Title: Lecture - Classical Physics, PHYS 612

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Subject: Mathematical physics, Other

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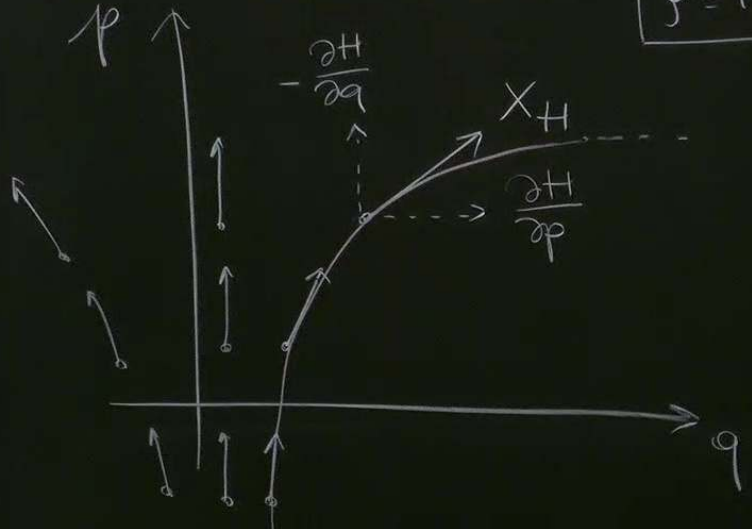
RECAP ($\partial_t H = 0$)

$\boxed{\mathcal{P} = T^*Q}$



$$\begin{cases} \dot{q}^i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q^i} \end{cases}$$

$$X_H^I = \begin{pmatrix} \partial H/\partial p_i \\ -\partial H/\partial q^i \end{pmatrix} = \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \right)$$

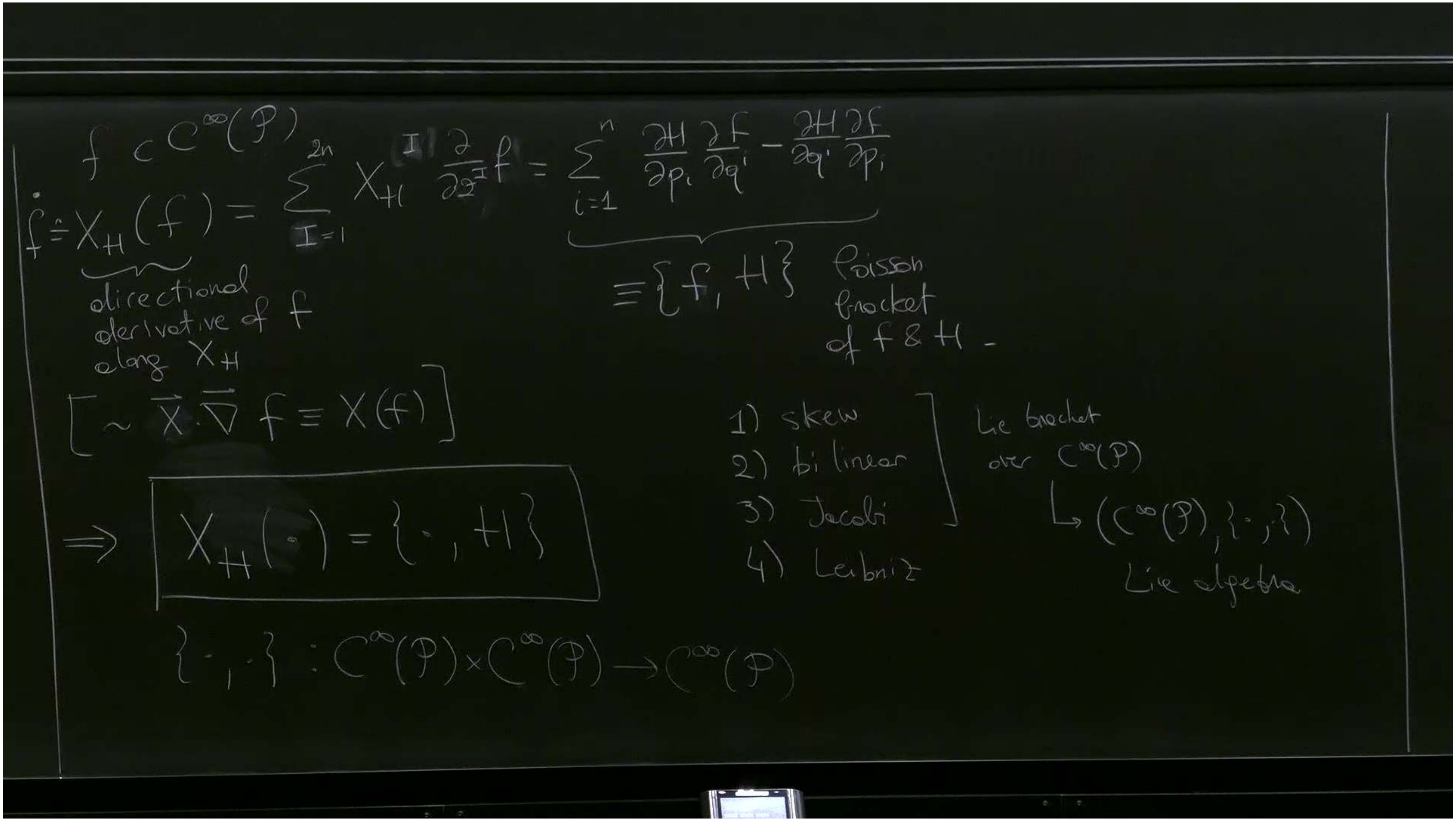
Hamiltonian
v.f.

$f \in C^\infty(\mathcal{P}$

$\underbrace{X_H}_{}(f) =$

directional
derivative of
along X_H

$[\sim \vec{X}\cdot\vec{\nabla} f$



$f \in C^{\infty}(\mathcal{P})$

$$\dot f \doteq \underbrace{X_H}(f) = \sum_{I=1}^{2n} X_H^{I} \frac{\partial}{\partial z^I} f = \underbrace{\sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i}}$$

directional derivative of f along X_H

$\equiv \{f, H\}$ Poisson bracket of f & H.

$$\left[\sim \vec{X}\cdot\vec{\nabla} f \equiv X(f)\right]$$

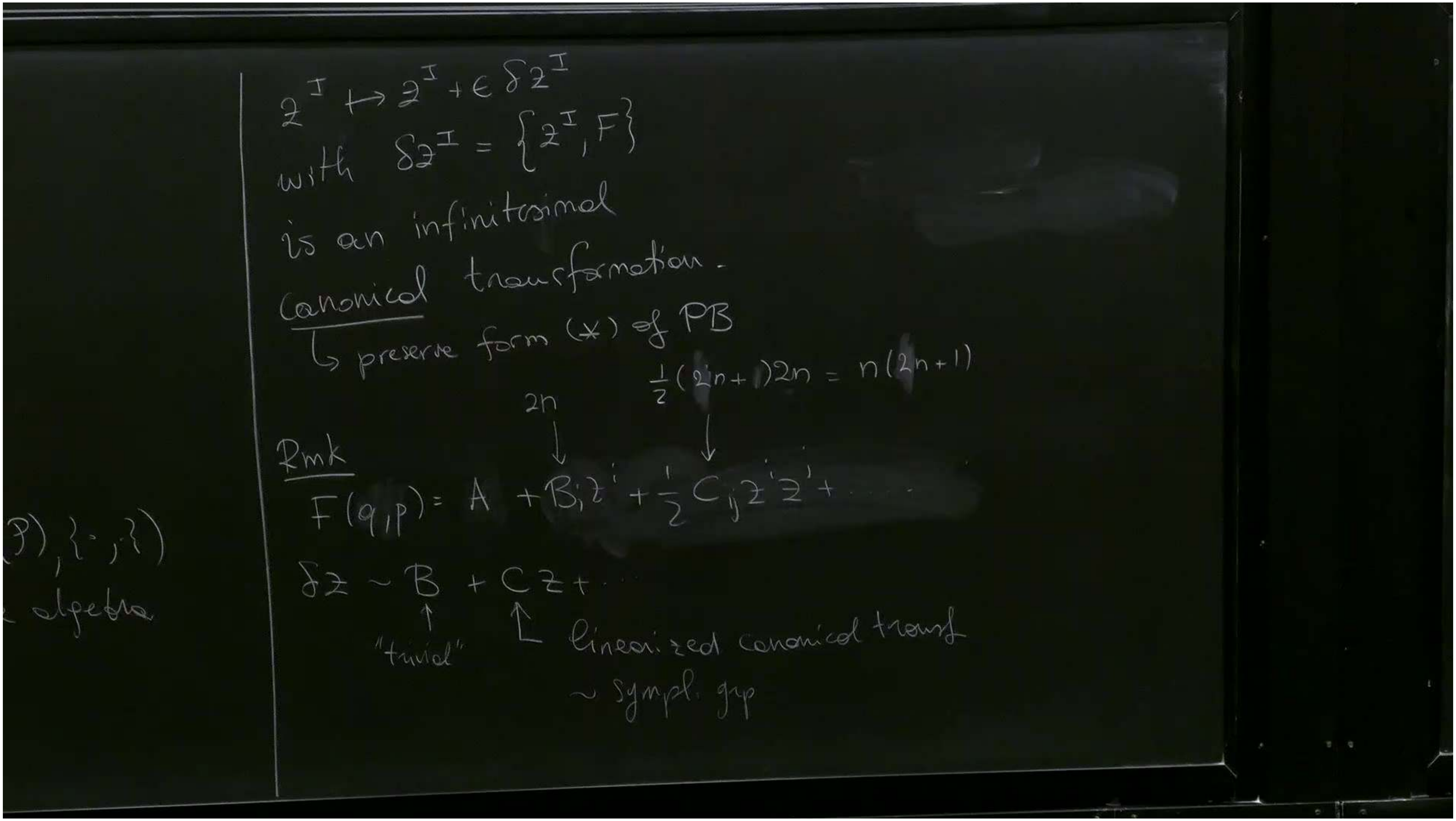
$$\Longrightarrow \boxed{X_H(\cdot) = \{\cdot, H\}}$$

$$\{\cdot,\cdot\} : C^{\infty}(\mathcal{P}) \times C^{\infty}(\mathcal{P}) \longrightarrow C^{\infty}(\mathcal{P})$$

1) skew
2) bi linear
3) Jacobi
4) Leibniz

] Lie bracket over $C^{\infty}(\mathcal{P})$

$\hookrightarrow (C^{\infty}(\mathcal{P}), \{\cdot,\cdot\})$ Lie algebra



$(\mathcal{P}), \{\cdot,\cdot\})$

algebra

$z^I \mapsto z^I + \epsilon\, \delta z^I$

with $\delta z^I = \{z^I, F\}$

is an infinitesimal

canonical transformation.

↳ preserve form (∗) of PB

$\frac{1}{2}(2n+\ )2n = n(2n+1)$

$2n$

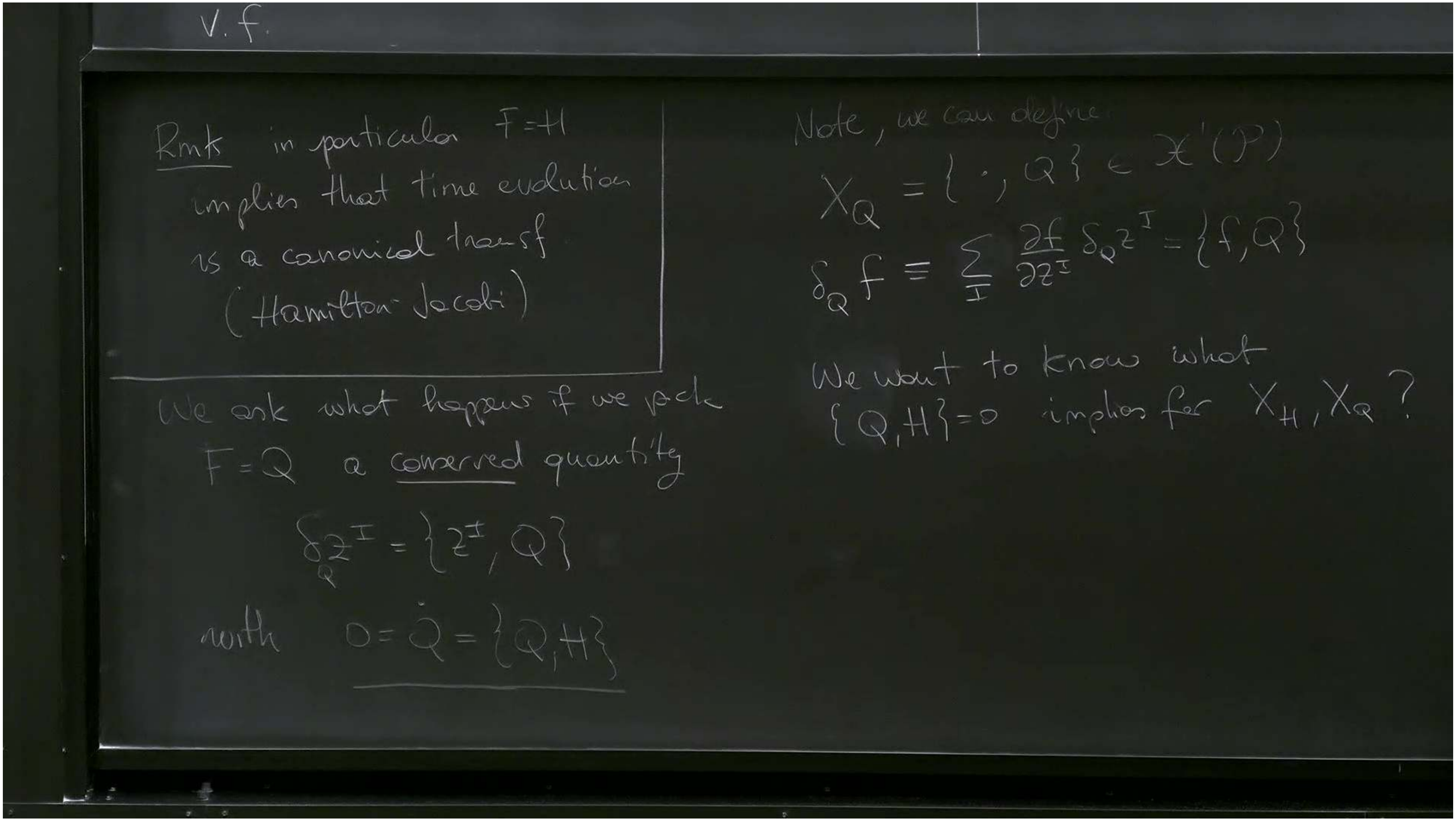
Rmk

$$F(q,p) = A + \overset{\overset{2n}{\downarrow}}{B_i z^i} + \frac{1}{2}\overset{\downarrow}{C_{ij}} z^i z^j + \cdots$$

$$\delta z \sim \underset{\substack{\uparrow \\ \text{"trivial"}}}{B} + C z + \cdots$$

↳ linearized canonical transf

~ sympl. grp



V. f.

Rmk in particular $F = H$
implies that time evolution
is a canonical transf
(Hamilton-Jacobi)

We ask what happens if we pick
$F = Q$ a conserved quantity

$$\delta_Q z^I = \{z^I, Q\}$$

with $0 = \dot{Q} = \{Q, H\}$

Note, we can define

$$X_Q = \{\cdot, Q\} \in \mathfrak{X}^1(\mathcal{P})$$

$$\delta_Q f \equiv \sum_I \frac{\partial f}{\partial z^I} \delta_Q z^I = \{f, Q\}$$

We want to know what
$\{Q, H\} = 0$ implies for X_H, X_Q ?

Mathematical aside

$(C^\infty(P), \{\cdot,\cdot\})$ is a Lie algebra

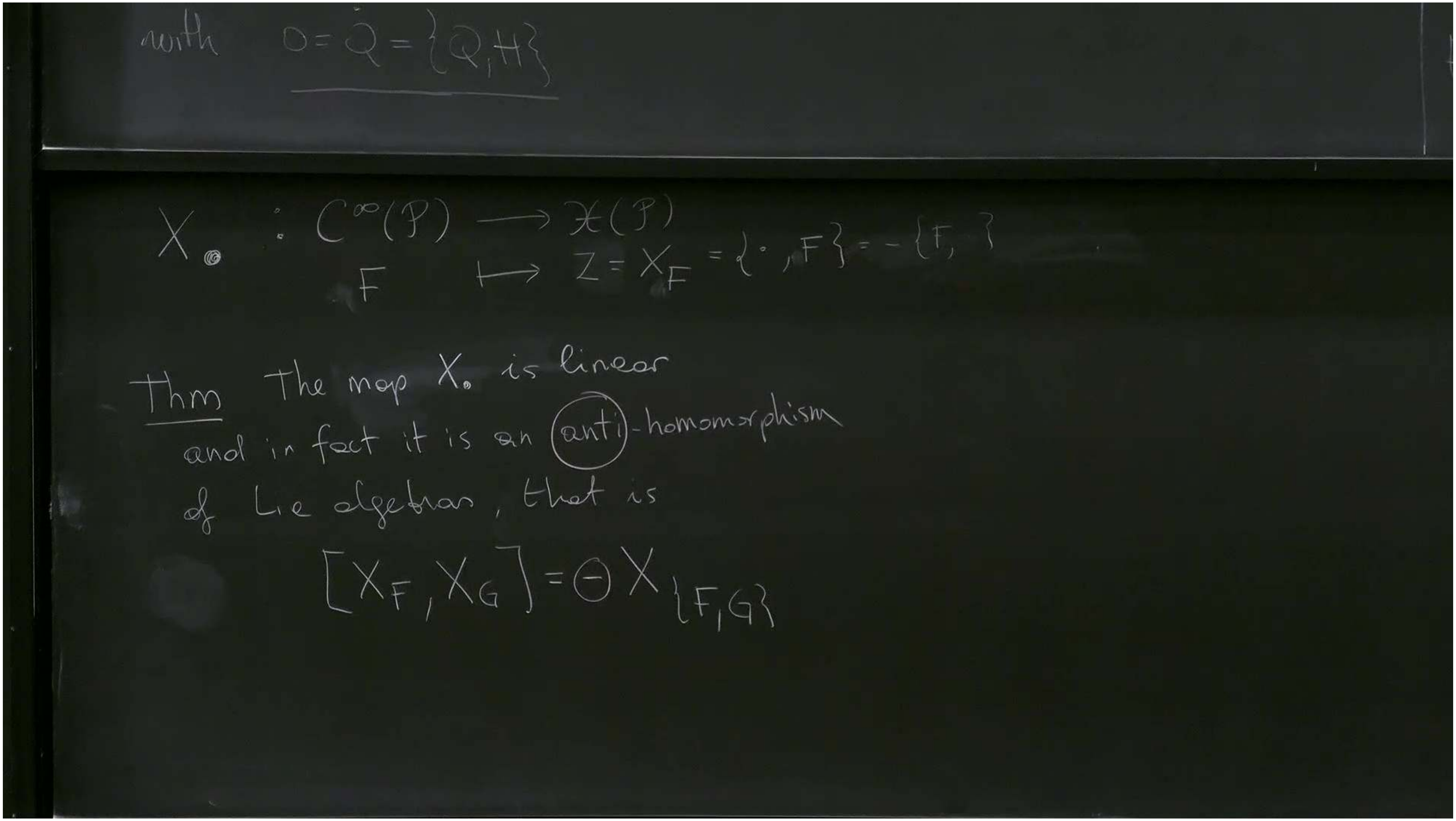
$(\underbrace{\mathfrak{X}(P)}_{\text{sp. of all v.f on } P}, [\cdot,\cdot])$ is another Lie alg.

where $[\cdot,\cdot] : \mathfrak{X}(P) \times \mathfrak{X}(P) \longrightarrow \mathfrak{X}(P)$

$(X \,,\, Y) \mapsto Z = [X,Y]$

defined by $Z(f) \equiv [X,Y](f) = X(Y(f)) - Y(X(f)) = Z^I \partial_I f + \cancel{A^{IJ} \partial_I \partial_J f}$

Exercise: use $\frac{\partial^2}{\partial z^I \partial z^J} = \frac{\partial^2}{\partial z^J \partial z^I}$ to show $Z^I = [X,Y]^I = X^J \partial_J Y^I - Y^J \partial_J X^I$

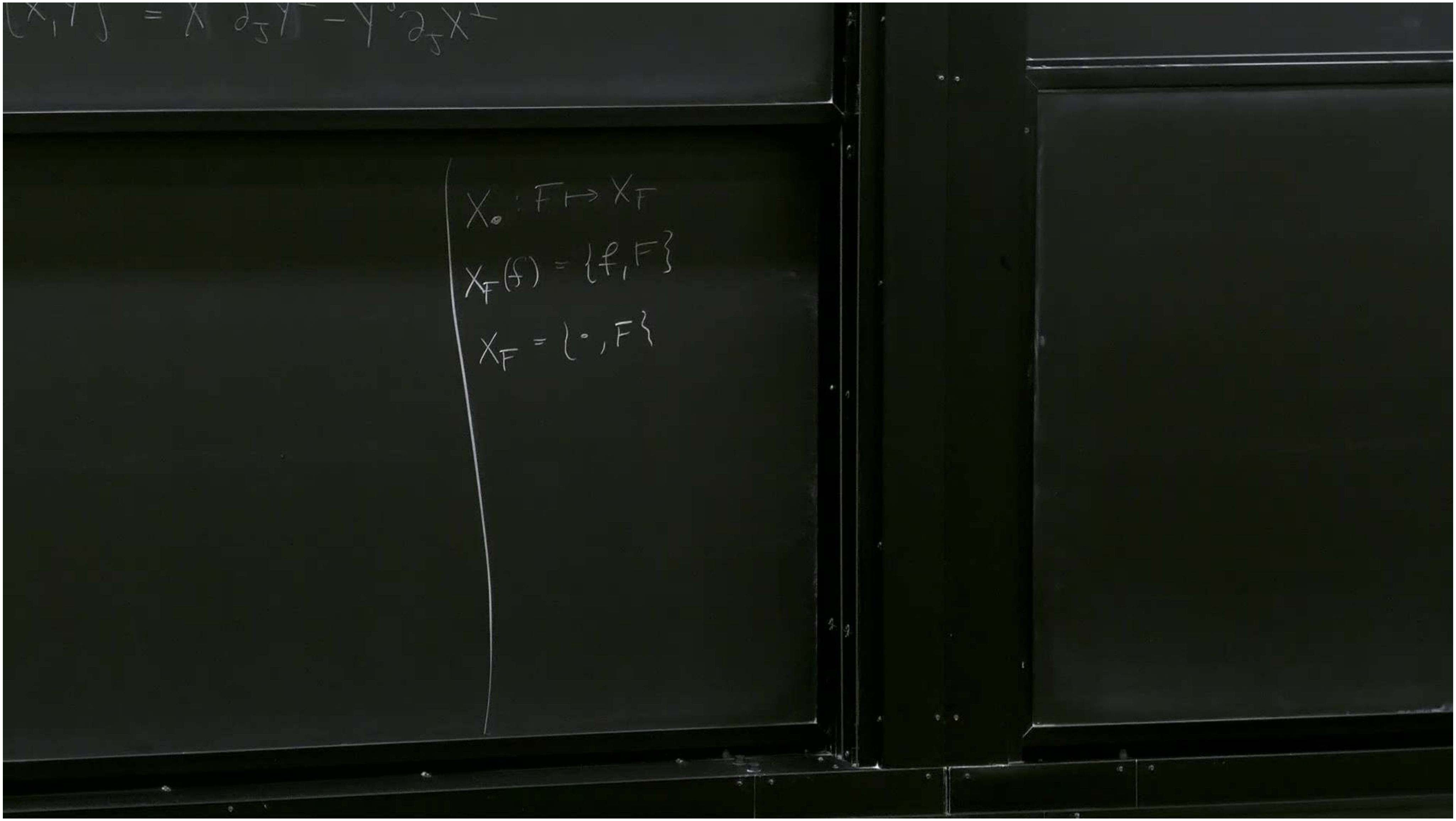


with $\underline{0 = \dot{Q} = \{Q, H\}}$

$$X_\bullet : C^\infty(\mathcal{P}) \longrightarrow \mathfrak{X}(\mathcal{P})$$
$$F \longmapsto Z = X_F = \{\cdot, F\} = -\{F, \ \}$$

<u>Thm</u> The map $X_\bullet$ is linear and in fact it is an anti-homomorphism of Lie algebras, that is

$$[X_F, X_G] = \ominus X_{\{F,G\}}$$



$[X, Y] = X \partial_J Y^I - Y^J \partial_J X^I$

$X_\bullet : F \mapsto X_F$

$X_F(f) = \{f, F\}$

$X_F = \{\cdot, F\}$

Proof

$$
\begin{aligned}
[X_F, X_G](f) &= X_F(X_G(f)) - X_G(X_F(f)) \\
&= X_F(\{f,G\}) - X_G(\{f,F\}) \\
&= \{\{f,G\},F\} - \{\{f,F\},G\} \\
\text{skew} \quad &= -\{\{G,f\},F\} - \{\{f,F\},G\} \\
\text{Jacobi} \quad &= \{\{F,G\},f\} \\
\text{skew} \quad &= -\{f,\{F,G\}\} = -X_{\{F,G\}}(f) \qquad \square
\end{aligned}
$$

)

})

G}

G}

G}(f) $\square$

Rmk

The image of $X_\bullet$ in $\mathfrak{X}(\mathcal{P})$ is called the space of Hamiltonian vector fields $\mathfrak{X}_{\mathrm{Ham}}(\mathcal{P}) \subsetneq \mathfrak{X}(\mathcal{P})$.

Being Hamiltonian is being special.

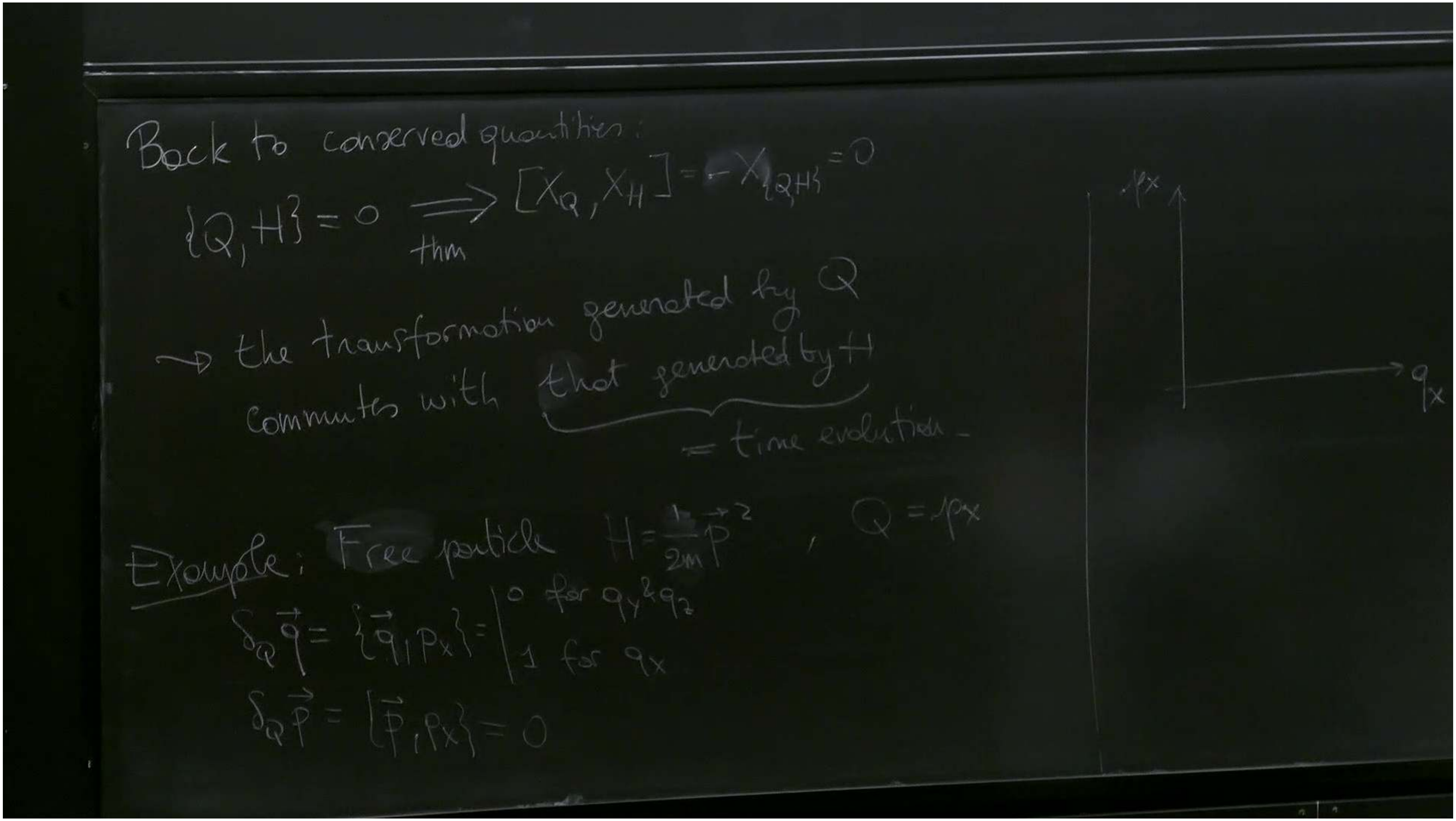
Namely: $Z \in \mathfrak{X}(\mathcal{P})$ Hamiltonian if $\exists F \in C^\infty(\mathcal{P})$ such that

$$Z = X_F$$

$$X_\bullet : F \mapsto X_F$$

$$X_F(f) = \{f, F\}$$

$$X_F = \{\cdot, F\}$$



Back to conserved quantities:

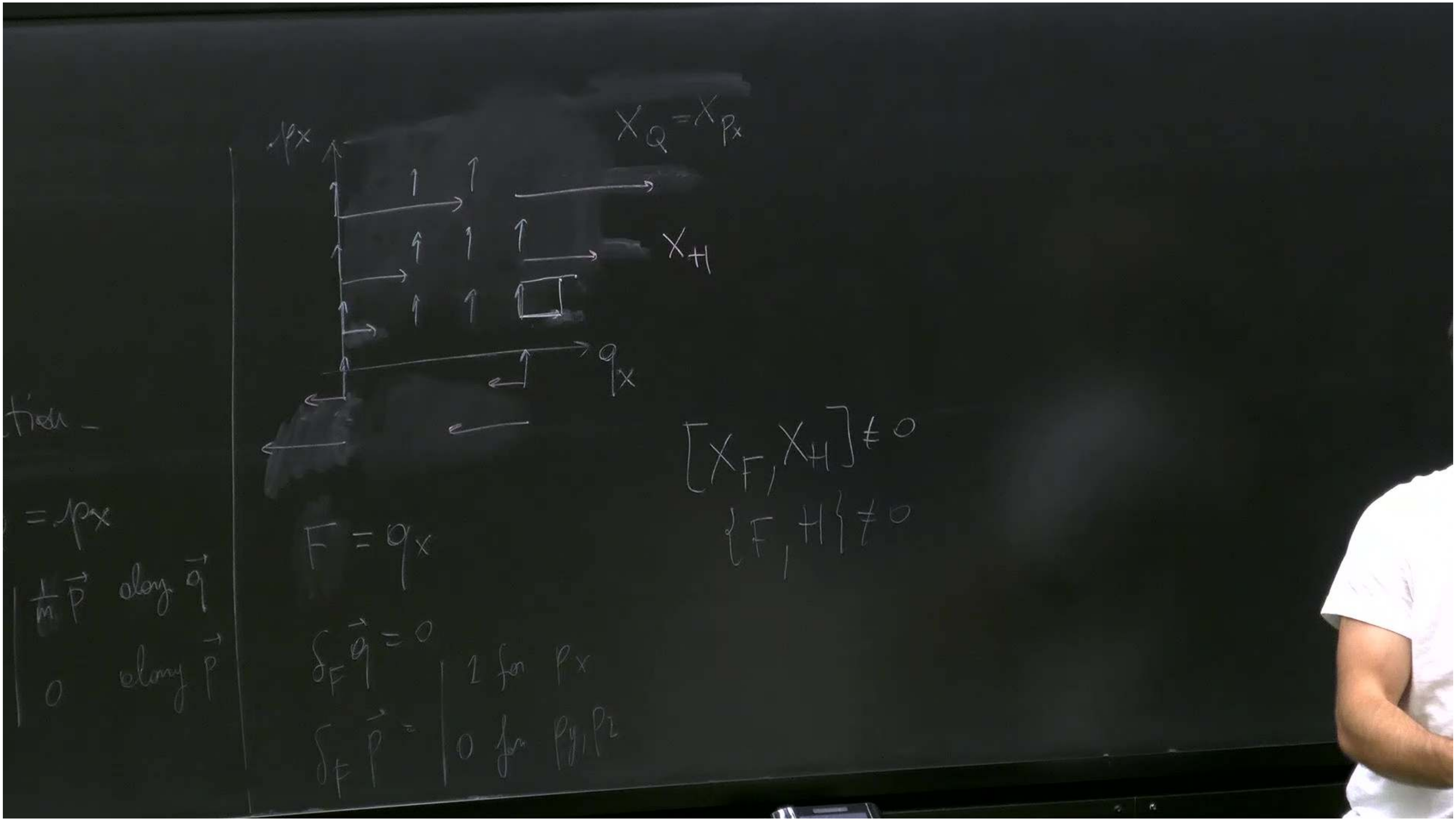
$$\{Q, H\} = 0 \underset{\text{thm}}{\Longrightarrow} [X_Q, X_H] = -X_{\{Q,H\}} = 0$$

→ the transformation generated by Q commutes with $\underbrace{\text{that generated by } H}_{= \text{ time evolution}}$.

Example: Free particle $H = \frac{1}{2m}\vec{p}^{\,2}$, $Q = p_x$

$$\delta_Q \vec{q} = \{\vec{q}, p_x\} = \begin{cases} 0 & \text{for } q_y \,\&\, q_z \\ 1 & \text{for } q_x \end{cases}$$

$$\delta_Q \vec{p} = \{\vec{p}, p_x\} = 0$$



tion_

$= p_x$

$\frac{1}{m}\vec{p}$ along $\vec{q}$

0 along $\vec{p}$

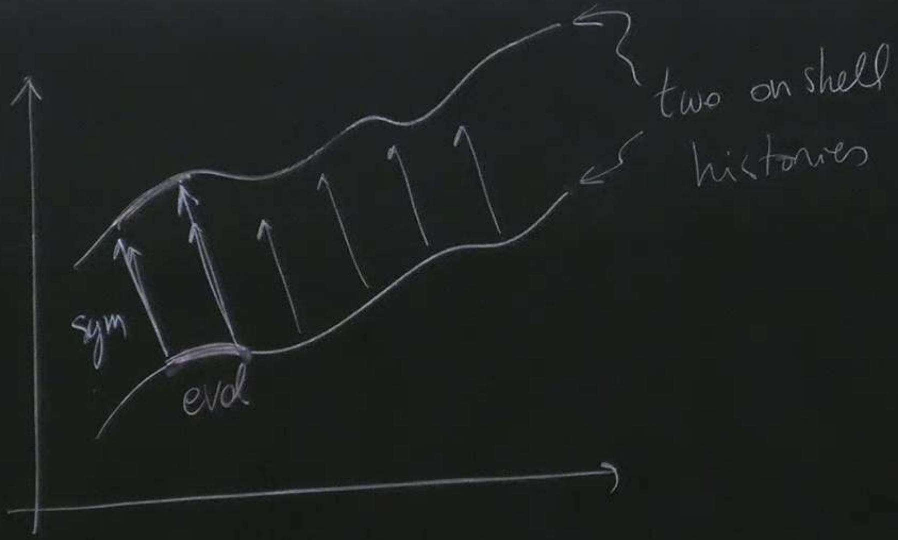
$F = q_x$

$\delta_F \vec{q} = 0$

$\delta_F \vec{p} = \begin{cases} 1 \text{ for } p_x \\ 0 \text{ for } p_y, p_z \end{cases}$

$[X_F, X_H] \neq 0$

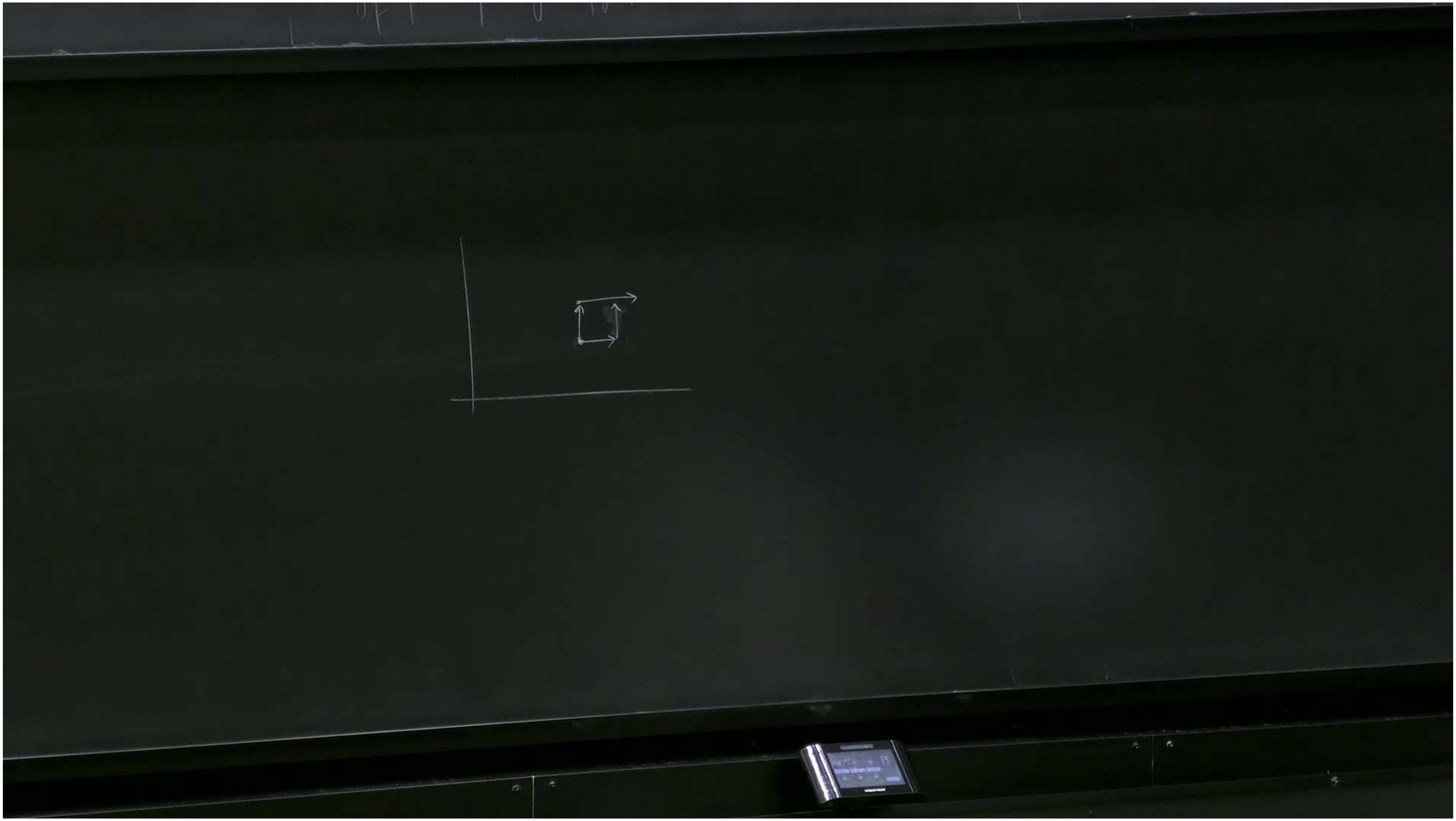
$\{F, H\} \neq 0$



For Q a conserved quantity it always closes

$$\{Q, H\} = 0$$

$$[X_Q, X_H] = 0$$



Rmk

We could still have

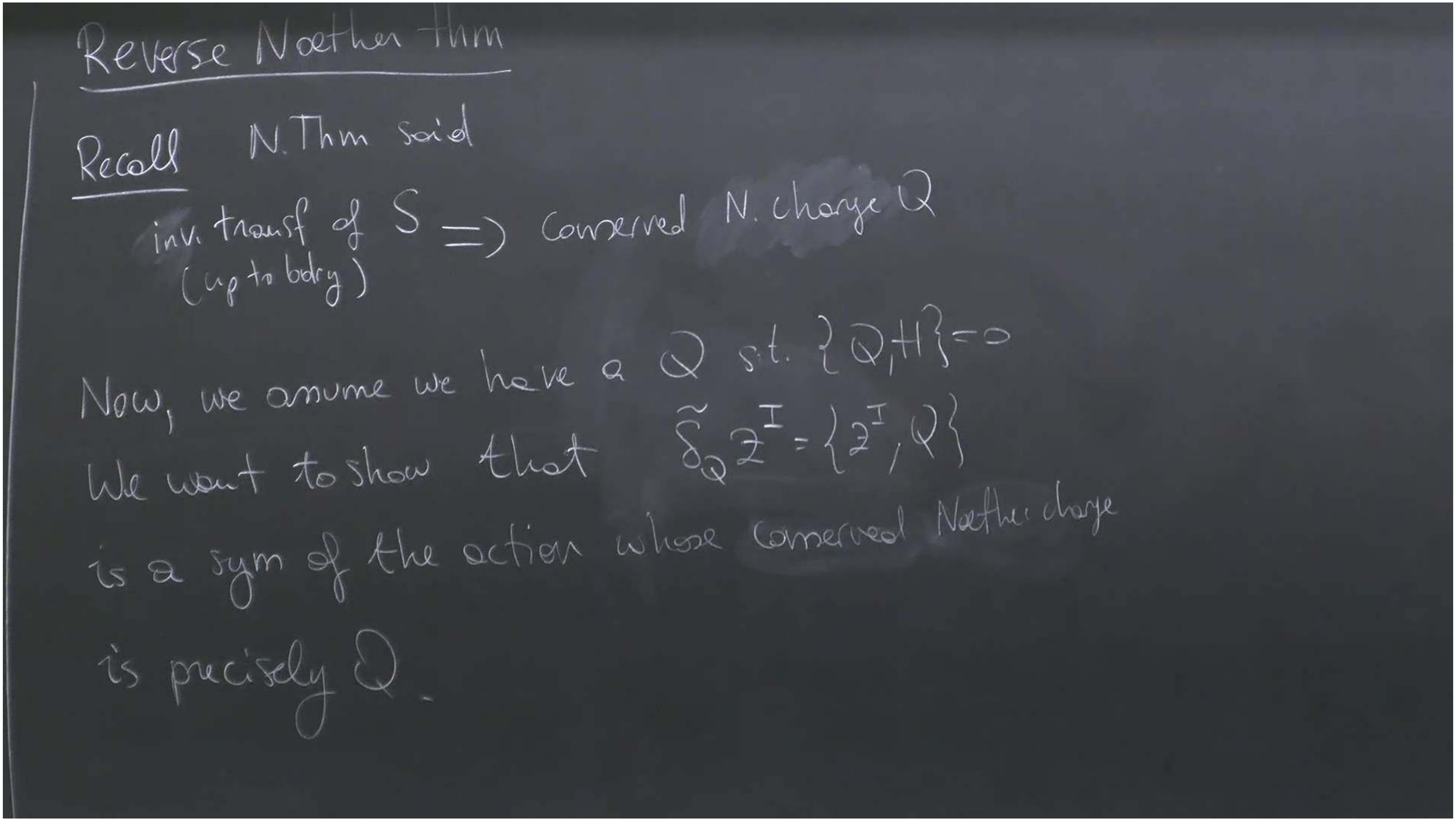
$$[X_F, X_G] = 0$$

with $\{F, G\} \neq 0$

But this means that

$$0 = X_{\{F,G\}} = \{\cdot, \{F,G\}\}$$

which is possible only if $\{F, G\} = \text{const}$



Reverse Noether thm

Recall N. Thm said

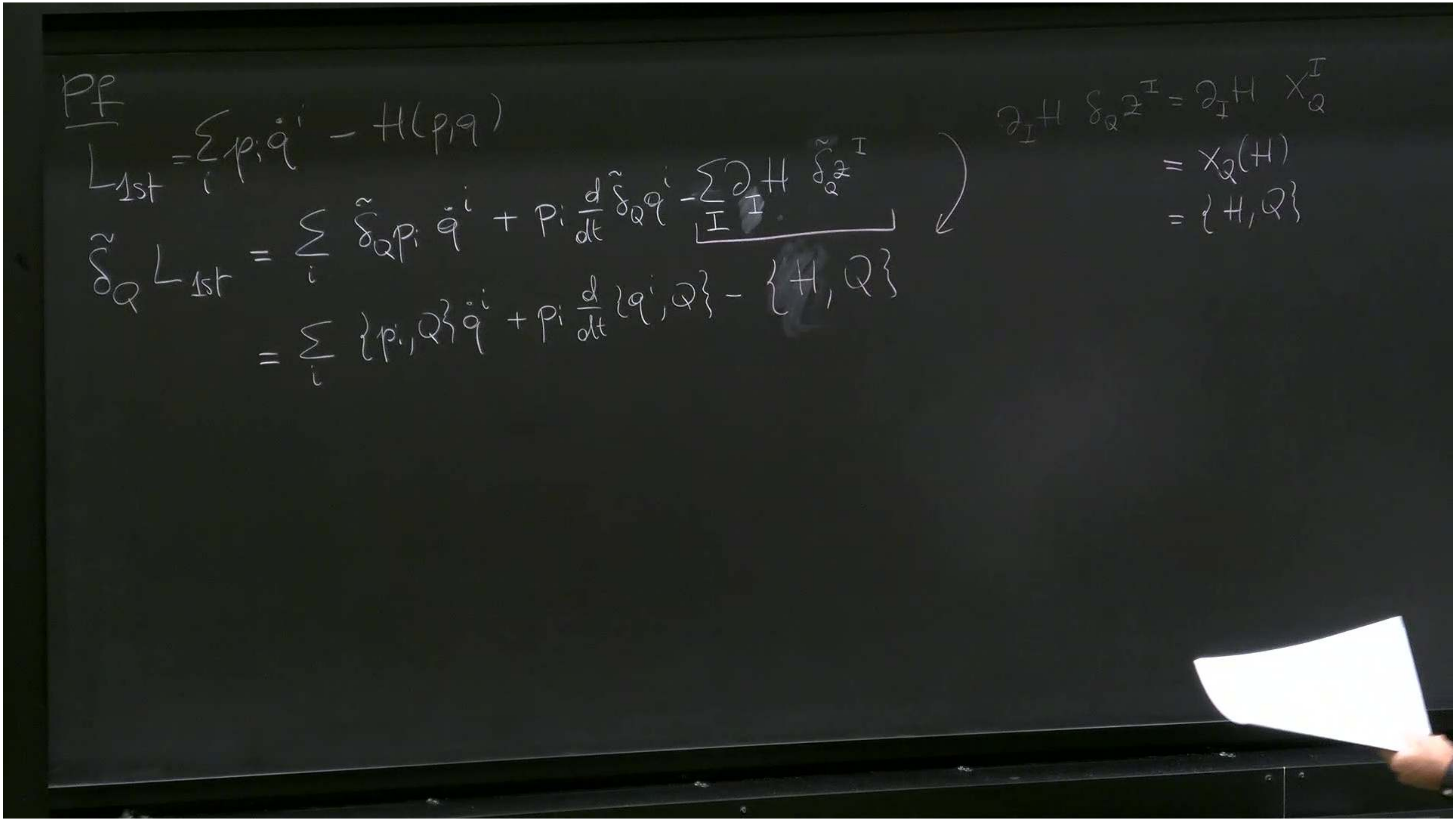
inv. transf of S $\Longrightarrow$ conserved N. charge Q
(up to bdry)

Now, we assume we have a Q s.t. $\{Q, H\} = 0$

We want to show that $\tilde{\delta}_Q z^I = \{z^I, Q\}$

is a sym of the action whose conserved Noether charge

is precisely Q.



Pf

$$L_{1st} = \sum_i p_i \dot{q}^i - H(p,q)$$

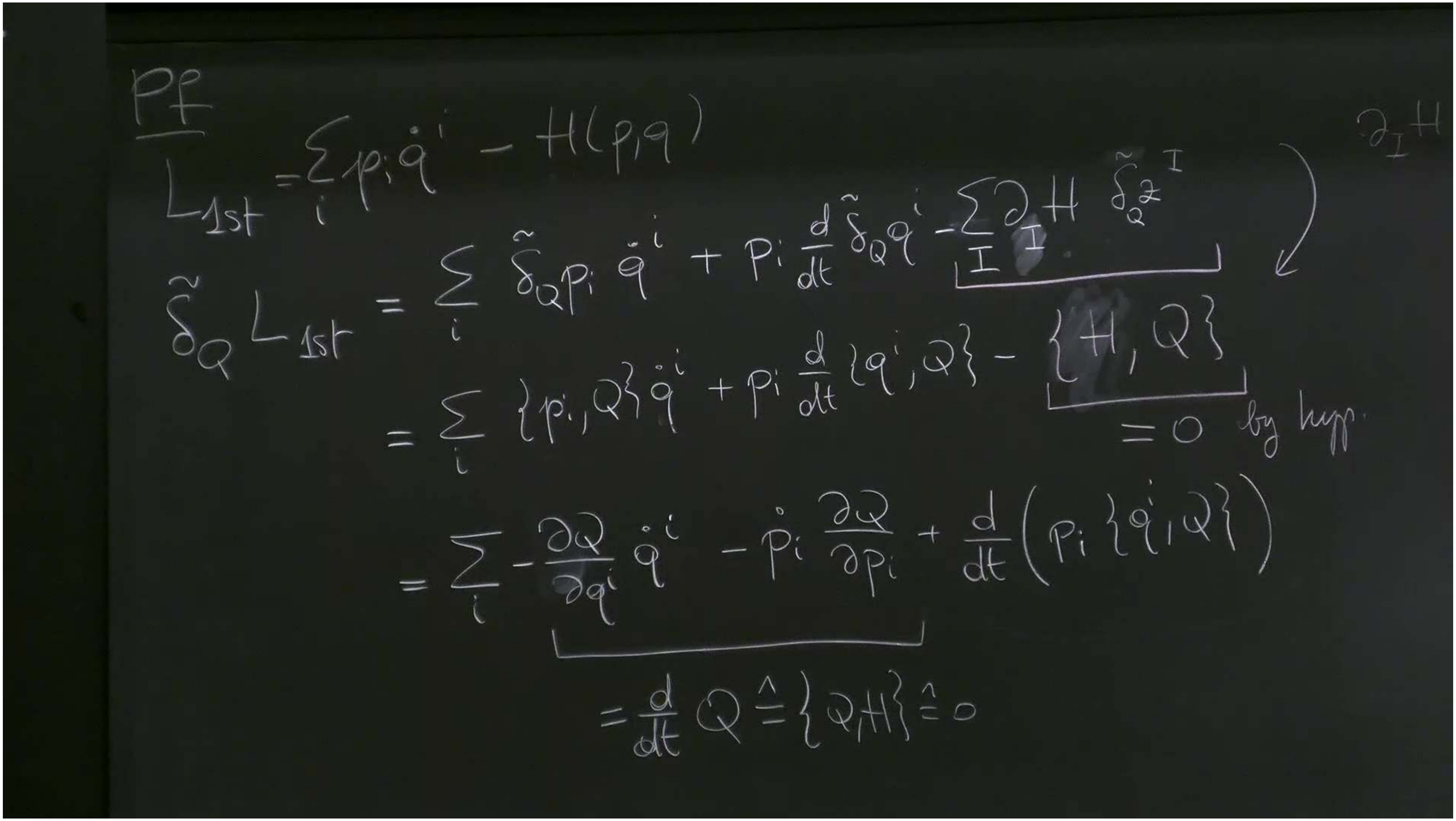
$$\tilde{\delta}_Q L_{1st} = \sum_i \tilde{\delta}_Q p_i \dot{q}^i + p_i \frac{d}{dt} \tilde{\delta}_Q q^i - \underbrace{\sum_I \partial_I H \tilde{\delta}_Q z^I}$$

$$= \sum_i \{p_i, Q\} \dot{q}^i + p_i \frac{d}{dt} \{q^i, Q\} - \{H, Q\}$$

$$\partial_I H \delta_Q z^I = \partial_I H X_Q^I$$

$$= X_Q(H)$$

$$= \{H, Q\}$$



Pf

$$L_{1st} = \sum_i p_i \dot{q}^i - H(p,q)$$

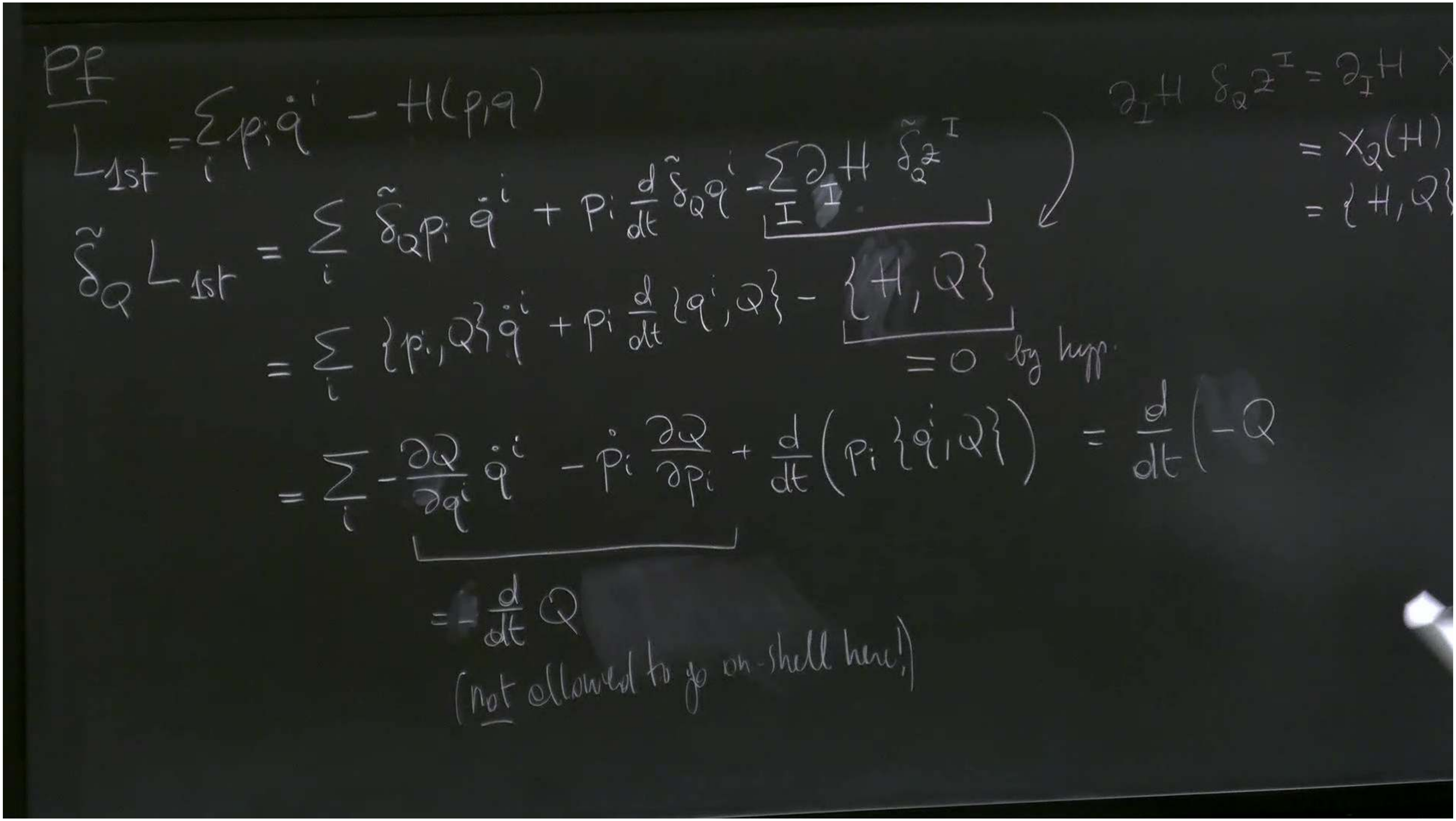
$$\partial_I H$$

$$\tilde{\delta}_Q L_{1st} = \sum_i \tilde{\delta}_Q p_i \, \dot{q}^i + p_i \frac{d}{dt} \tilde{\delta}_Q q^i - \underbrace{\sum_I \partial_I H \, \tilde{\delta}_Q z^I}$$

$$= \sum_i \{p_i, Q\} \dot{q}^i + p_i \frac{d}{dt} \{q^i, Q\} - \underbrace{\{H, Q\}}_{= 0 \text{ by hyp.}}$$

$$= \sum_i \underbrace{-\frac{\partial Q}{\partial q^i} \dot{q}^i - \dot{p}_i \frac{\partial Q}{\partial p_i}} + \frac{d}{dt}\left(p_i \{q^i, Q\}\right)$$

$$= \frac{d}{dt} Q \hat{=} \{Q, H\} \hat{=} 0$$



Pf

$$L_{1st} = \sum_i p_i \dot{q}^i - H(p,q)$$

$$\tilde{\delta}_Q L_{1st} = \sum_i \tilde{\delta}_Q p_i \, \dot{q}^i + p_i \frac{d}{dt} \tilde{\delta}_Q q^i - \underbrace{\sum_I \partial_I H \, \tilde{\delta}_Q z^I}$$

$$\partial_I H \, \delta_Q z^I = \partial_I H \, X$$

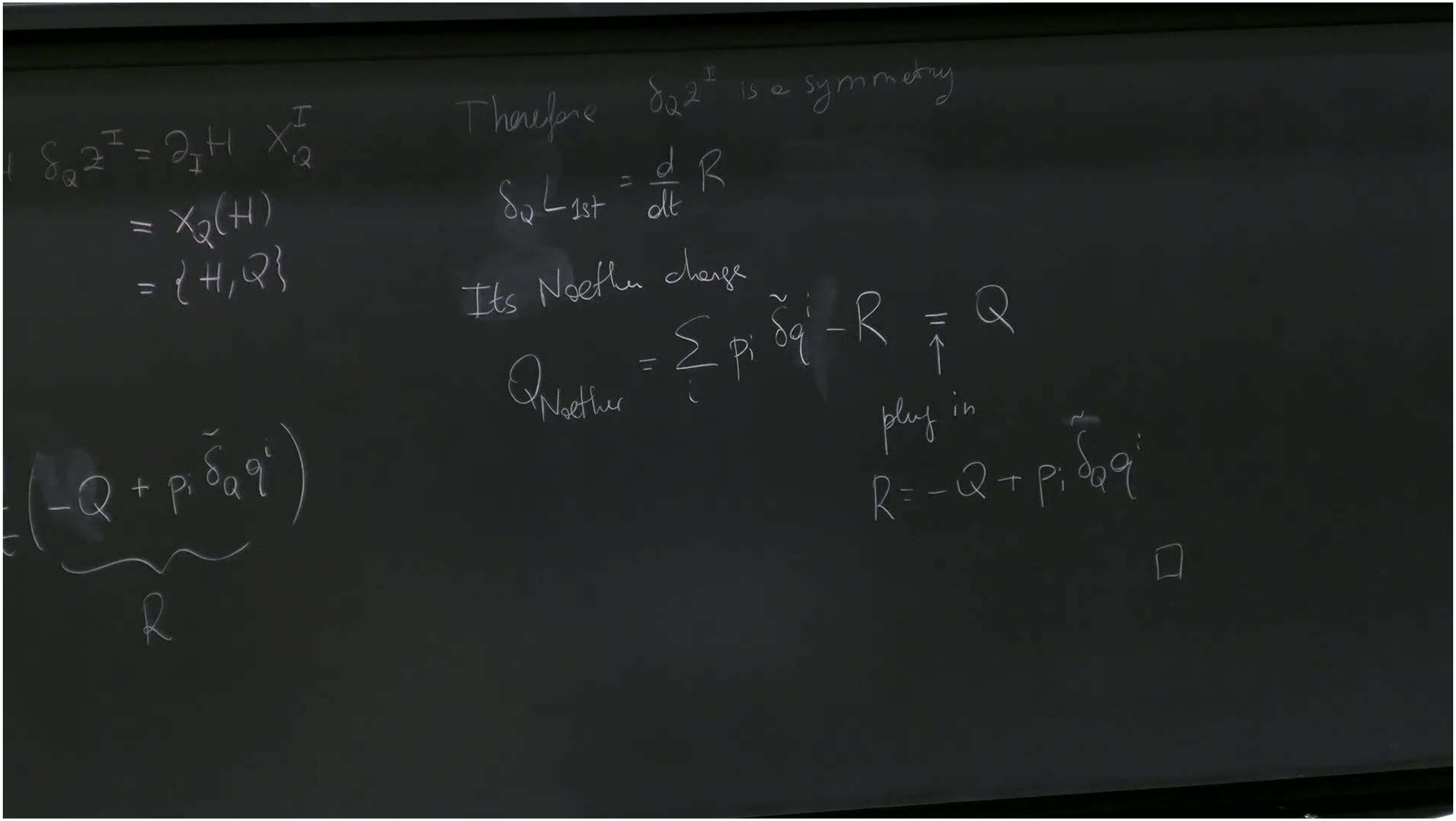
$$= X_Q(H)$$

$$= \{ H, Q \}$$

$$= \sum_i \{p_i, Q\} \dot{q}^i + p_i \frac{d}{dt} \{q^i, Q\} - \underbrace{\{H, Q\}}_{= 0 \text{ by hyp.}}$$

$$= \sum_i \underbrace{- \frac{\partial Q}{\partial q^i} \dot{q}^i - \dot{p}_i \frac{\partial Q}{\partial p_i}}_{= -\frac{d}{dt} Q} + \frac{d}{dt}\left(p_i \{q^i, Q\} \right) = \frac{d}{dt}\Big(-Q$$

(not allowed to go on-shell here!)



$$\delta_Q z^I = \partial_I H \; X_Q^I$$

$$= X_Q(H)$$

$$= \{H, Q\}$$

$$\Big(\underbrace{-Q + p_i \tilde{\delta}_Q q^i}_{R}\Big)$$

Therefore $\delta_Q z^I$ is a symmetry

$$\delta_Q L_{1st} = \frac{d}{dt} R$$

Its Noether charge

$$Q_{Noether} = \sum_i p_i \tilde{\delta} q^i - R \underset{\uparrow}{=} Q$$

plug in

$$R = -Q + p_i \tilde{\delta}_Q q^i$$

$\square$