Title: Lecture - Classical Physics, PHYS 612

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RECAP

$L = L(q, v, t)$

} Legendre transform to

$H = H(q, p, t)$

where $p = P(q, v, t) = \frac{\partial L}{\partial v}$

Then Euler-Lagrange eqs

$$\frac{d}{dt}\frac{\partial L}{\partial v}\Big|_{\bar{q}} - \frac{\partial L}{\partial q}\Big|_{\bar{q}} = 0$$

which are 2nd order ODE

become

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which are 2nd order ODE

$\eta : \mathbb{R} \longrightarrow Q$

become the Hamilton's eqs

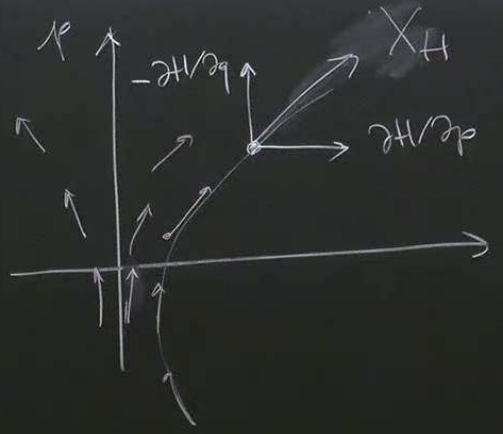
$$\begin{cases} \dot{q} = \left.\frac{\partial H}{\partial p}\right|_{(q,\pi)} \\ \dot{\pi} = -\left.\frac{\partial H}{\partial q}\right|_{(q,\pi)} \end{cases}, \quad z = \begin{pmatrix} q \\ \pi \end{pmatrix} \qquad \begin{pmatrix} \eta \\ \pi \end{pmatrix} : \mathbb{R} \longrightarrow T^*Q$$

First order action principle

$$S_{1st}[z(t)] = \int dt \left(p\dot{q} - H(p, q, t) \right) \qquad \text{w/ b.c. } \delta q = 0$$

Geometrical interpretation in terms of a Hamiltonian vector field

$$X_H(q, p) = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$



$X_\bullet : C^\infty(P) \longrightarrow \mathfrak{X}(P)$

to be studied later!

$H \longmapsto X_H$

Geometrization of Hamilton's eqs

$F : P \times \mathbb{R} \longrightarrow \mathbb{R}$

$(z, t) \longmapsto F(z, t)$

History $\zeta : \mathbb{R} \longrightarrow P = T^*Q$

$t \longmapsto z = \zeta(t) = \begin{pmatrix} q(t) \\ p(t) \end{pmatrix}$

$\bar{\zeta}$ on-shell history, i.e. $\bar{\zeta}$ satisfies eom.

$\dot{\bar{\zeta}}(t) = X_H(\bar{\zeta}(t), t)$

$f_{\bar{\zeta}}(t) = F(\bar{\zeta}(t), t)$

$$\frac{d}{dt} f_{cs} \equiv \frac{d}{dt} F\Big|_\xi = \frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial p_i}\dot{p}_i\Big|_\xi$$

$$= \frac{\partial F}{\partial t} + \sum_i \underbrace{\frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i}}$$

by def: Poisson bracket of F & H:

$$\equiv \{F, H\}$$

Rmk Ehrenfest thm: $i\hbar \frac{d}{dt}\langle \hat{F} \rangle = i\hbar \langle \partial_t \hat{F} \rangle + \langle [\hat{F}, \hat{H}] \rangle$

vs Hamilton's eq: $\frac{d}{dt} F = \partial_t F + \{F, H\}$

Axiomatization of P.B.

$\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P)$

$(F, G) \mapsto \{F, G\}$

s.t. $\forall a \in \mathbb{R}$, $F, G, H \in C^\infty(P)$ it satisfies

1) skew $\{F, G\} + \{G, F\} = 0$

2) ($\mathbb{R}$)-linear $\{aF + G, H\} = a\{F, H\} + \{G, H\}$

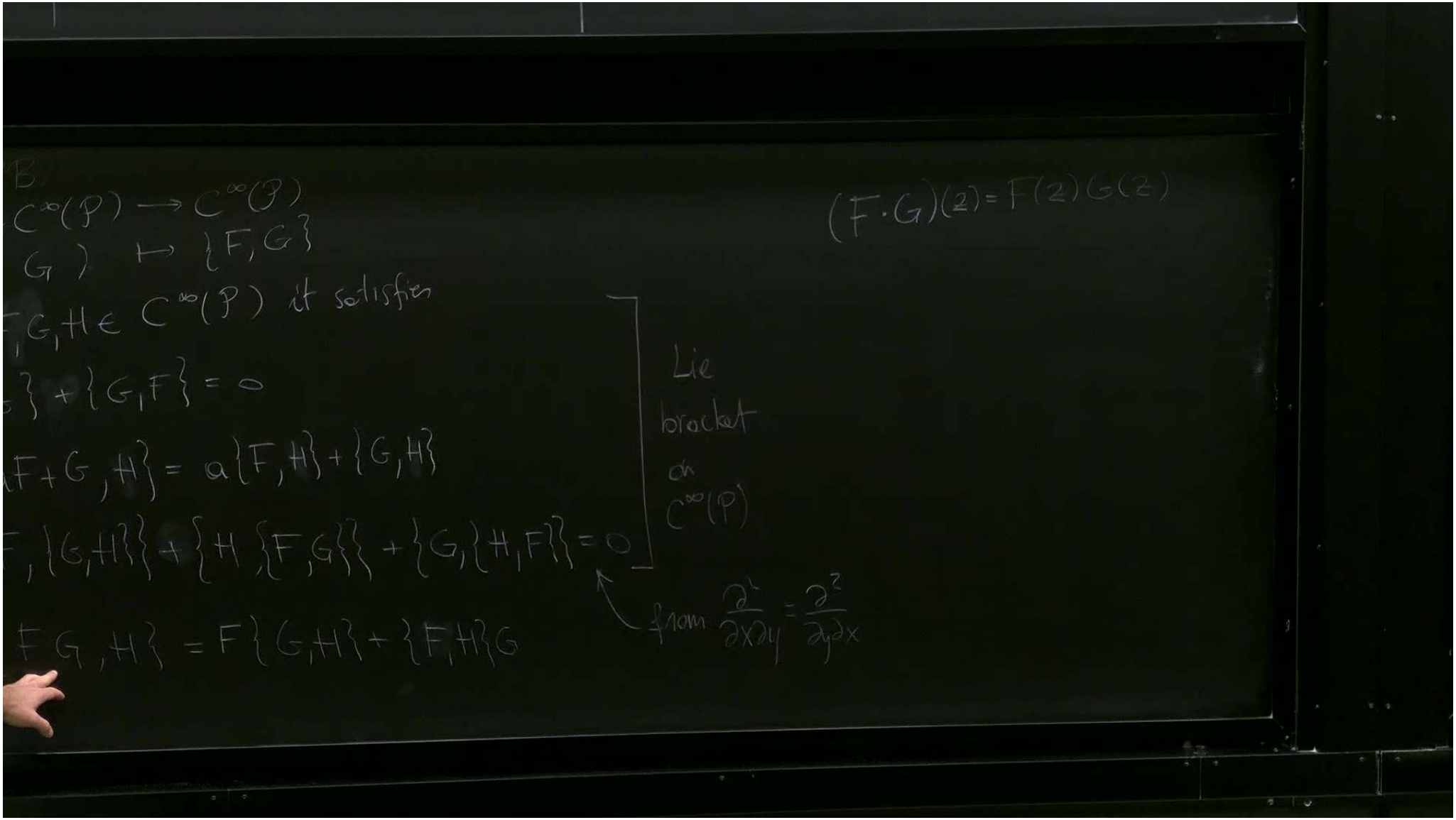
3) Jacobi $\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0$

4) Leibniz $\{FG, H\} = F\{G, H\} + \{F, H\}G$

Lie bracket on $C^\infty(P)$

from $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$

$(F \cdot G)(z) = F(z$


B
$C^\infty(P) \longrightarrow C^\infty(P)$
$G) \mapsto \{F,G\}$
$F,G,H \in C^\infty(P)$ it satisfies
$\} + \{G,F\} = 0$
$aF+G,H\} = a\{F,H\} + \{G,H\}$
$F,\{G,H\}\} + \{H,\{F,G\}\} + \{G,\{H,F\}\} = 0$
$FG,H\} = F\{G,H\} + \{F,H\}G$
$(F\cdot G)(z) = F(z)G(z)$
Lie bracket on $C^\infty(P)$
from $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$

Thm if $\partial_t H = 0$ then it is conserved, i.e.

$$\frac{d}{dt} H\big|_{\bar{\xi}} = 0$$

Pf $\frac{d}{dt} H\big|_{\bar{\xi}} = \underbrace{\partial_t H}_{=0 \text{ by hyp.}} + \{H, H\}\big|_{\bar{\xi}} = 0$ b.c. of (1) ∎

Rmk by Leg. tr. $\partial_t H = 0$ iff $\partial_t L = 0$

(recall Noether for time transl.)

Generalized energy

$$E = H\big|_{\bar{\xi}}$$

$$= Pv - L\big|_{\bar{\eta}}$$

Rmk

$$X_H = -\{H, \cdot\} = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

$F \in C^\infty(P)$

$X_H(F) = -\{H, F\} = \{F, H\}$.

directional der along X_H

Generalized energy

$$E = H|_{\bar{\eta}}$$

$$= Pv - L|_{\bar{\eta}}$$

Rmk

$$X_H = -\{H, \cdot\} = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

$$F \in C^\infty(P)$$

$$\underbrace{X_H(F)}_{\text{directional der. along } X_H} = -\{H, F\} = \{F, H\}.$$

CANONICAL TRANSF

Question define new coords on $\mathcal{P}$

$$\begin{cases} Q^i = Q^i(q,p) \\ P_i = P_i(q,p) \end{cases}$$

When are these still "canonical"?

That is such that:

$$\{f, g\}_{(q,p)} = \{F, G\}_{(Q,P)}$$

when $f(q,p) = F(Q(q,p), P(q,p))$

$g(\ ) = G(\ )$

$$\{f,g\}_{(q,p)} = \sum_a \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}$$

$$= \sum_a \sum_{i,j} \left(\frac{\partial F}{\partial Q^i}\frac{\partial Q^i}{\partial q^a} + \frac{\partial F}{\partial P_i}\frac{\partial P_i}{\partial q^a} \right)\left(\frac{\partial G}{\partial Q^j}\frac{\partial Q^j}{\partial p_a} + \frac{\partial G}{\partial P_j}\frac{\partial P_j}{\partial p_a} \right) - (f \leftrightarrow g)$$

$$= \sum_{i,j} \{Q^i, Q^j\}_{(q,p)} \frac{\partial F}{\partial Q^i}\frac{\partial G}{\partial Q^j}$$

$$+ \{P_i, P_j\}_{(q,p)} \frac{\partial F}{\partial P_i}\frac{\partial G}{\partial P_j}$$

$$+ \{Q^i, P_j\}_{(q,p)} \left(\frac{\partial F}{\partial Q^i}\frac{\partial G}{\partial P_j} - \frac{\partial F}{\partial P_i}\frac{\partial G}{\partial Q^j} \right)$$

$$\overset{!}{=} \{F,G\}_{(Q,P)}$$

it works $\forall F, G$ if

$$\{Q^i, Q^j\}_{(q,p)} = 0$$

$$\{P_i, P_j\}_{(q,p)} = 0$$

$$\{Q^i, P_j\}_{(q,p)} = \delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

$(q,p) \longrightarrow (Q,P)$ canonical transf.

just as

$$\{q^a, q^b\}_{(q,p)} = 0$$

$$\{p_a, p_b\}_{(q,p)} = 0$$

$$\{q^a, p_b\}_{(q,p)} = \delta^a_b$$

Rmk

more general than

$q \longrightarrow Q(q)$

from Lagrangian formalism!

Rmk $n=1$ $\quad P \sim \mathbb{R}^2 \Rightarrow (q,p) = z$

$$\{z^i, z^j\} = \begin{pmatrix} \{q,q\} & \{q,p\} \\ \{p,q\} & \{p,p\} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

↑ symplectic matrix

$(q,p) \longrightarrow (Q,P)$

canonical transf.

1 if $i=j$

0 otherwise.

Rmk

more general than

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↑ symplectic matrix

$\mathbb{R}^2$ metric $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in Cartesian coords

$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ in polar coords

To get a grip on this, look at infinitesimal canonical transf:

$$\begin{cases} Q^i(q,p) = q^i + \epsilon\, \delta q^i(q,p) \\ P_i(\ \ ) = p_i + \epsilon\, \delta p_i(\ \ ) \end{cases}$$

Then

$$\delta^i_j = \{Q^i, P_j\}_{(q,p)} = \delta^i_j + \epsilon\{q^i, \delta p_j\} + \epsilon\{\delta q^i, p_j\} + O(\epsilon^2)$$

$$= \delta^i_j + \epsilon\underbrace{\left(\frac{\partial \delta p_j}{\partial p_i} + \frac{\partial \delta q^i}{\partial q^j}\right)}_{\doteq 0} + O(\epsilon^2)$$

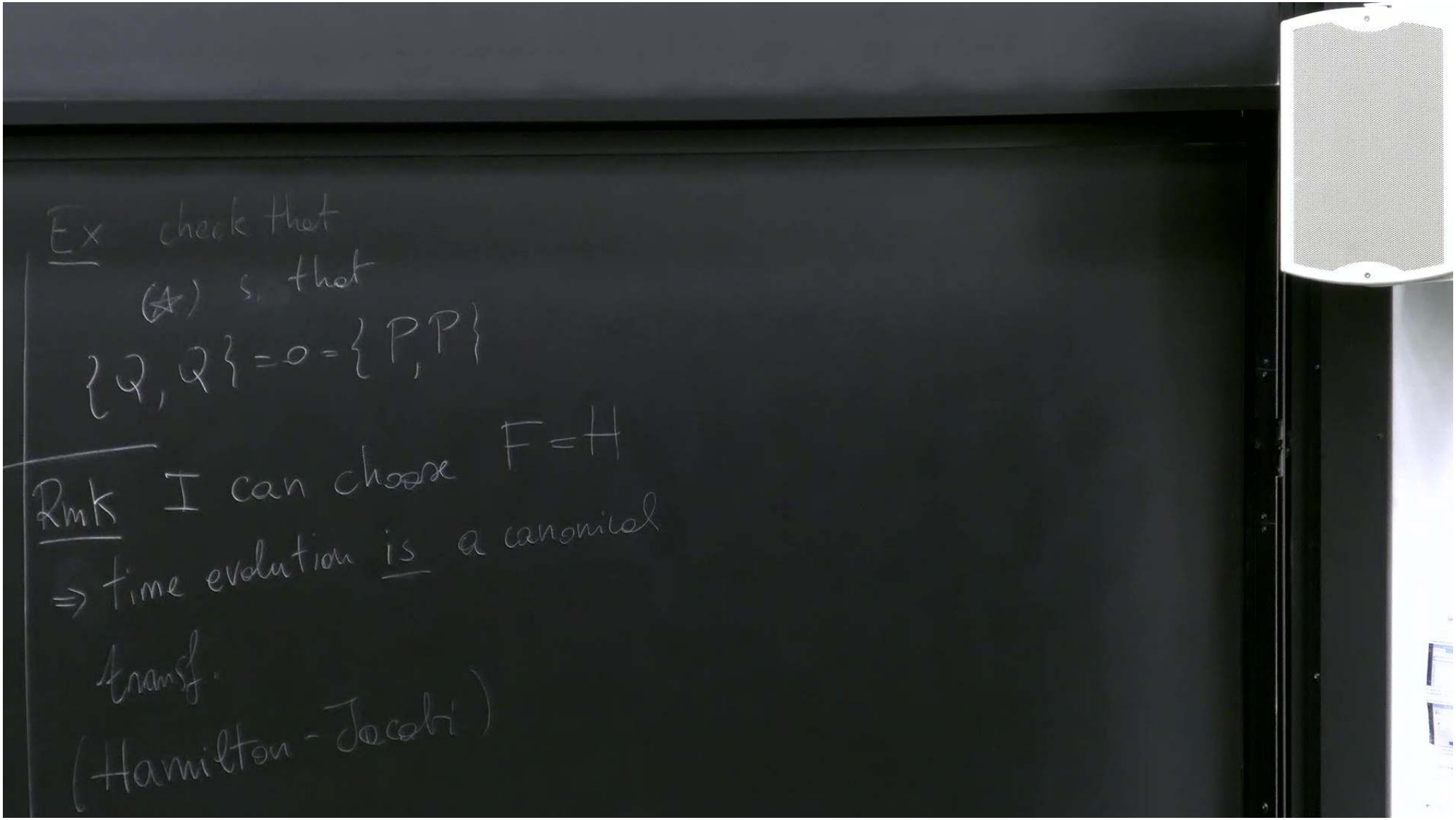
This is solved by

$$\begin{cases} \delta q^i = \dfrac{\partial F}{\partial p_i} = \{q, F\} \\ \delta p_i = -\dfrac{\partial F}{\partial q^i} = \{p, F\} \end{cases}$$

for any $F = F(q, p)$

$\Rightarrow$ Functions on phase space generate, via Poisson bracket, canonical transf

i.e. $X_F = \{\cdot, F\}$ is an infinitesimal canonical transf.


Ex check that
(*) is that
$\{Q,Q\}=0=\{P,P\}$
Rmk I can choose $F=H$
$\Rightarrow$ time evolution is a canonical
transf.
(Hamilton-Jacobi)