**Title:** Principle and constructive theories of physical probability

**Speakers:** Simon Saunders

**Collection/Series:** Lee's Fest: Quantum Gravity and the Nature of Time

**Date:** June 03, 2025 - 11:30 AM

**URL:** https://pirsa.org/25060039

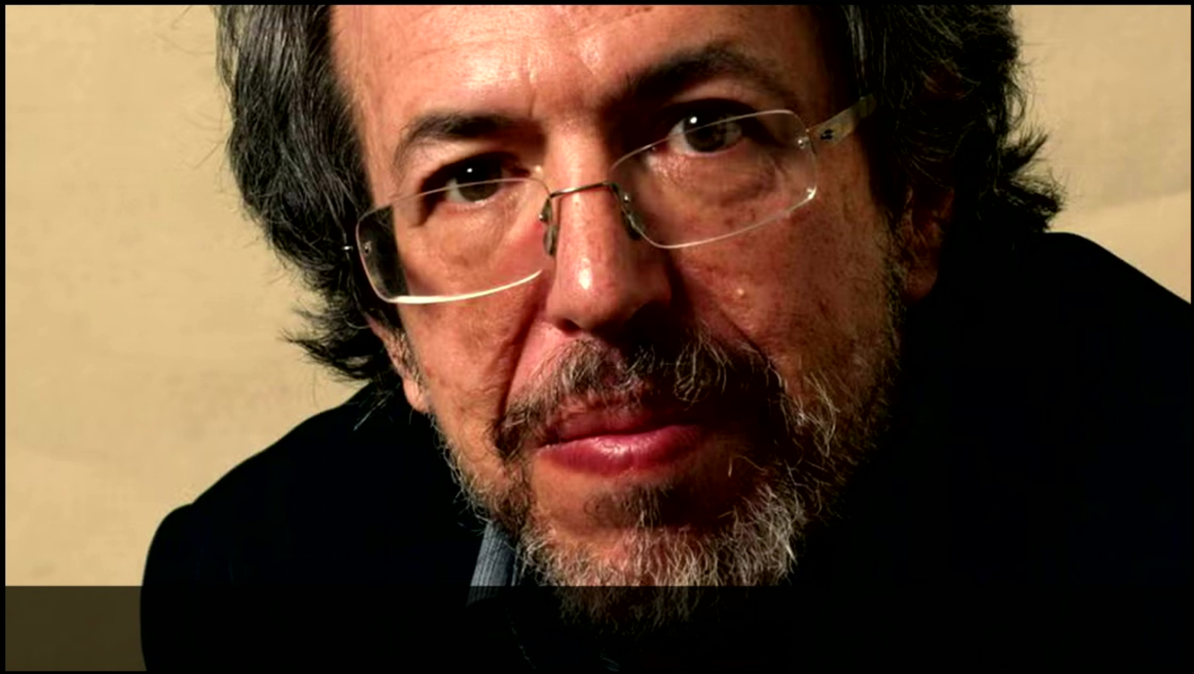
**Abstract:**

I show how to make sense of physical probability even for a single atom at a single time in an otherwise empty universe. The ideas are in https://arxiv.org/abs/2404.12954 and http://arxiv.org/abs/2505.06983. As applied to the EPRB setup, probability defined in this way is perfectly local.

# Principle and constructive theories of physical probability, and Bell inequalities

Lee's Fest PI

Simon Saunders, Oxford



# Principle theory

Parameter independence:

$$p_{a,b}(s|\lambda) = p_{a,b'}(s|\lambda)$$

Conditional outcome independence

$$p_{a,b}(s|\lambda) = p_{a,b}(s|t,\lambda)$$

1. Boolean: some space $\Sigma$ of elements $X \in \Sigma$ equipped with disjointness, union, and complement, defining a Boolean algebra
2. Synchronic: the probability $\mu[X]$ of $X$ depends only on the instantaneous physical state and $\Sigma$
3. Formal: $0 \leq \mu[X] \leq 1$; $\mu[\Sigma] = 1, \mu[\emptyset] = 0$; $\mu$ is additive for disjoint $X, Y$.
4. Intrinsic: $\mu[X]$ can only be changed if there is a change in $X$.

- Some space of ELEMENTS $X \in \Sigma$ ('beables')
- Probability defined by PHYSICAL STATE and $\Sigma$
- Probability of X can be changed only if there is a CHANGE in X


Full specification of what happens in 3 makes events in 2 irrelevant for predictions about 1 in a locally causal theory (Bell Nouvelle Cuisine)

"A theory will be said to be locally causal if the probabilities attached to values of local beables in a space-time region 1 are unaltered by specification of values of local beables in a space-like separated region 2, given a full specification of local beables in a space-time region 3" (Bell)

# Principle theory

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Physical state independence?

$$p_{a,b}(s|\lambda) = p_{a,b}(s|\lambda')$$

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# Constructive theory

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4. Intrinsic: $\mu[X]$ can only be changed if there is a change in $X$.

- Take 'the physical state' as the (vector) quantum state $\psi \in \mathcal{H}$
- Consider the vectors in an orthogonal expansion $\psi = \varphi_1 + \ldots + \varphi_n$ as elements of $\Sigma$
- Write $\mu[X]$ as $\mu_{\psi_\Sigma}[X]$
- If $\varphi, \eta \in \Sigma$ are orthogonal, then $\mu_{\psi_\Sigma}[\varphi + \eta] = \mu_{\psi_\Sigma}[\varphi] + \mu_{\psi_\Sigma}[\eta]$ .
- If $U\varphi = \varphi$, then $\mu_{U\psi_\Sigma}[\varphi] = \mu_{\psi_\Sigma}[\varphi]$

Theorem (Short (2023, Saunders (2025): vectors in $\Sigma$ of equal norm have equal physical probability

$$\varphi, \eta \in \Sigma, \|\varphi\| = \|\eta\| \Rightarrow \mu_{\psi_\Sigma}[\varphi] = \mu_{\psi_\Sigma}[\eta].$$

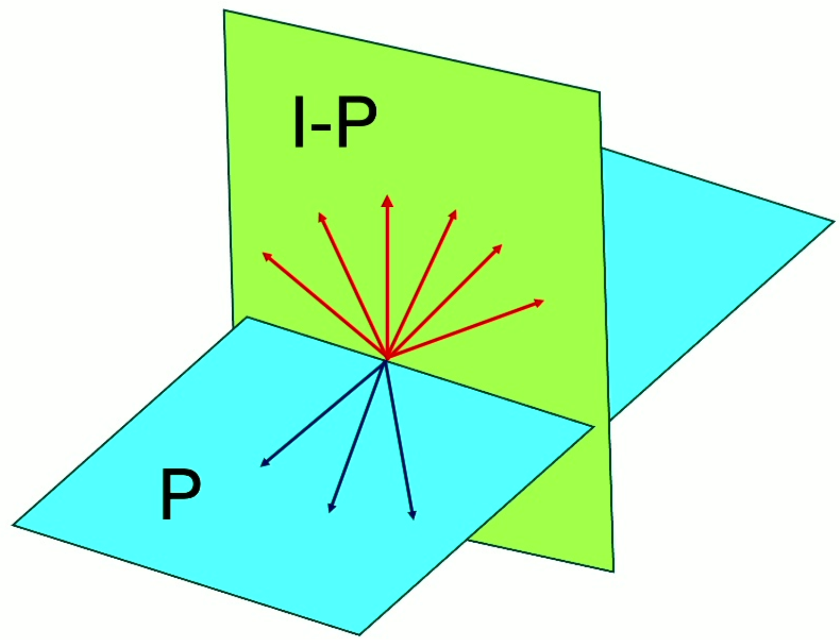
# Constructive theory

Frequentism: the probability of P in ensemble $\Lambda$ is the fraction of $\Lambda$ with P

- Take 'the physical state' as the (vector) quantum state $\psi \in \mathcal{H}$, where $\dim(\mathcal{H}) = \infty$
- Expand $\psi$ in terms of orthogonal vectors of equal norm ('microstates')
- For any* projector P on $\mathcal{H}$, choose an expansion into microstates all* diagonalizing P.

$$\psi = \overbrace{\xi_1 + \cdots + \xi_m}^{\text{eigenvalue 1}} + \overbrace{\xi_{m+1} + \cdots + \xi_n}^{\text{eigenvalue 0}}.$$

$$\frac{\|P\psi\|^2}{\|\psi\|^2} = \frac{\|P(\xi_1 + \ldots + \xi_n)\|^2}{\|\xi_1 + \ldots + \xi_n\|^2} = \frac{\|(\xi_1 + \ldots + \xi_m)\|^2}{\|\xi_1 + \ldots + \xi_n\|^2} = \frac{m}{n}$$



Constructive theory
$\lambda$-MANY

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- Expand $\psi$ in terms of orthogonal vectors of equal norm ('microstates')
- For any* projector P on $\mathcal{H}$, choose an expansion into microstates all* diagonalizing P.
- Treat the equal-norm vectors ('microstates') in such an expansion as an ensemble, denote $\Lambda_{\psi}^{n}$, an 'event space', and define probabilities as in frequentism

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# Constructive theory
# $\lambda$-MANY

Parameter independence:

$$p_{a,b}(s|\lambda) = p_{a,b'}(s|\lambda)$$

Conditional outcome independence:

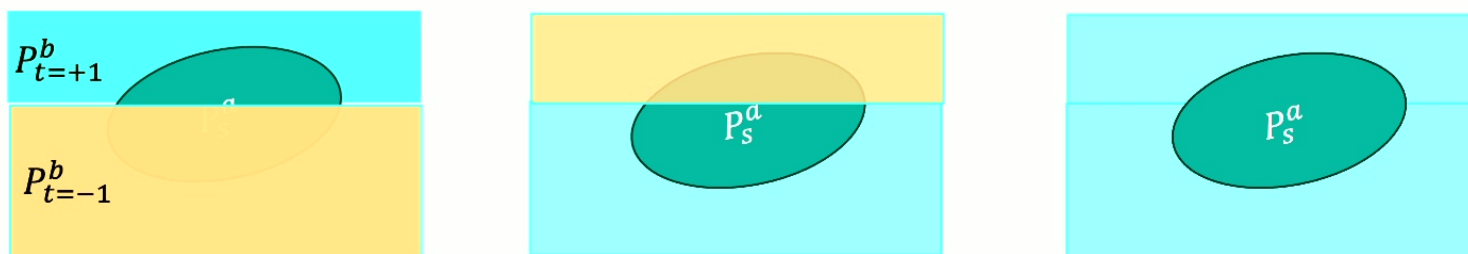
$$p_{a,b}(s|\lambda) = p_{a,b}(s|t,\lambda)$$

Complete outcome independence?

$$p_{a,b}(s|\lambda) = p_{a,b}(s|t=+1,-1,\lambda)$$

$$p_{a,b}(s|\lambda) = \frac{\text{number of microstates in } [P_s^a \otimes I]}{\text{number of microstates in } [I \otimes I]}$$

$$p_{a,b}(s|t,\lambda) = \frac{\text{number of microstates in } [P_s^a \otimes P_t^b]}{\text{number of microstates in } [I \otimes P_t^b]}$$



Constructive theory
$\lambda$-MANY

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$$p_{a,b}(s|\lambda) = p_{a,b'}(s|\lambda)$$

✓

Conditional outcome independence:

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✗

Complete outcome independence?

$$p_{a,b}(s|\lambda) = p_{a,b}(s|t=+1,-1,\lambda)$$

✓

$$p_{a,b}(s|\lambda) = \frac{\text{number of microstates in } [P_s^a \otimes I]}{\text{number of microstates in } [I \otimes I]}$$

$$p_{a,b}(s|t,\lambda) = \frac{\text{number of microstates in } [P_s^a \otimes P_t^b]}{\text{number of microstates in } [I \otimes P_t^b]}$$

- New derivation of the Born rule without assuming non-contextuality
- Demonstration that no-collapse quantum mechanics hosts a notion of physical probability

Conclusion: $\lambda$-MANY is perfectly free of action-at-a-distance

# Principle theory

Parameter independence: ($LOC$)

$$p_{a,b}(s|\lambda) = p_{a,b'}(s|\lambda)$$

Conditional outcome independence: ($LOC \wedge UNIQUE$)

$$p_{a,b}(s|\lambda) = p_{a,b}(s|t,\lambda)$$

Complete outcome independence? ($LOC \wedge \neg UNIQUE$)

$$p_{a,b}(s|\lambda) = p_{a,b}(s|t = +1, -1, \lambda)$$

$\lambda$- independence ($LOC \wedge RET$)

$$p_{a,b}(s|\lambda) = p_{a,b}(s|\lambda')$$

Bell inequalities ($BELL$)

$$(LOC \wedge UNIQUE \wedge \neg RET) \rightarrow BELL$$

Logically equivalent

$$(UNIQUE \wedge \neg BELL \wedge \neg RET) \rightarrow \neg LOC$$


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The ‘many world interpretation’ seems to me an extravagant, and above all an extravagantly vague, hypothesis. I could almost dismiss it as silly. And yet…it may have something distinctive to say in connection with the ‘Einstein Podolsky Rosen puzzle’, and it would be worthwhile, I think, to formulate some precise version of it to see if this is really so. (Six possible worlds of quantum mechanics)

To Lee, with affection, admiration, and differences
(lest one of us be superfluous!)

A. Short (2023) 'Probability in many worlds theories'
S.S. (2024) 'Finite frequentism explains quantum probability'
S. S. (2025a) 'Physical probability and locality in no-collapse quantum theory'
S.S. (2025b) 'Principle and constructive theories of physical probability, and Bell inequalities' (to appear in *Everett and Locality*, A. Ney (ed), OUP).