Title: One of us would be superfluous

Speakers: Carlo Rovelli

Collection/Series: Lee's Fest: Quantum Gravity and the Nature of Time

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Abstract:

I will tell stories. About Lee's physics. About his insights that have shaped many of the ideas that we take for granted today. From the birth of Loop Quantum Gravity to the discreteness of physical space.

to
my friend

with love
and
gratitude

Jim Baggott:

Lee, you and Carlo have done so much physics together.
And yet you so often disagree on so many things.
How is this possible?

Lee:

If we agreed on everything,
one of us would be superfluous.

Chapter 1
Yale 1988

Knot Theory and Quantum Gravity

Carlo Rovelli
Sezione di Roma, Instituto Nazionale di Fisica Nucleare, University of Rome, Rome, Italy

and

Lee Smolin[a]
Department of Physics, Yale University, New Haven, Connecticut 06520
(Received 13 June 1988)

A new representation for quantum general relativity is described, which is defined in terms of functionals of sets of loops in three-space. In this representation exact solutions of the quantum constraints may be obtained. This result is related to the simplification of the constraints in Ashtekar's new formalism. We give in closed form the general solution of the diffeomorphism constraints and a large class of solutions of the full set of constraints. These are classified by the knot and link classes of the spatial three-manifold.

PACS numbers: 04.60.+n, 02.40.+m

Despite the failure of standard perturbative quantizations, many people have argued that quantum general relativity may still exist because strong-coupling effects at short distances contradict the assumption, which underlies perturbation theory, that quantum geometry may be understood in terms of small fluctuations around a classical background spacetime.[1] One approach to the investigation of this hypothesis is canonical quantization, in which the splitting of the metric into a classical background part and a fluctuating quantum part is not made.[2] In the canonical formulation of general relativity, for the case of closed space Σ, the Hamiltonian is weakly vanishing and in the quantum theory the dynamics is expressed by the quantum constraint equations.

In this Letter we describe a new representation of canonical quantum general relativity, called the loop representation, in which exact, nonperturbative, solutions to the constraint equations may be obtained.[3] In particular, we describe here the following results.

(1) The entire space of states annihilated by the spatial diffeomorphism constraints $D_a(x)$ is found in terms of an explicit countable basis. The elements of this basis are in one-to-one correspondence with the generalized link classes of the 3D manifold Σ. These are the equivalence classes, under Diff(Σ), the identity-connected component of the diffeomorphism group of Σ, of sets of piecewise differentiable loops in Σ.

(2) Among these states are some which are also annihilated by the Hamiltonian constraint $\mathcal{C}(x)$, and are thus exact physical quantum states of the gravitational field. Included in these is a sector whose basis is in one-to-one correspondence with the subset of the generalized link classes of Σ which are based on sets of smooth, nonintersection loops. These are the well studied ordinary link classes, whose classification is the subject of knot theory.[4]

The loop representation is a development of Ashtekar's reformulation of general relativity[5] and is motivated by the discovery[6] of a set of solutions of the Wheeler-DeWitt equation related to loops. In Ref. 7 it was first introduced by means of a functional transform from the self-dual representation.[5] Here, following Isham's ideas,[8] we define directly the loop representation as the quantization of a suitable Poisson algebra of nonlocal classical observables.

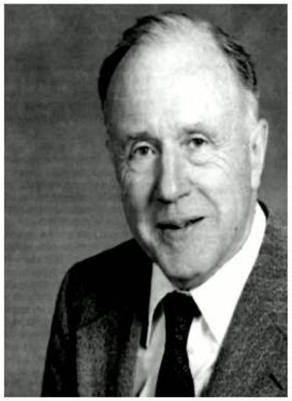
We proceed by describing the loop representation, we then explain why it is a quantization of general relativity, and, finally, we describe how the solutions are found.

Let Σ be a compact three-manifold, of arbitrary topology, without metric or connection structure. Let $\mathcal{L}_\Sigma$ be the space of piecewise differentiable, closed, parametrized, nondegenerate curves in Σ (called in what follows, loops, and denoted by greek letters $\gamma, \eta, \ldots$) and let $\mathcal{M}_\Sigma$ be the space of the (unordered) set of elements of $\mathcal{L}_\Sigma$ (called multiple loops and denoted $\{\gamma\}, \{\eta\}, \ldots$). Let $\mathcal{S}$ be the space of complex-valued functions $\mathcal{A}[\{\gamma\}]$ on $\mathcal{M}_\Sigma$ which (1) are invariant under reparametrization and inversion of the loops, and (2) satisfy, for any γ and η with a common base point, the equation $\mathcal{A}[\gamma \# \eta] + \mathcal{A}[\gamma^{-1} \# \eta] = \mathcal{A}[\{\gamma, \eta\}]$, where $\gamma^{-1}(s) \equiv \gamma(1-s)$ ($s \in [0,1]$ is the loop parameter) and $\gamma \# \eta$ is the loop made by going once round γ and then once round η before closing. (As in spin network formalism,[9] this is an implementation in the loop space of the fundamental two-spinor identity, $\delta_A^B \delta_C^D - \delta_A^D \delta_C^B = \epsilon_{AC} \epsilon^{BD}$.)

On this space $\mathcal{S}$ there exists an algebra of regulated linear operators which is a representation of a complete, observable algebra for general relativity. The algebra, called the $\tilde{\mathcal{T}}$ algebra, is graded by the nonnegative integers. The zero elements are defined for every loop γ by

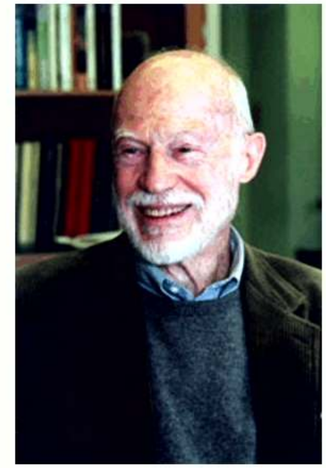
$$\tilde{T}^0[\gamma]\mathcal{A}[\{\eta\}] = \mathcal{A}[\gamma \cup \{\eta\}].$$

These are a kind of lowering operator. For $n \geq 1$ the operators are denoted $\tilde{T}_\epsilon^{n, a_1, \ldots, a_n}[\gamma](s_1, \ldots, s_n)$ (a_i



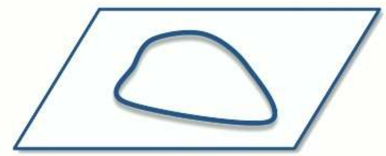
$$\left((g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd})\frac{\delta}{\delta g_{ab}}\frac{\delta}{\delta g_{cd}} - \sqrt{\det g}\ R[g]\right)\Psi[g_{ab}] = 0$$

$$D_a\frac{\delta}{\delta g_{cd}}\Psi[g_{ab}] = 0$$



Change variables $\Psi[g] \to \Psi[A]$

γ :



Jacobson Smolin solutions of the WdW eq. $\Psi_\gamma[A] = \mathrm{Tr}\left[e^{\oint_\gamma A}\right]$

Change variables again $\Psi[g] \to \Psi[A] \to \Psi[\gamma]$

$$\Psi[\gamma] = \int [DA]\ \mathrm{Tr}\left[e^{\oint_\gamma A}\right]\Psi[A]$$

Solution of the second equ. $\Psi[\gamma] = \Psi[k[\gamma]]$

Chapter 1
Yale 1988

VOLUME 61, NUMBER 10 PHYSICAL REVIEW LETTERS 5 SEPTEMBER 1988

Knot Theory and Quantum Gravity

Carlo Rovelli

Sezione di Roma, Instituto Nazionale di Fisica Nucleare, University of Rome, Rome, Italy

and

Lee Smolin[(a)]

Department of Physics, Yale University, New Haven, Connecticut 06520

(Received 13 June 1988)

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North-Holland

LOOP SPACE REPRESENTATION OF QUANTUM GENERAL RELATIVITY

Carlo ROVELLI

INFN Sezione di Roma, Università di Roma, Roma, Italy and
Dipartimento di Fisica, Università di Trento, Trento, Italy

Lee SMOLIN

Department of Physics, Yale University, New Haven, CT 06511, USA and
Department of Physics, Syracuse University, Syracuse, NY 13210, USA and
Dipartimento di Fisica, Università di Trento, Trento, Italy

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(Revised 30 March 1989)

We define a new representation for quantum general relativity, in which exact solutions of the quantum constraints may be obtained.

The representation is constructed by means of a noncanonical graded Poisson algebra of classical observables, defined in terms of Ashtekar's new variables. The observables in this algebra are nonlocal and involve parallel transport around loops in a three-manifold Σ. The theory is quantized by constructing a linear representation of a deformation of this algebra. This representation is given in terms of an algebra of linear operators defined on a state space which consists of functionals of sets of loops in Σ. The construction is general and can be applied also to Yang–Mills theories.

The diffeomorphism constraint is defined in terms of a natural representation of the diffeomorphism group. The hamiltonian constraint, which contains the dynamics of quantum gravity, is constructed as a limit of a sequence of observables which incorporates a regularization prescription. We give the general solution of the diffeomorphism constraint in closed form. It is spanned by a countable basis which is in one-to-one correspondence with the diffeomorphism equivalence classes of multiple loops, which are a generalization of the link classes studied in knot theory. Then we explicitly construct, in closed form, a large space of solution of the entire set of constraints, including the hamiltonian constraint. These turn out to be classified by the ordinary knot and link classes of Σ.

The space of solutions that we find is a sector of the physical states space of nonperturbative quantum general relativity. The failure of perturbation theory is thus shown to be not relevant to the problem of the existence of a nontrivial physical state space in quantum gravity. The relationship between this new loop representation and the self-dual representation of Ashtekar is illuminated by means of a functional transform between states in the two representations. Questions of the completeness of the solution space, the meaning of the physical operators and the physical inner product, are discussed, but not, so far, resolved.

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(North-Holland)

Chapter 2
Goa 1988

Chapter 3
Trento 1992

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Weaving a Classical Metric with Quantum Threads

Abhay Ashtekar,[(1)] Carlo Rovelli,[(2)] and Lee Smolin[(1)]

[(1)]*Department of Physics, Syracuse University, Syracuse, New York 13244-1130*
[(2)]*Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260*
and Dipartimento di Fisica, Università di Trento, Trento, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Padova, Padova, Italy
(Received 14 April 1992)

Results that illuminate the physical interpretation of states of nonperturbative quantum gravity are obtained using the recently introduced loop variables. It is shown that (i) while local operators such as the metric at a point may not be well defined, there do exist *nonlocal* operators, such as the area of a given two-surface, which can be regulated diffeomorphism invariantly and which are finite *without* renormalization; (ii) there exist quantum states which approximate a given metric at large scales, but such states exhibit a discrete structure at the Planck scale.

PACS numbers: 04.60.+n

It is by now generally accepted that perturbative approaches to quantum gravity fail because they assume that space-time geometry can be approximated by a smooth continuum at all scales. What is needed are nonperturbative approaches which can *predict*—rather than assume—what the true nature of the microstructure of this geometry is. In such an approach, background fields such as a classical metric or a connection cannot play a fundamental role; quantum theory must be formulated in a diffeomorphism-invariant fashion. An important task in these programs is then to introduce techniques needed to describe geometry and to "explain" from first principles how smooth geometries can arise on macroscopic scales.

Over the past five years, two avenues have been pursued to test if quantum general relativity can exist nonperturbatively. The first is based on numerical simulations [1], while the second is based on canonical quantization [2–6]. This Letter concerns the second approach. While the canonical approach itself was introduced by Dirac in the late 1950s, the recent work departs from the early treatment in two important ways: (i) It is based on a new canonically conjugate pair, the configuration variable being a connection [2,5]; and (ii) it uses a new representation in which quantum states arise as suitable functions on the space of closed loops on a (spatial) 3-manifold [2,3,6]. The new ingredients have led to technical as well as conceptual simplifications which, in turn, have led to a variety of new results. In particular, these methods have opened up bridges between quantum gravity and other areas in mathematics and physics such as knot theory, Chern-Simons theory, and Yang-Mills theory.

The purpose of this Letter is to report on the picture of quantum geometry that arises from the use of the loop variables. To explore the geometry nonperturbatively, we must first introduce operators that carry the metric information and regulate them in such a way that the final operators do *not* depend on any background structure introduced in the regularization. We will show that such operators do exist and that they are finite without renormalization. Using these operators, we seek nonperturbative states which can approximate a given classical geometry up terms $O(l_P/L)$, where l_P is the Planck length and L is a macroscopic length scale, lengths being defined by the given metric. We find that such states do exist but that they exhibit a discrete structure at the Planck scale l_P. Such a result was anticipated on general grounds since the 1930s. Indeed, there exist a number of quantum gravity programs that *begin* by postulating discrete structures at the Planck scale and then attempt to recover from it the known macroscopic physics [7]. The key difference is that, in our approach, discreteness is *arrived at* by combining general relativity with quantum mechanics using loop variables.

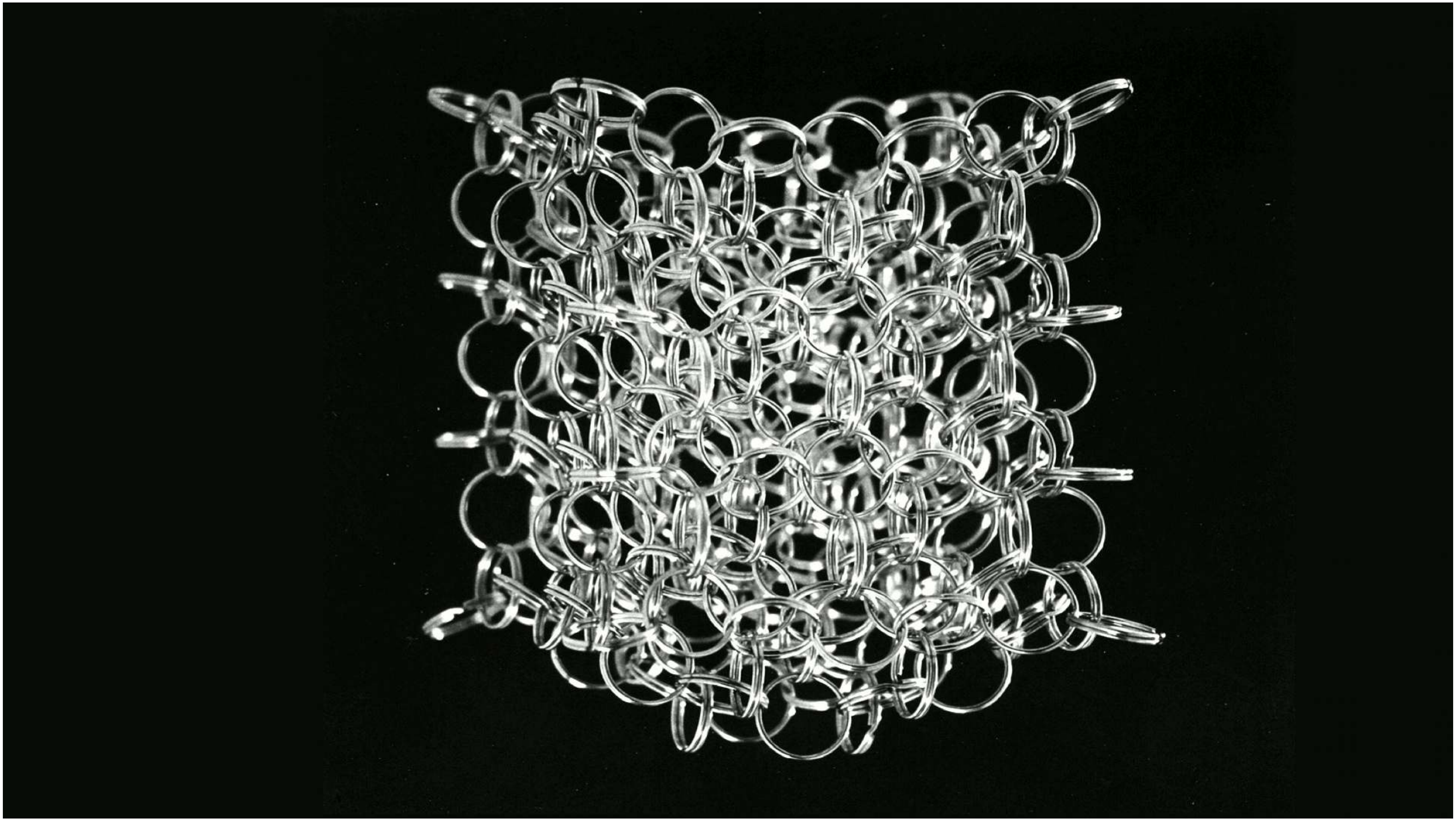
In this Letter, we will only sketch the main ideas involved; details will appear elsewhere [8].

Let us begin with the classical phase space. The configuration variable A_a^i is a complex SU(2) connection and its conjugate momentum $\tilde{E}_i^a$, the mathematical analog of the electric field in Yang-Mills theory, is a triad with density weight 1 [5]. (Throughout we will let a, $b, \dots$ denote the spatial indices and $i, j, \dots$, the internal indices. A tilde over a letter will denote a density weight 1.) The first step is the introduction of loop variables [6] which are manifestly SU(2)-gauge-invariant functions on the phase space. The configuration variables are the Wilson loops: Given a closed loop γ on the 3-manifold Σ, we set

$$T[\gamma] = \tfrac{1}{2} \operatorname{Tr} \mathcal{P} \exp G \oint_\gamma A_a \, dl^a , \tag{1}$$

where G is Newton's constant. (Throughout, we use the 2-dimensional representation of the gauge group to evaluate traces.) Variables with momentum dependence are constructed by inserting $\tilde{E}_i^a$ at various points on the loop before taking the trace. Thus, for example, the loop variable quadratic in momenta is given by

$$T^{aa'}[\gamma](y,y') = \tfrac{1}{2} \operatorname{Tr}\left[\left(\mathcal{P} \exp G \int_{y'}^{y} A_a \, d\gamma^a \right) \tilde{E}^a(y') \times \left(\mathcal{P} \exp G \int_{y}^{y'} A_a \, d\gamma^a \right) \tilde{E}^{a'}(y) \right] , \tag{2}$$



VOLUME 72, NUMBER 4 PHYSICAL REVIEW LETTERS 24 JANUARY 1994

The Physical Hamiltonian in Nonperturbative Quantum Gravity

Carlo Rovelli[1,*] and Lee Smolin[2,†]

[1]*Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260 and Dipartimento di Fisica, Università di Trento, Trento, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Padova, Padova, Italy*

[2]*Center for Gravitational Physics and Geometry, Pennsylvania State University, University Park, Pennsylvania 16802-6360*

(Received 5 August 1993)

A quantum Hamiltonian which evolves the gravitational field according to time as measured by constant surfaces of a scalar field is defined through a regularization procedure based on the loop representation, and is shown to be finite and diffeomorphism invariant. The problem of constructing this Hamiltonian is reduced to a combinatorial and algebraic problem which involves the rearrangements of lines through the vertices of arbitrary graphs. This procedure also provides a construction of the Hamiltonian constraint as a finite operator on the space of diffeomorphism invariant states as well as a construction of the operator corresponding to the spatial volume of the Universe.

PACS numbers: 04.60.Gw

One of the main problems of nonperturbative quantum gravity has been how to realize physical time evolution in the absence of a fixed background spacetime geometry [1]. One solution to this problem, which has been often discussed, is to use a matter degree of freedom to provide a physical clock [2,3], and represent evolution as change with respect to it. In this Letter we show that it is possible to explicitly implement this proposal in the full theory of quantum general relativity, using the nonperturbative approach based on the loop representation [4–7]. We use a scalar field as a clock as suggested in several recent papers [8], and we show that it is possible to construct the Hamiltonian operator $\hat{H}$ that gives the evolution in this clock time.

We construct the Hamiltonian operator $\hat{H}$ by using regularization techniques recently introduced [6,9] for diffeomorphism invariant theories. The main result that we obtain is that the operator $\hat{H}$, although constructed through a regularization procedure that breaks diffeomorphism invariance, is nevertheless diffeomorphism invariant, background independent, and (as we have argued elsewhere [9] is implied by these conditions) finite. It follows that $\hat{H}$ is well defined on the space, $\mathcal{V}$, of the diffeomorphism invariant states of the gravitational field. As $\mathcal{V}$ is spanned by the basis given by the generalized knot classes [4] (diffeomorphism equivalence classes of finite sets of loops in Σ, the three-dimensional space manifold), $\hat{H}$ is represented by an infinite dimensional matrix in knot space. We present here a procedure for computing all the matrix elements of the Hamiltonian $\hat{H}$ in knot space. This procedure is purely combinatorial and algebraic. Thus, our main result is the reduction of the problem of computing the physical evolution of the quantum gravitational field with respect to a clock to a problem in graph theory and combinatorics.

We begin by introducing the scalar field $T(x)$, whose three-surfaces of constant values may be taken, under appropriate circumstances, to represent time [8]. We denote the physical regime in which this can be done (in which the scalar field grows monotonically everywhere on Σ) as the clock regime. The formalism developed here is meaningful only within this regime. If we denote its conjugate momentum by $\pi(x)$, the Hamiltonian constraint is

$$\mathcal{C}(x) = \frac{1}{2\mu}\pi^2 + \frac{\mu}{2}\tilde{\tilde{q}}^{ab}\partial_a T \partial_b T + \mathcal{C}_{\rm grav}\,, \qquad (1)$$

where μ is a constant. The gravitational contribution has the standard form $\mathcal{C}_{\rm grav} = \mathcal{C}_{\rm Einst} + \Lambda_q$ where $\mathcal{C}_{\rm Einst} = \epsilon_{ijk} \times \tilde{E}^{ai}\tilde{E}^{bj}F^k_{ab}$ and $q = \det(q_{ab})$. Here Λ is the cosmological constant, and all other symbols have the usual meaning in the Ashtekar formalism [10]. We then restrict the freedom of choosing the time coordinate by fixing the gauge $\partial_a T = 0$. This implies that the lapse is $N(x) = a/\pi(x)$ for some constant a and that all of the infinite number of Hamiltonian constraints $\mathcal{C}(x)$ turn out to be gauge fixed, except one, which is

$$\int_\Sigma \frac{\mathcal{C}}{\pi} = (2\mu)^{-1/2}\int_\Sigma \pi + \int_\Sigma \sqrt{-\mathcal{C}_{\rm grav}} = 0\,. \qquad (2)$$

In the quantum theory the diffeomorphism invariant states are then of the form $\Psi[\{\alpha\}, T]$, where $\{\alpha\}$ indicates a generalized knot class and the real number T is the constant value of the time. These states satisfy a Schrödinger-type equation $i\hbar(d\Psi/dT) = \sqrt{2\mu}\hat{H}\Psi$ where $\hat{H}$ is the quantum operator corresponding to the observable

$$H = \int_\Sigma \sqrt{-\mathcal{C}_{\rm grav}}\,. \qquad (3)$$

We now proceed to construct the quantum operator $\hat{H}$. We regularize the integral by writing it as a limit of a sum, and, in addition, we regularize each operator product. We write

$$\hat{H} = \lim_{L\to 0, A\to 0, \delta\to 0} \sum_I L^3 \sqrt{-\hat{\mathcal{C}}^{L,\delta,A}_{{\rm Einst}\,I} - \Lambda \hat{q}^L_I}\,, \qquad (4)$$

where we have divided the spatial manifold Σ into cubes of size L according to an arbitrary set of fixed Euclidean coordinates, and the sum is over these cubes, labeled I. The quantities δ and A are parameters involved in the

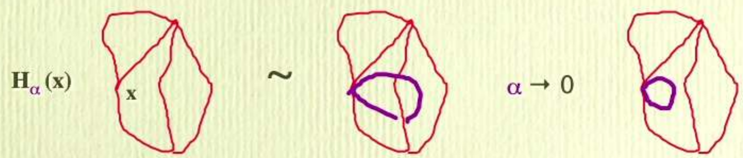
Chapter 4
Pittsburgh–Penn State 1994

Dynamics

- **H Ψ = 0**

- **regularization ➞ in the limit in which the cut of removed**
  H is <u>finite</u> on diff-invariant states

- **H = B EE,** regularization: $B = \lim_{\alpha \to 0} U_\alpha(A)$



The theory is naturally ultraviolet finite

Far from being inconsistent, Gr and QFT conspire to solve each others problems

ELSEVIER

Nuclear Physics B 442 (1995) 593-619

NUCLEAR PHYSICS B

Discreteness of area and volume in quantum gravity

Carlo Rovelli [a,1], Lee Smolin [b,2]

[a] *Department of Physics, University of Pittsburgh, Pittsburgh, PA 15260, USA*

[b] *Center for Gravitational Physics and Geometry, Department of Physics, Pennsylvania State University, University Park, PA 16802-6360, USA*

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Abstract

We study the operator that corresponds to the measurement of volume, in non-perturbative quantum gravity, and we compute its spectrum. The operator is constructed in the loop representation, via a regularization procedure; it is finite, background independent, and diffeomorphism-invariant, and therefore well defined on the space of diffeomorphism invariant states (knot states). We find that the spectrum of the volume of any physical region is discrete. A family of eigenstates are in one to one correspondence with the spin networks, which were introduced by Penrose in a different context. We compute the corresponding component of the spectrum, and exhibit the eigenvalues explicitly. The other eigenstates are related to a generalization of the spin networks, and their eigenvalues can be computed by diagonalizing finite dimensional matrices. Furthermore, we show that the eigenstates of the volume diagonalize also the area operator. We argue that the spectra of volume and area determined here can be considered as predictions of the loop-representation formulation of quantum gravity on the outcomes of (hypothetical) Planck-scale sensitive measurements of the geometry of space.

1. Introduction

In spite of recent progress, research in quantum gravity [1] has produced few precise physical predictions, against which the theory might be, at least in principle, experimentally tested. In this paper, we show that, under certain assumptions, predictions for the spectra of certain geometric quantities can be derived from the quantum theory of

[1] E-mail address: rovelli@vms.cis.pitt.edu.
[2] E-mail address: smolin@phys.psu.edu.

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Chapter 5
Verona 1994

Can we compute eigenvalues non perturbatively?

Yes!

Physical geometry is discrete at the Planck scale,
As a direct result of taking GR and QT together.

Chapter 6
Oxford–Verona 1995

PHYSICAL REVIEW D VOLUME 52, NUMBER 10 15 NOVEMBER 1995

Spin networks and quantum gravity

Carlo Rovelli*
Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

Lee Smolin†
Center for Gravitational Physics and Geometry, Department of Physics, Pennsylvania State University, University Park, Pennsylvania 16802-6360
and School of Natural Science, Institute for Advanced Study, Princeton, New Jersey 08540

(Received 5 May 1995)

We introduce a new basis on the state space of nonperturbative quantum gravity. The states of this basis are linearly independent, are well defined in both the loop representation and the connection representation, and are labeled by a generalization of Penrose's spin networks. The new basis fully reduces the spinor identities [SU(2) Mandelstam identities] and simplifies calculations in nonperturbative quantum gravity. In particular, it allows a simple expression for the exact solutions of the Hamiltonian constraint (Wheeler-DeWitt equation) that have been discovered in the loop representation. The states in this basis diagonalize operators that represent the three-geometry of space, such as the area and the volume of arbitrary surfaces and regions, and therefore provide a discrete picture of quantum geometry at the Planck scale.

PACS number(s): 04.60.Ds, 75.25.+z

I. INTRODUCTION

The loop representation [1,2] is a formulation of quantum field theory suitable when the degrees of freedom of the theory are given by a gauge field, or a connection. This formulation has been used in the context of continuum and lattice gauge theory [3], and it has found a particularly effective application in quantum gravity [2,4], because it allows a description of the diffeomorphism invariant quantum states in terms of knot theory [2,5], and, at the same time, because it partially diagonalizes the quantum dynamics of the theory, leading to the discovery of solutions of the dynamical constraints [2,6]. Recent results in quantum gravity based on the loop representation include the construction of a finite physical Hamiltonian operator for pure gravity [7] and fermions [8], the computation of the physical spectra of area [9] and volume [10], and the development of a perturbation scheme that may allow transition amplitudes to be explicitly computed [7,11,12]. A mathematically rigorous formulation of quantum field theories whose configuration space is a space of connections, inspired by the loop representation, has been recently developed [13,14] and the kinematics of the theory is now on a level of rigor comparable to that of constructive quantum field theory [15]. This approach has also produced interesting mathematical spinoffs such as the construction of diffeomorphism invariant generalized measures on spaces of connections [14] and could be relevant for a constructive field theory approach to non-Abelian Yang-Mills theories.

Applications of the loop representation, however, have been burdened by complications arising from two technical nuisances. The first is given by the Mandelstam identities, because of which the loop states are not independent and form an overcomplete basis. The second is the presence of a certain sign factor in the definition of the fundamental loop operators T^n for $n > 1$. This sign depends on the global connectivity of the loops on which the operator acts and obstructs a simple local graphical description of the operator's action. In this work, we describe an elegant way to overcome both of these complications. This comes from using a particular basis, which we denote as spin network basis, since it is related to the spin networks of Penrose [16]. The spin network basis has the following properties: (i) It solves the Mandelstam identities; (ii) it allows a simple and entirely local graphical calculus for the T^n operators; (iii) it diagonalizes the area and volume operators. The spin network basis states, being eigenstates of operators that correspond to measurement of the physical geometry, provide a physical picture of the three-dimensional quantum geometry of space at the Planck-scale level.

The main idea behind this construction, long advocated by Loll [17], is to identify a basis of independent loop states in which the Mandelstam identities are completely reduced. We achieve such a result by exploiting the fact that all irreducible representations of SU(2) are built by symmetrized powers of the fundamental representation. We will show that in the loop representation this translates into the fact that we can suitably antisymmetrize all loops overlapping each other, without losing generality. More precisely, the (suitably) antisymmetrized loop states span, but do not overspan, the kinematical state space of quantum gravity.

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*Electronic address: rovelli@vms.cis.pitt.edu
†Electronic address: smolin @phys.psu.edu

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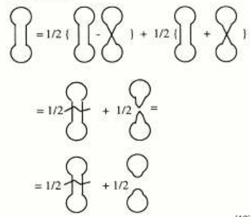
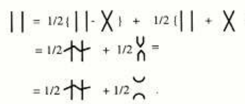
trivalent loops. For instance, in the example above the intersection between α and γ in the loop $\alpha\cdot\gamma\cdot\beta\cdot\gamma^{-1}$ is trivalent because γ and γ^{-1} form a single set of overlapping loop segments (a single rope) emerging from the intersection. We will deal with nontrivalent intersections in the Appendix.

We may hope to reduce the degeneracy by replacing every overlapping segment with a suitable antisymmetrized combination, plus "tails" that can be got rid of by means of the retracing identity. In the example considered above, for instance, we can reduce the state $\langle\alpha\cdot\gamma\cdot\beta\cdot\gamma^{-1}|$ to a linear combination of the two states defined in (11)



(12)

So we may hope that any time we have two parallel lines, we could use the spinor identity as follows:



(13)

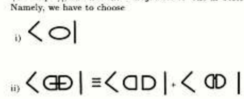
Unfortunately, this does not work. To understand why, consider a loop α and an open segment γ that starts and ends in two different points of α. Denote by α_1 and α_2 the two segments in which the two intersections with γ partition α. Then we have, due to the spinor and retracing identities,

$$\langle\alpha| - \langle\alpha_1\cdot\gamma^{-1}\cup\alpha_2\cdot\gamma| + \langle\alpha_1\cdot\gamma\cdot\alpha_2\cdot\gamma| = 0, \quad (14)$$

namely

(15)

If we want to pick two independent linear combinations, we have to choose the *symmetric* combination $\langle\alpha| + \langle\alpha_1\cdot\gamma^{-1}\cup\alpha_1\cdot\gamma|$, and not the *antisymmetric* one as before. Namely, we have to choose



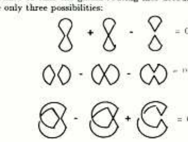
(16)

Thus, to pick the independent combination of loop states, we have to antisymmetrize the rope in one case, but we have to symmetrize it in the other case. In general, the choice between symmetrization and antisymmetrization can only be worked out by writing out explicitly the full pattern of rootings in the multiple loop. In other words, Eq. (13) is in general wrong if taken as a calculation rule that can be used in dealing with any loop state. More precisely, at every intersection, the spinor identity provides a linear relation between the three multiple loops obtained by replacing the intersection with the three possible rootings through the loop

(17)

but the sign in front of each term depends on the global routing of the loops.

There is a simple way out of this difficulty, which does allow us to get rid of the spinor identities among trivalent loops simply by antisymmetrization. In order to determine the correct signs of the various terms in Eq. (17), we have to take the global routing into account. There are only three possibilities:



(18)

counts the terms of the symmetrization. Let us analyze this state in some detail. For every link l with color p_l, there are p parallel propagators $U_l(A)$ along the link l, each one in the spin-$\frac{1}{2}$ representation, that enter the definition of $\psi_S(A)$. Let us indicate tensors' indices explicitly; we introduce spinor indices $A, B, \ldots$ with value 0, 1. The connection A has components $A_A{}^B$, which form an sl(2,C) matrix, and its parallel propagator along a link l is a matrix $U_{lA}{}^B$ in the SL(2,C) group. Since SL(2,C) is the group of matrices with unit determinant, we have

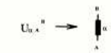
$$\det U_l + \tfrac{1}{2}\epsilon_{AB}\epsilon^{CD}U_{lC}{}^{A}U_{lD}{}^{B} = 1, \quad (34)$$

where ϵ_{AB} is the totally antisymmetric two-dimensional object defined by

$$\epsilon_{01} = \epsilon^{01} = 1. \quad (35)$$

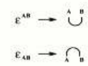
One can write $\psi_S(A)$ explicitly in terms of the parallel propagators $U_{lA}{}^B$, the objects ϵ_{AB} and ϵ^{AB} and the Kroneker delta δ^B_A. Thus, any spin-network state can be expressed by means of a certain tensor expression formed by sl(2,C) tensors, ϵ and δ objects.

Penrose has described in [33] a graphical notation for tensor expressions of this kind. This notation is going to play a role in what follows, so we begin by recalling its main ingredients. We indicate two-index tensors with thick lines, with the indices at the open ends of the line, respecting the distinction between upper indices, indicated by lines pointing up and lower indices, corresponding to lines pointing down. More precisely, we indicate the matrix of the parallel propagator $U_{\alpha}{}^B_A$ of an (open or closed) curve α as a vertical bold line as in



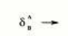
(36)

where the label α is understood unless needed for clarity; we indicate the antisymmetric tensors as in



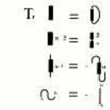
(37)

and the Kroneker δ as in



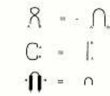
(38)

Finally, we indicate the sum over repeated indices by connecting the open ends of the lines where the indices are. We then have, for instance,



(39)

Also



(40)

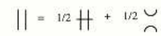
The most interesting relation is the identity

$$\delta_A{}^B\delta_C{}^D - \delta_A{}^D\delta_C{}^B = \epsilon^{BD}\epsilon_{AC}, \quad (41)$$

which becomes

(42)

which is of course related to the loop representation spinor identity. Because of this last relation, in the Penrose diagram of any loop state we can use the graphical relation



(43)

where the bar indicates symmetrization, on any (true) intersection or overlapping loop.

Now, consider a generic (multiple) loop state in the connection representation; this is given as a product of terms, each of which is the trace of a product of matrices. We can represent these traces in terms of the corresponding graphical tensor diagram, which will result as a set of closed lines. We adopt the additional convention of drawing lines that form a rope as nearby parallel lines, and of reproducing the intersections of the original loops as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Portugal
Greece
Türkiye
Turkmenistan
Kyrgyzstan
China
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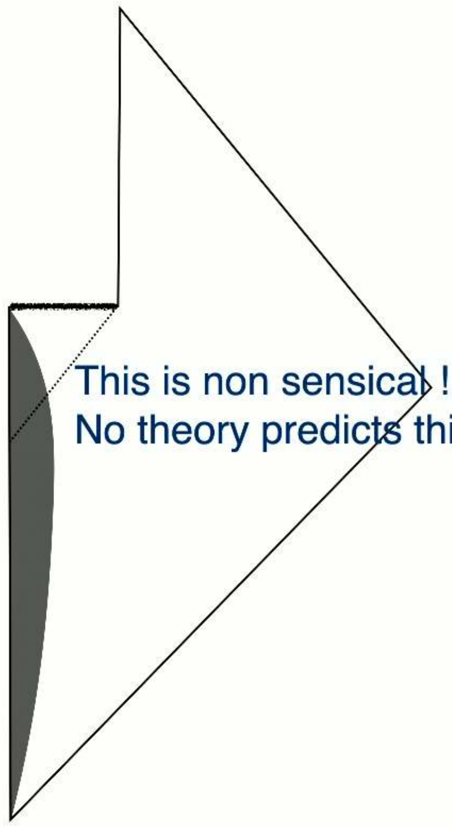
Japan
Syria
Iraq
Iran
Afghanistan
Pakistan
Nepal
India
Morocco
Tunisia
Algeria
Libya
Egypt
Saudi Arabia
Oman
Western Sahara
Mauritania
Mali
Niger
Chad
Sudan
Yemen
Burkina Faso
Guinea
Nigeria
Ghana
Ethiopia
Somalia
Kenya
Gabon
DRC
Tanzania
Angola
Zambia
Mozambique
Namibia
Zimbabwe
Botswana
Madagascar
South Africa
Myanmar (Burma)
Thailand
Vietnam
Philippines
Malaysia
Indonesia
Papua New Guinea
Australia
United States
Mexico
Cuba
Puerto Rico
Guatemala
Nicaragua
Venezuela
Guyana
Suriname
Colombia
Ecuador
Peru
Brazil
Bolivia
Paraguay
Chile
Uruguay
Argentina
North Atlantic Ocean
South Atlantic Ocean
Indian Ocean

Thanks for the fabulous journey, my friend



This is non sensical !
No theory predicts this !