Title: Flatness, Non-flatness, and Orientations in Ponzano-Regge

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Abstract:

The original spinfoam model, the Ponzano-Regge spinfoam model of 3D Euclidean gravity with zero cosmological constant, has two properties in seeming contradiction: (1.) Its connection formulation consists in the imposition of exact flatness, and (2.) Its sum-over-spins formulation has, as its leading order large spin asymptotics, Regge calculus, modified to include an additional local discrete orientation variable for each tetrahedron, which, when fixed inhomogeneously, leads to critical point equations for the edge lengths which do not imply flatness, but rather allow spikes. We explore possible resolutions to this paradox which may also have relevance for the semiclassical regime of both Euclidean and Lorentzian 4D spinfoams, with both zero and non-zero cosmological constant, in which a similar sum over local orientations appears.

JONATHAN ENGLE (FL. ATLANTIC U.)

LEE'S FEST: QUANTUM GRAVITY AND THE NATURE OF TIME.

FLATNESS, NON-FLATNESS, AND ORIENTATIONS IN PONZANO-REGGE

Personal Note

I went Penn State in 2001 for my Ph.D. because Lee, Abhay Ashtekar, and Jorge Pullin were there.

When I arrived, Lee and Jorge were gone! Lee gone to found PI, and Jorge to start the quantum gravity at LSU.

So, unfortunately I missed out on Lee's influence at that time.

However, I have visited PI many times, and there is really no place in the world like it – constant discussions, new ideas being developed.

The environment you have helped create at PI is really wonderful and I thank you for what you have done here, Lee!

OUTLINE

Tension in Ponzano-Regge:

a. In connection formulation: Manifest flatness

b. Large spin asymptotics: Locally oriented Regge!

c. Equations of motion for fixed inhomogeneous local orientations: Non-flatness! Doesn't match classical equation of motion!

Possible resolutions? Something to learn?

a. Contradiction seems to arise only when model diverges and so is ill-defined anyway. So, strictly speaking, there is no contradiction. Satisfactory?

b. Is connection possibly sensitive to orientation, so that connection is flat even though geometry is not?

Analogous issues in 4D spin-foams

a. Locally oriented Regge asymptotics again lead to wrong E.O.M.

b. Do Lessons from Ponzano-Regge help?

c. `Force' homogeneous orientation? (proper/causal/Feynman Spin-foam amplitude? Related to Lee and Fotini's pre-SF causal SN evolution?)

From spin to connection formulation: Manifest flatness

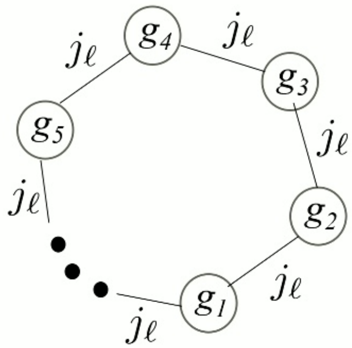
Given a 3D triangulation Δ with edges ℓ, triangles t, and tetrahedra σ,

$$W_{PR} = \sum_{\{j_\ell\}}{}' \prod_\ell (-1)^{2j_\ell}(2j_\ell+1) \prod_t (-1)^{j_1+j_2+j_3} \prod_\sigma \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}$$

N.B. $2j_\ell, j_1+j_2+j_3 \in \mathbb{N}$, so signs are well-defined!

where $\{j_\ell\}' := \{j_\ell\}_{\ell \in \mathrm{int}\Delta} \subset \mathbb{N}/2$. Assume, for simplicity, no boundary. Then

$$W_{PR} = \sum_{\{j_\ell\}} \int \left(\prod_t \mathrm{d}g_t\right) \prod_\ell (2j_\ell+1)$$



$$= \int \left(\prod_t \mathrm{d}g_t\right) \prod_\ell \sum_j (2j+1)\mathrm{Tr}_j(h_\ell)$$

$$= \boxed{\int \left(\prod_t \mathrm{d}g_t\right) \prod_\ell \delta(h_\ell)} \text{ “} = \int \mathcal{D}\omega\, \delta(F(\omega)) = \int \mathcal{D}\omega \mathcal{D}e \exp\left(i \int e \wedge F(\omega)\right) = \int \mathcal{D}\omega \mathcal{D}e \exp\left(iS[e,\omega]\right) \text{”}$$

also shows first order formalism underlying Ponzano-Regge.

Large spin asymptotics: Locally oriented Regge!

Setting $j_\ell = \lambda j^o_\ell$ $(\in \mathbb{N}/2)$ for j^o_ℓ fixed.

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \underset{\lambda\to\infty}{\sim} \frac{1}{\sqrt{3\pi \mathrm{V}}} \cos\left(\sum_{a=1}^{6} j_a \Theta_a + \frac{\pi}{4}\right)$$

[Ponzano and Regge (1968);
Dowdall, Gomes, and Hellmann (2009);
Christodoulou, Långvik, Riello, Röken, and Rovelli (2012)]

where V is the volume of the tetrahedron with edge lengths λj_a and
Θ_a is the *external* dihedral angle at edge a (angle between the normals to the two triangles at a).

implies

(This choice to express the -1's as exponentials is a generalization of that in Chistodoulou et al. and agrees for their triangulation.)

$$W_{PR} \sim {\sum_{\{j_\ell\}}}' \prod_\ell (e^{i\pi})^{2j_\ell}(2j_\ell+1) \prod_t (e^{-i\pi})^{j_1+j_2+j_3} \prod_\sigma \sum_{\mu_\sigma=\pm 1} \frac{1}{\sqrt{12\pi \mathrm{V}(\sigma)}} \exp i\mu_\sigma \left(\sum_{\ell\in\sigma} j_\ell \Theta_\ell(\sigma) + \frac{\pi}{4}\right)$$

$$= \sum_{\{j_\ell\}'} \sum_{\{\mu_\sigma\}} \left(\prod_\sigma \frac{1}{\sqrt{12\pi \mathrm{V}(\sigma)}}\right) \exp i\left(S_{R,\mu} + \frac{\pi}{4}\sum_\sigma \mu_\sigma\right)$$

From spin to connection formulation: Manifest flatness

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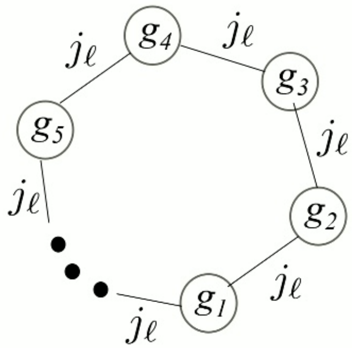
N.B. $2j_\ell, j_1+j_2+j_3 \in \mathbb{N}$, so signs are well-defined!

where $\{j_\ell\}' := \{j_\ell\}_{\ell\in \mathrm{int}\Delta} \subset \mathbb{N}/2$. Assume, for simplicity, no boundary. Then

$$W_{PR} = \sum_{\{j_\ell\}} \int \left(\prod_t \mathrm{d}g_t\right) \prod_\ell (2j_\ell+1)$$



$$= \int \left(\prod_t \mathrm{d}g_t\right) \prod_\ell \sum_j (2j+1)\mathrm{Tr}_j(h_\ell)$$

$$= \boxed{\int \left(\prod_t \mathrm{d}g_t\right) \prod_\ell \delta(h_\ell)} \text{ “} = \int \mathcal{D}\omega\,\delta(F(\omega)) = \int \mathcal{D}\omega\mathcal{D}e \exp\left(i\int e\wedge F(\omega)\right) = \int \mathcal{D}\omega\mathcal{D}e \exp\left(iS[e,\omega]\right)\text{”}$$

also shows first order formalism underlying Ponzano-Regge.

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where

$$\boxed{S_{R,\mu} := \sum_{\ell} j_\ell \left(\left(2 - |T_\ell| + \sum_{\sigma\in\Sigma_\ell} \mu_\sigma \right) \pi - \sum_{\sigma\in\Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) \right)}$$

Here T_ℓ and Σ_ℓ respectively denote the set of triangles and tetrahedra containing ℓ, and $\theta_\ell(\sigma) = \pi - \Theta_\ell(\sigma)$ is the *internal* dihedral angle in σ at ℓ (angle inside σ between the planes of the two triangles at ℓ) **The sign μ_σ appearing here is the discrete analogue of sgn**$(\det(e))$**..**

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for $\mu_\sigma \equiv +1$

$$\boxed{S_{R,+1} = \sum_{\ell\in\mathrm{int}\Delta} j_\ell \left(2\pi - \sum_{\sigma\in\Sigma_\ell} \Theta_\ell(\sigma) \right) + \sum_{\ell\in\partial\Delta} j_\ell \left(\pi - \sum_{\sigma\in\Sigma_\ell} \Theta_\ell(\sigma) \right) = S_{\mathrm{Regge}}}$$

Exactly the Regge action, including correct boundary terms, for a general triangulation!

Equations of motion for fixed local orientations: Non-flatness!

Varying the internal j_ℓ:

$$\boxed{\sum_{\sigma\in\Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) = \left(2 + \sum_{\sigma\in\Sigma_\ell} (\mu_\sigma - 1) \right) \pi}$$

giving flatness, $\sum_{\sigma\in\Sigma_\ell} \theta_\ell(\sigma) = 2\pi$, **only** for $\mu \equiv 1$.

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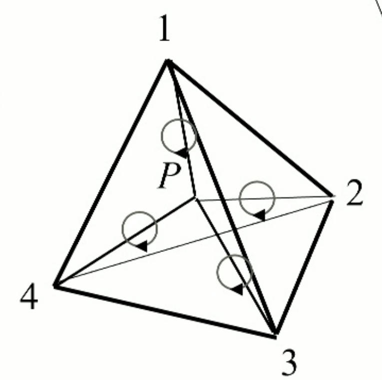
Simplest example: 4-1 Pachner move triangulation

For $\mu_1 = \mu_2 = \mu_3 = \mu_4 = +1$:

$$2\pi = \theta_{\ell_1}(\sigma_2) + \theta_{\ell_1}(\sigma_3) + \theta_{\ell_1}(\sigma_4)$$
$$2\pi = \theta_{\ell_2}(\sigma_1) + \theta_{\ell_2}(\sigma_3) + \theta_{\ell_2}(\sigma_4), \text{ etc.}$$

Flatness around all 4 internal ℓ_a, as expected.



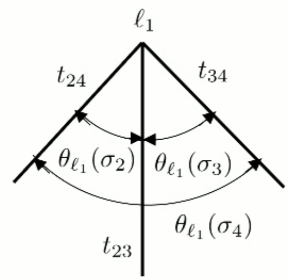
For $\mu_1 = \mu_2 = \mu_3 = +1,\ \mu_4 = -1$:

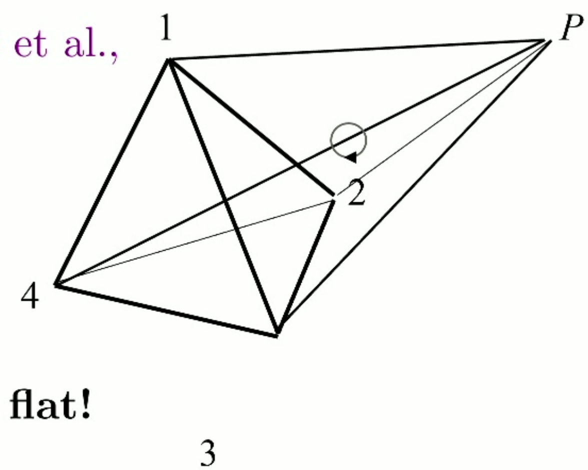
$$0 = \theta_{\ell_1}(\sigma_2) + \theta_{\ell_1}(\sigma_3) - \theta_{\ell_1}(\sigma_4)$$
$$0 = \theta_{\ell_2}(\sigma_1) + \theta_{\ell_2}(\sigma_3) - \theta_{\ell_2}(\sigma_4)$$
$$0 = \theta_{\ell_3}(\sigma_1) + \theta_{\ell_3}(\sigma_2) - \theta_{\ell_3}(\sigma_4)$$
$$2\pi = \theta_{\ell_4}(\sigma_1) + \theta_{\ell_4}(\sigma_2) + \theta_{\ell_4}(\sigma_3)$$

Flatness around ℓ_4,
but **not around** ℓ_1, ℓ_2, ℓ_3!

Following Christodoulou et al., we call this a **Spike**.

E.g., in plane $\perp \ell_1$:



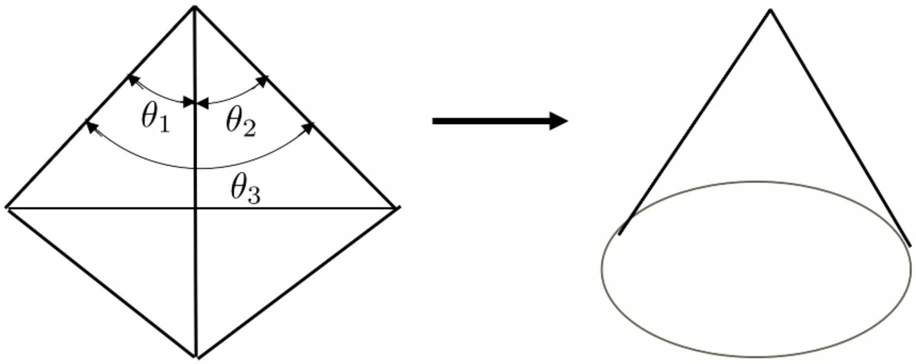


Not flat!

Equations of motion for fixed local orientations: Non-flatness!

2D analogue:

$\theta_1 + \theta_2 = \theta_3$



Conical singularity - not flat!

Key point: If interior dihedral angles around a hinge **don't sum to** 2π, then the geometry in a neighborhood of the hinge is not embeddable into $\mathbb{R}^n$ and so is **not flat!**

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Flatness or curved spikes? Possible resolutions:

1. **Spikes generally correspond to bubbles for which model is ill-defined**

   - **In connection formulation**, Redundant δ's: **Divergence**
   - **In spin formulation**, unbounded sums over internal spins in spikes: **Divergence**

   Because both formulations are ill-defined in this case, there is no strict mathematical contradiction. Does this satisfy us?

2. **Is the connection at spikes flat, even if geometry is not?**

   - Could $\sum_{\sigma\in\Sigma_\ell}\theta_\ell(\sigma) = \left(2 - \sum_{\sigma\in\Sigma_\ell}(\mu_\sigma - 1)\right)\pi$ somehow be the condition for flatness for the **spin-connection** determined by the **triad** e, which knows about orientation?
   - **Is the spin-connection even sensitive to the orientation of the triad?** Consider $\tilde{e}^i_a = \mu e^i_a$. Then $$\omega(\tilde{e})^{ij}_a = 2\tilde{e}^{b[i}\partial_{[a}\tilde{e}^{j]}_{b]} + \tilde{e}_{ak}\tilde{e}^{bi}\tilde{e}^{dj}\partial_{[d}\tilde{e}^k_{b]} = \cdots = 2\mu(\partial_b\mu)e^{b[j}e^{i]}_a + \omega(e)^{ij}_a$$

     In coordinate patch (x, y, z), if $\mu = \mathrm{sgn}(x)$, then $\mu\partial_b\mu = 2\mathrm{sgn}(x)\delta(x)\partial_b x = 0$ **if** we regularize $\mathrm{sgn}(x)$ symmetrically. Then $\omega(\mu e) = \omega(e)$, **so it seems ω is not sensitive to μ.**

Analogous tension in continuum! Perhaps start here!

First order formulation

$$S[e,\omega] := \int e \wedge F(\omega) \quad \Rightarrow \text{E.O.M.}$$

- $d_\omega e = 0 \quad \Rightarrow \quad \omega = \omega(e)$
- $F(\omega) = 0$ **Flatness**

Second order formulation

$$S[e] := S[e,\omega(e)] = \int e \wedge F(\omega(e)) = \int \mu(x) R[g_{ab}] \sqrt{\det g(x)}\, d^3x =: S[g]$$

where $\mu(x) := \operatorname{sgn}(\det(e(x)))$ and $g_{ab}(x) := e_a^i(x) e_{bi}(x)$.

$$\begin{aligned}
\delta S[g] &= \int \mu\sqrt{g}\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right)\delta g^{\mu\nu} d^3x + \int \mu \partial_\alpha \left(\tilde{V}^\alpha_{\mu\nu}\delta g^{\mu\nu} + \tilde{W}^{\alpha\beta}_{\mu\nu}\partial_\beta \delta g^{\mu\nu}\right) d^3x \\
&= \int \mu\sqrt{g} G_{\mu\nu}\delta g^{\mu\nu} d^3x - \int (\partial_\alpha \mu)\, \tilde{V}^\alpha_{\mu\nu}\delta g^{\mu\nu} d^3x + \int \left(\partial_\beta \left(\tilde{W}^{\alpha\beta}_{\mu\nu}\partial_\alpha \mu\right)\right)\delta g^{\mu\nu} d^3x \\
&= \int \mu\sqrt{g}\left(G_{\mu\nu}\delta g^{\mu\nu} - g^{-1/2}\mu \tilde{V}^\alpha_{\mu\nu}\partial_\alpha\mu + g^{-1/2}\mu\partial_\beta\left(\tilde{W}^{\alpha\beta}_{\mu\nu}\partial_\alpha \mu\right)\right)\delta g^{\mu\nu} d^3x
\end{aligned}$$

$\Rightarrow$ E.O.M. $G_{\mu\nu}$ can be distributional where μ changes sign!

Correct E.O.M. (flatness) only where μ is homogeneous!

Related issues in 4D spinfoams:

a) Large spin asymptotics also gives locally oriented Regge.

b) Tetrad gravity EOM are equivalent to GR only with homogeneity of orientation!!

Resolutions?

a) Might resolution to paradoxes in 3D case also give insight to solving above problems?

b) If not, `force' one homogeneous orientation?

   i. `Proper' vertex [Engle, Zipfel 2012-2016], `causal'/`Feynman' spin-foam propagator [Livine, Oriti 2003,2004], **possibly related to** `causal evolution of spin networks' [Markopoulou, Smolin 1997]. **Fixing of time orientation?**

   ii. Support from requiring projection onto kernel of Constraint operator in LQC [Ashtekar, Campiglia, Henderson 2010] and full LQG [Thiemann, Zipfel 2014].

c) Modification to yield homogeneity of orientation at least in non-degenerate regions [Rovelli, Wilson-Ewing 2012]