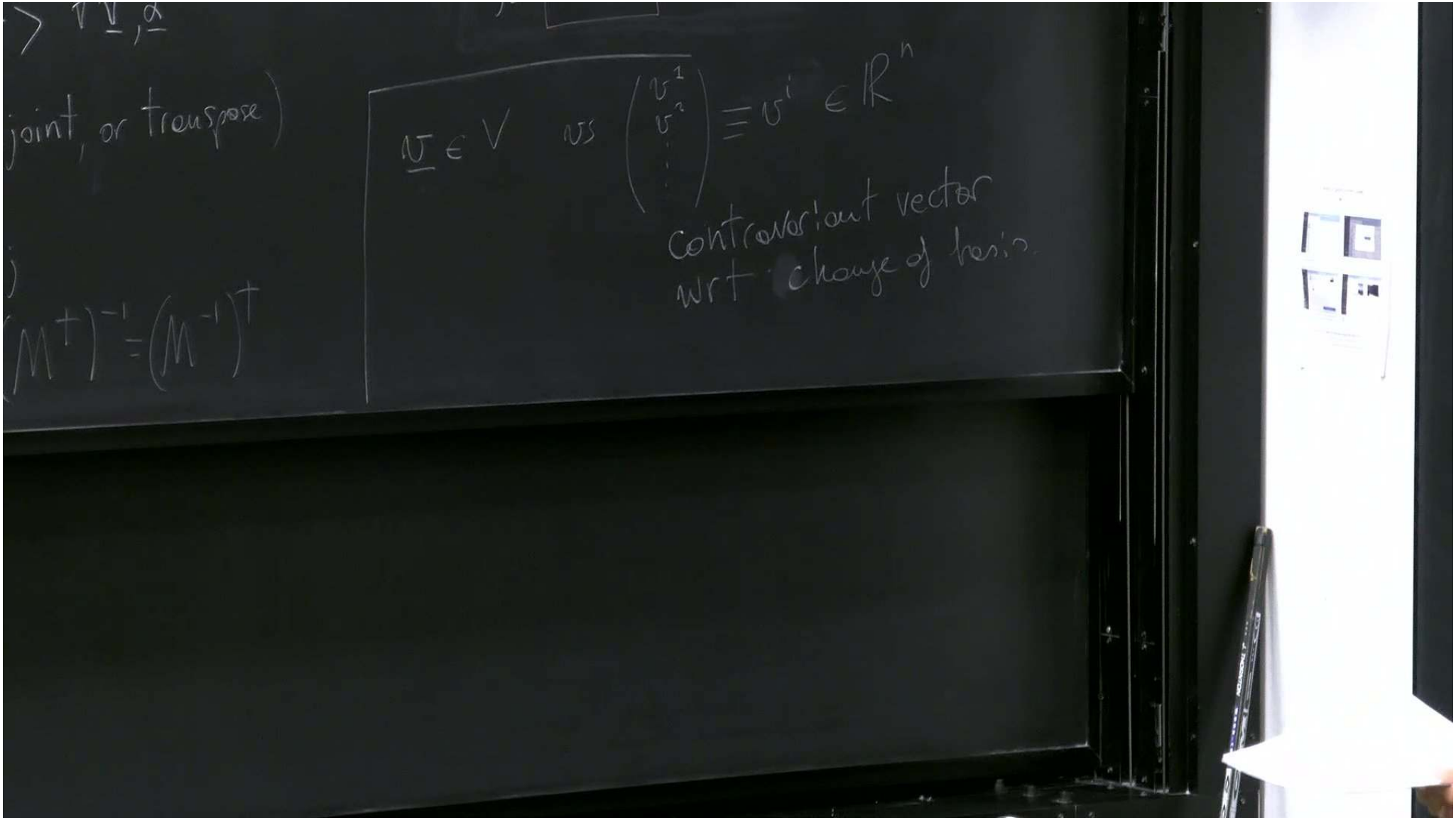
Define $M^\dagger$ by $\langle M^\dagger \underline{\alpha}, \underline{v} \rangle \equiv \langle \underline{\alpha}, M\underline{v} \rangle \quad \forall \underline{v}, \underline{\alpha}$

$M^\dagger : V^* \longrightarrow V^*$ (adjoint, or transpose)

$\rightsquigarrow \underline{f}'^i$ defined by $\langle \underline{f}'^i, \underline{e}'_j \rangle = \delta^i_j$

is given by $(M^\dagger)^{-1} \underline{f}^i = \underline{f}'^i$, $(M^\dagger)^{-1} = (M^{-1})^\dagger$


Me:
transforms
Ket"
$\langle \underline{\alpha}, M\underline{v} \rangle \quad \forall \underline{v}, \underline{\alpha}$
V^* (adjoint, or transpose)
$\rangle = \delta^i_j$
$(M^\dagger)^{-1} = (M^{-1})^\dagger$
$\Rightarrow v'^i \equiv \langle \underline{f}'^i, \underline{v} \rangle$
$= \langle (M^{-1})^\dagger \underline{f}^i, \underline{v} \rangle$
$= \langle \underline{f}^i, M^{-1}\underline{v} \rangle$
$= \sum_{j=1}^{n} (M^{-1})^i_j v^j$
$\underline{v} \in V$ vs $\begin{pmatrix} v^1 \\ v^2 \\ \vdots \end{pmatrix} \in \mathbb{R}^n$


joint, or transpose
$\underline{v} \in V$ vs $\begin{pmatrix} v^1 \\ v^2 \\ \vdots \end{pmatrix} \equiv v^i \in \mathbb{R}^n$
contravariant vector
wrt change of basis

$$\alpha_i' = \sum_{j=1}^{n} \alpha_j \Lambda^j{}_i \equiv \sum_{j=1}^{n} \Lambda^j{}_i \alpha_j$$

↑ change of basis

"covariant vector"

Rmk keeping track of index position
I know where (co) vectors live and
how they transform under a change of basis.
It also allows me to remove the Σ symbol