Title: Lecture - Combinatorial QFT, CO 739-002

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Collection/Series: Combinatorial QFT, CO 739-002, September 4 - December 2, 2025

Subject: Mathematical physics, Quantum Fields and Strings

Date: September 23, 2025 - 2:00 PM

URL: https://pirsa.org/25090006

No session on Sept 25

Recap:

* Each iso class of a graph with s edges can be represented by $\dfrac{(2s)!}{|\mathrm{Aut}\, G|}$ $[2s]$-labeled graphs

* there are

$$(2s-1)!! \times \frac{(2s)!}{\prod_{k\geq 1} (k!)^{n_k}\, n_k!}$$

Thm 10

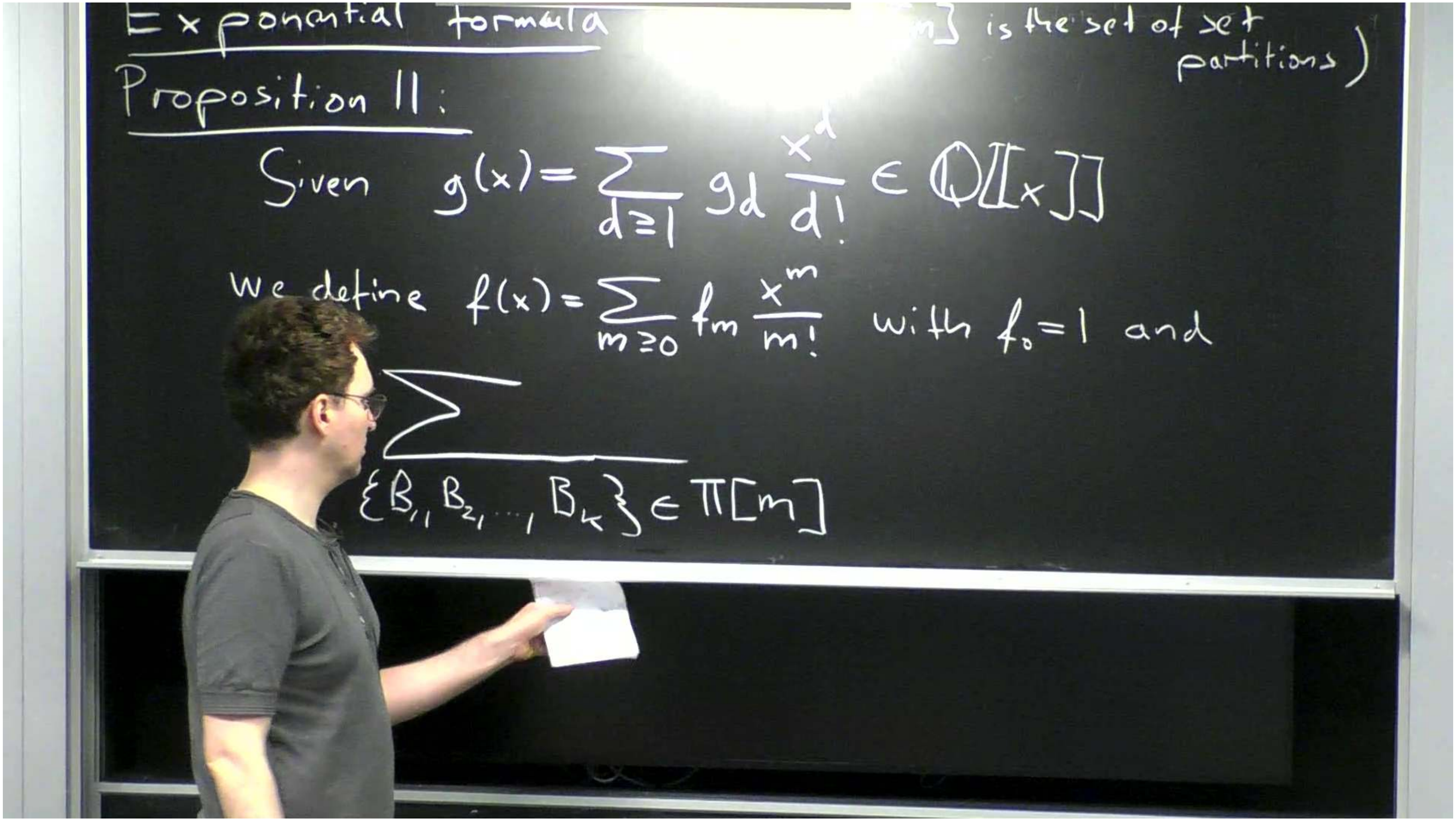
H-labeled graphs with

$n_1, n_2, n_3, \ldots$ vertices of degree $1, 2, 3, \ldots$

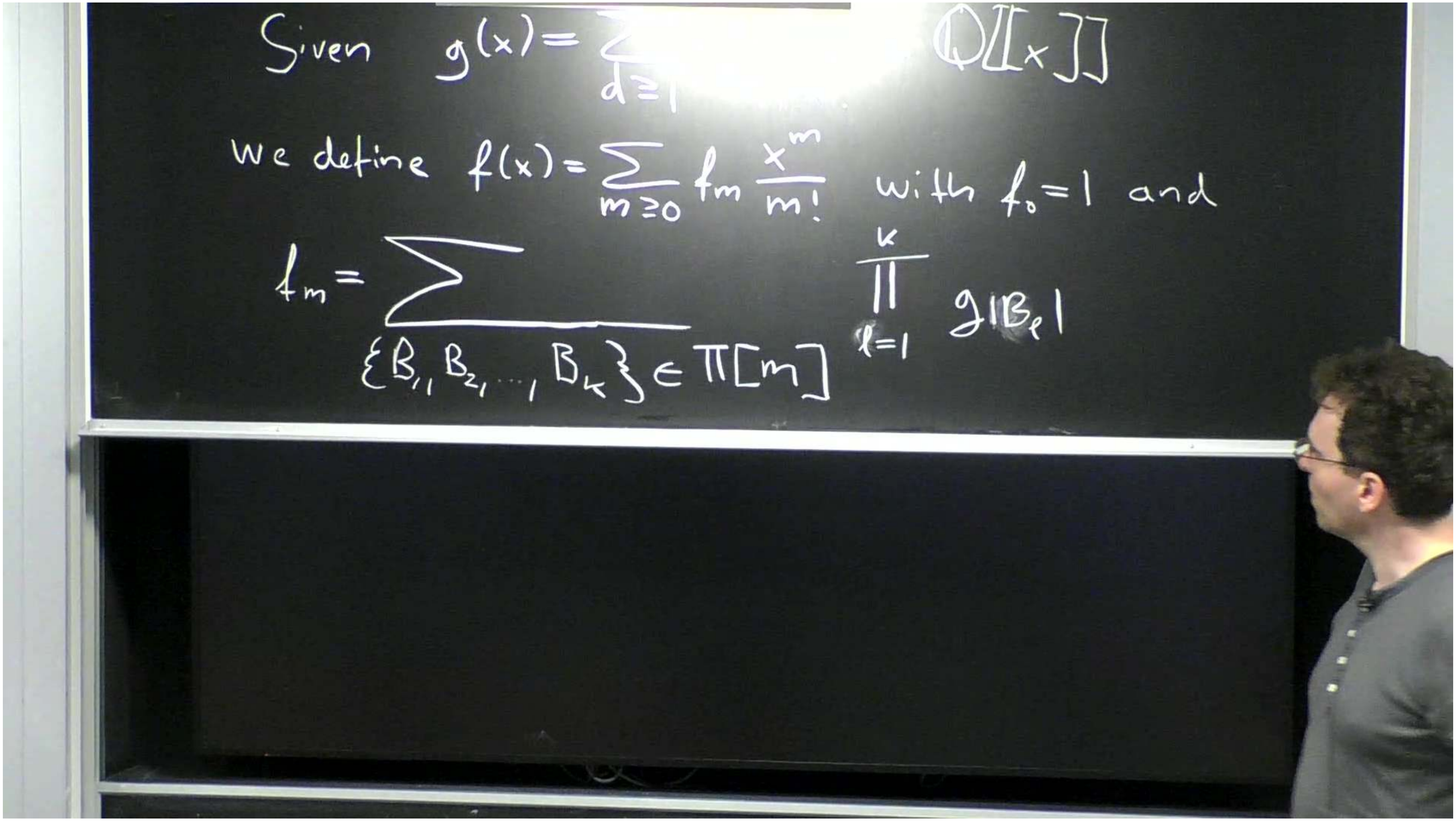
s.t. $|H| = 2s = \sum_{k\geq 1} k\, n_k$

Exponential formula

Proposition II:



Exponential formula
n] is the set of set partitions)
Proposition 11:
Given $g(x)=\sum_{d\geq 1} g_d \frac{x^d}{d!} \in \mathbb{Q}[[x]]$
We define $f(x)=\sum_{m\geq 0} f_m \frac{x^m}{m!}$ with $f_0=1$ and
$\sum_{\{B_1, B_2, \dots, B_k\}\in \Pi[m]}$


Given $g(x)=\sum_{d\geq 1}$ $\mathbb{Q}[[x]]$
We define $f(x)=\sum_{m\geq 0} f_m \frac{x^m}{m!}$ with $f_0=1$ and
$f_m=\sum_{\{B_1,B_2,\dots,B_k\}\in\Pi[m]} \prod_{\ell=1}^{k} g_{|B_\ell|}$

$m \geq 0$ ... $f_0 = 1$ and

$$f_m = \sum_{\{B_1, B_2, \dots, B_k\} \in \Pi[m]} \prod_{\ell=1}^{k} g_{|B_\ell|}, \text{ then } \boxed{f(x) = \exp(g(x))}$$

Proof. Let $\Pi[m](n_1, n_2, \dots) \subset \Pi[m]$ set partions of $\{1, 2, \dots, m\}$

Contains set part. with n_1 blocks of size 1, n_2 blocks of s. 2

$$f_m = \sum_{\substack{n_1, n_2, \dots, n_m \geq 0 \\ \sum_{k=1} k n_k = m}} |\Pi[m](n_1, n_2, \dots)| \prod_{k=1}^{m} g_k^{n_k}$$

$$|\Pi[m](n_1, n_2, \dots)| = \frac{m!}{\prod_{k=1} (k!)^{n_k} n_k!}$$

$$\sum_{m\geq 0} \frac{x^m}{m!} f_m = \sum_{n_1,n_2,\dots} x^{\sum_k k n_k} \frac{1}{\cancel{m!}} \frac{\cancel{m!}}{\prod_{k\geq 1}(k!)^{n_k} n_k!} \prod_{k\geq 1} g_k^{n_k}$$

$$= \exp\left(\frac{g_1}{1!}x\right)\exp\left(\frac{g_2}{2!}x^2\right)\dots \qquad \square$$

Interpretation: Generating function for set partitions.

Generating function of H-labeled graphs

$[x^s] f(x) =$ coefficient of x^s in the power series expansion of $f(x)$.

$k \geq 1$

Theorem 12:

$$\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{G \in \mathcal{G}[m]} w^{|E_G|} \prod_{v \in V_G} \lambda_{|v|} =$$

(the set of $\{1, \ldots, m\}$-labeled graphs)

$$= \sum_{s=0}^{\infty} w^s (2s-1)!! \, [x^{2s}] \exp\left(\sum_{d \geq 1} \lambda_d \frac{x^d}{d!} \right)$$

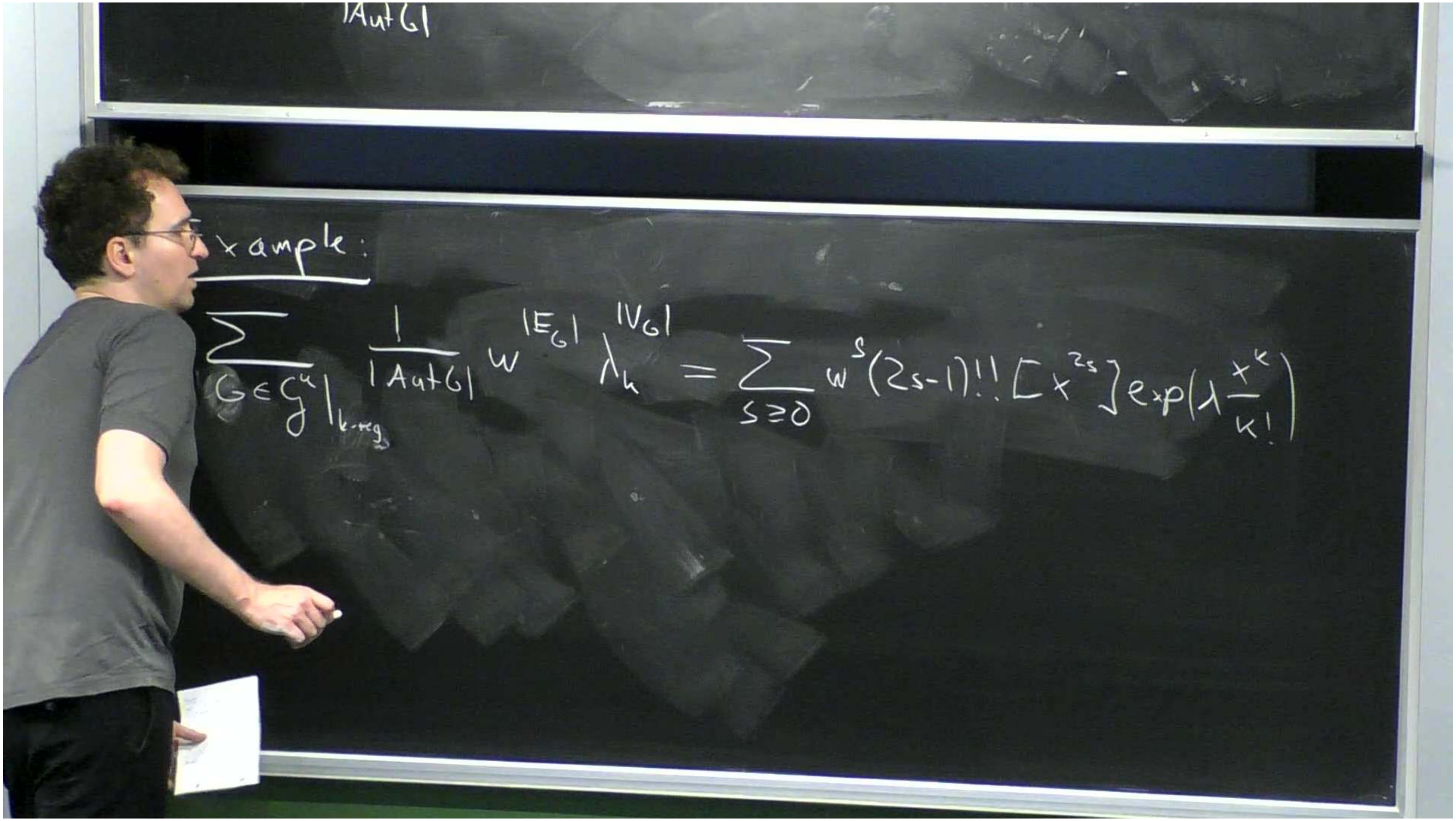
Proof: Use Theorem 10 + Prop. 11.

Unlabeled version:

Def: $\mathcal{G}^u := \bigcup_{m \geq 0} \mathcal{G}[m] \big/ \mathrm{Sym}[m]$ $\left(\frac{m!}{|\mathrm{Aut}\, G|} = \#\text{labelings} \right)$

Theorem 12*:

$$\sum_{G \in \mathcal{G}^u} \underbrace{(\#\text{labelings of } G) \frac{1}{m!}}_{= \frac{1}{|\mathrm{Aut}\, G|}} w^{|E_G|} \prod_{v \in V_G} \lambda_{|v|} = (*)$$


$|\mathrm{Aut}\,G|$
Example:
$\sum_{G\in\mathcal{G}^u|_{k\text{-reg}}} \frac{1}{|\mathrm{Aut}\,G|} w^{|E_G|} \lambda_k^{|V_G|} = \sum_{s\geq 0} w^s (2s-1)!! [x^{2s}] \exp\left(\lambda \frac{x^k}{k!}\right)$

Example:

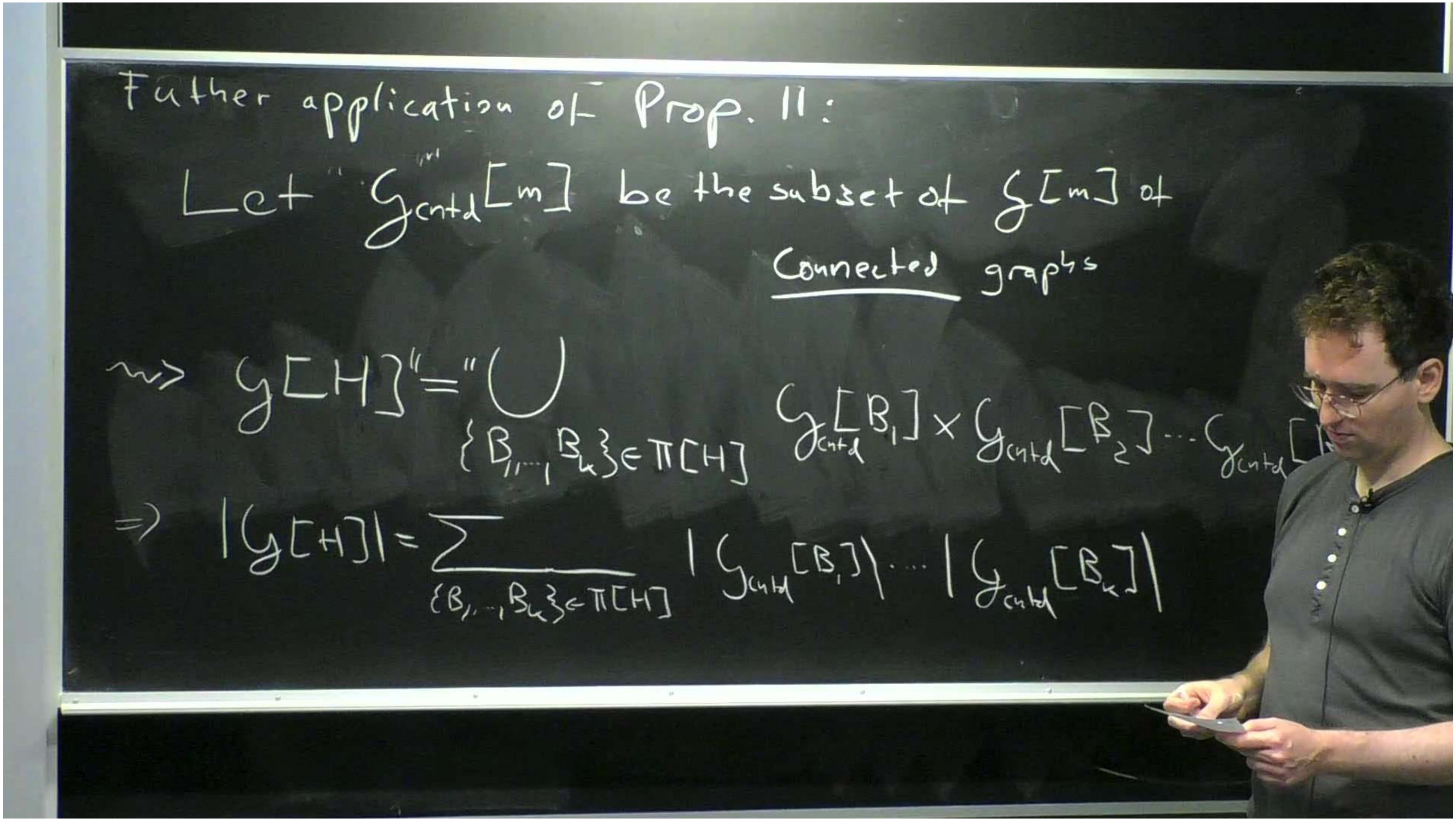
$$\sum_{G\in \mathcal{G}^u|_{k\text{-reg}}} \frac{1}{|\mathrm{Aut}\,G|} w^{|E_G|} \lambda_k^{|V_G|} = \sum_{s\ge 0} w^s (2s-1)!!\,[x^{2s}]\exp\left(\lambda \frac{x^k}{k!}\right)$$

# of k-valent vtcs of G

$$\sum_{G\in \mathcal{G}^u|_{1,k\text{-deg.}}} \frac{1}{|\mathrm{Aut}\,G|} w^{|E_G|} \lambda_k^{n_k(G)} j^{n_1(G)} = \sum_{s=0}^{\infty} w^s (2s-1)!!\,[x^{2s}]\exp\left(\frac{\lambda_k x^k}{k!} + jx\right)$$

Unlabeled version:

Def: $\mathcal{G}^u := \bigcup_{m\ge 0} \mathcal{G}[m] / \mathrm{Sym}[m]$ $\quad \left(\frac{m!}{|\mathrm{Aut}\,G|} = \#\text{ labelings}\right)$


Futher application of Prop. 11:
Let "$\mathcal{G}_{cntd}[m]$" be the subset of $\mathcal{G}[m]$ of
Connected graphs
$\rightsquigarrow \mathcal{G}[H] \text{``}=\text{''} \bigcup_{\{B_1,\dots,B_k\}\in\Pi[H]} \mathcal{G}_{cntd}[B_1]\times\mathcal{G}_{cntd}[B_2]\cdots\mathcal{G}_{cntd}$
$\Rightarrow |\mathcal{G}[H]| = \sum_{\{B_1,\dots,B_k\}\in\Pi[H]} |\mathcal{G}_{cntd}[B_1]|\cdots|\mathcal{G}_{cntd}[B_k]|$

$$\Rightarrow \quad |\mathcal{G}[H]| = \sum_{\{B_1,\dots,B_k\}\in\Pi[H]} |\mathcal{G}_{cntd}[B_1]| \cdots |\mathcal{G}_{cntd}[B_k]|$$

$$f_m = |\mathcal{G}[m]| \qquad g_d = |\mathcal{G}_{cntd}[d]|$$

$$\Rightarrow \quad f(x) = \sum_{m\geq 0} \frac{|\mathcal{G}[m]|}{m!} x^m = \exp(g(x)) =$$

$$= \exp\left(\sum_{d\geq 1} \frac{|\mathcal{G}_{cntd}[d]|}{d!}\right)$$

$d \geq 1$ $\quad d! \,)$

$$= (*)$$

Theorem 12**:

$$\sum_{m \geq 0} \frac{1}{m!} \sum_{G \in \mathcal{G}[m]_{cntd}} \omega^{|E_G|} \prod_{v \in V_G} \lambda_{|v|} =$$

(4-1)

$$= \log\left(\sum_{s \geq 0} \omega^{s} (2s-1)!! [t^{2s}] \exp\left(\sum_{d \geq 1} \lambda_d \frac{t^d}{d!} \right) \right)$$

convergence is guaranteed

Theorem 12***: Do unlabeled connected graphs?

Asymptotic expansions

Fix some limit $z \to \infty$