Title: Canonical purifications revisited

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Abstract:

I will use algebras to revisit and generalize the theory of canonical purifications, explain the general concept of "CRT sewing without gravity," and explain how this technology allows the influential-but-complicated "Araki-Yamagami theorem" from the '80s to be proved with 10x less work using a QI-motivated trick. Based on work with Caminiti and Capeccia.

Canonical purifications revisited

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Perimeter Institute, QIQG 2025

Based on work with Jackie Caminiti and Federico Capeccia

Big picture

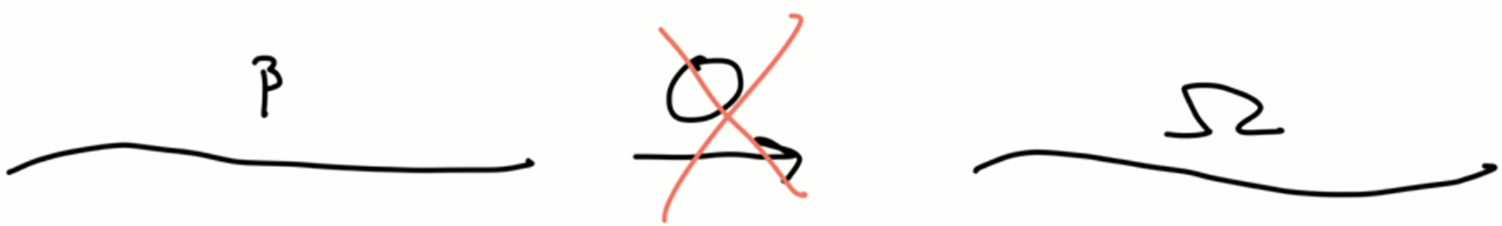
- Quantum info makes intractable problems tractable and hard problems easy.
- Personally, the problems I find hardest are ones in mathematical physics — I don't have a great intuition for many of the basic theorems about mathematical QFT.
- Some complicated, trick-riddled proofs can be made massively simpler and (to me) more intuitive using ideas from QI.
- I'll tell you about one today.

An overview of the problem [1/2]

- When can one physical state be realized as a local excitation of another?

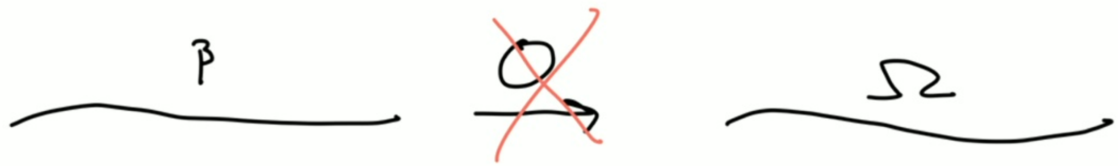
$$\langle \ldots \rangle_{\omega_2} = \langle O\omega_1 | \ldots | O\omega_1 \rangle$$

- Not possible, e.g., for global Minkowski vacuum vs. thermal:



An overview of the problem [2/2]

- But this should be possible locally:



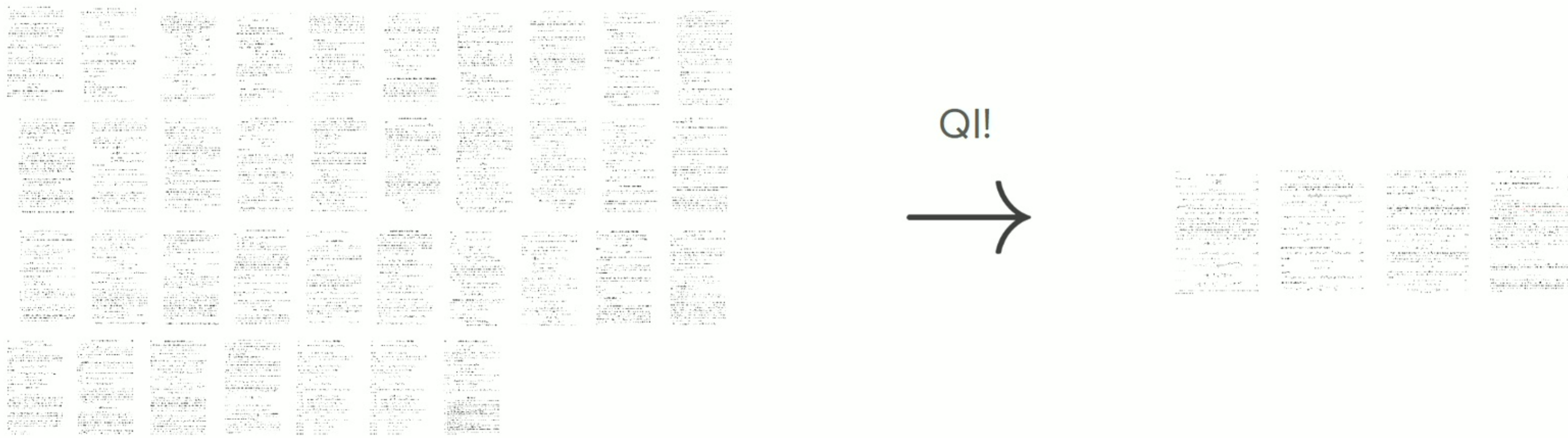
- Physically intuitive, but how do you actually show it? This isn't an obvious calculation — you need to do some mathematical physics.

A bit about the mathematical physics literature

- In free field theory, a general framework for this "local excitation" problem was worked out by Araki and Yamagami in 1982.
- Using this, Verch proved the physically intuitive statement that "any two (Hadamard) states can be excited into one another within compact regions."
- Beautiful literature, but very technically involved.
- To give you a taste, let me show you the Araki-Yamagami proof.

Araki-Yamagami [Compressed]

This talk in one slide



The tool: canonical purification

- If you have a density matrix, it's often useful to purify:

$$\rho_A \to |\psi\rangle_{AR} \qquad \mathrm{tr}_R |\psi\rangle\langle\psi| = \rho_A .$$

- There are infinitely many ways to do this, but an easy one is

$$\sum_j p_j |j\rangle\langle j| \to \sum_j \sqrt{p_j} |j\rangle |j\rangle$$

- Generalizes TFD:

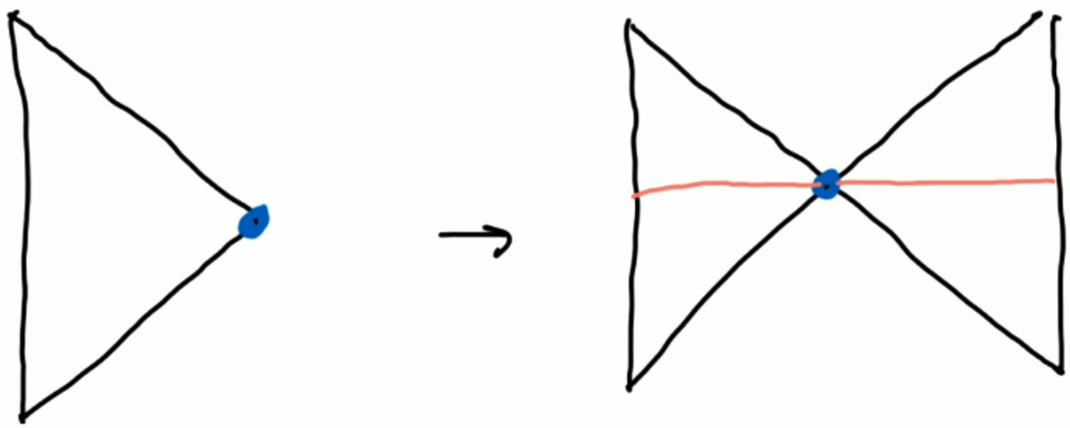
$$e^{-\beta H} \to \sum_n e^{-\beta E_n/2} |n\rangle |n\rangle$$

- This is called a **canonical purification**.

Nice purifications in holography [2/2]

- To make a pure state, need a complete spacetime.

- Just double your entanglement wedge and sew it together:
  [Engelhardt & Wall '17, '18]



Canonical purifications and sewing

- For the TFD, canonical purification matches CRT-sewing.
- Path integral arguments tell you that they're generally the same.
  [Dutta & Faulkner '19]
- Lots of nice applications, often connecting to information-theoretic properties of canonical purifications.
  [Bousso et al '19; Marolf '19; Bueno & Casini '20ab; Zou et al '20; Engelhardt et al '21, 22; Hayden et al '21; Akers et al '21; Akers et al '22ab; Dutta et al '22; Parrikar & Singh '23]

Limitations of the tool

- Not every physical question can be answered using density matrices.
- General physical systems are described using algebras, and there are important settings where you can't address your question by regulating.
  - Crossed product entropy [Witten '21, ...]
  - Subalgebra/subregion duality [Leutheusser & Liu '22]
  - Closed-universe quantum gravity [Witten '23]
  - Quantum field theory in curved spacetime (QFTCS)

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- In an appendix to their 2019 paper, Dutta and Faulkner outlined a general framework for connecting canonical purification to the "GNS purification" beyond the density matrix regime.

- We'll come back to that in a little bit, but the main point of this talk is that by taking that connection seriously, we can make massive simplifications to formal problems in QFT.

Part Two

The problem of sectors

Sectors in QFT

- We usually think about doing QFT in a Hilbert space.
- But even in Minkowski, there are a few different Hilbert spaces you could choose — typically the **thermal sector**

$$\mathcal{H}_\beta$$

  including the vacuum sector as a special case.
- These Hilbert spaces are totally inequivalent! You can't map

$$U : \mathcal{H}_{\beta_1} \to \mathcal{H}_{\beta_2}$$

The GNS framework

- In general, have an **algebra of quantum fields** living on a spacetime, without reference to a Hilbert space.

- Free field theory:

$$\langle \phi(x) \mid (\nabla^2 - m^2)\phi = 0 \text{ and } [\phi(x), \phi(y)] = -iE(x,y) \rangle$$

- You just choose a set of correlation functions and construct a Hilbert space in which they are realized:

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle_\omega \to \langle \omega | \phi_1(x_1) \dots \phi_n(x_n) | \omega \rangle$$

Sector equivalence [1/2]

- In this kind of general framework, there's a serious **isomorphism problem**.
- When do two sectors describe the same physics?

$$\mathcal{H}_{\omega_1} \sim \mathcal{H}_{\omega_2}$$

- If you don't know how to answer this question, you end up hopelessly confused.

An answer in free field theory

- In free field theory, the answer to this question is known since the '80s.
  [Powers & Stormer '70; Araki '71; Van Daele '71; Araki & Yamagami '82]

- Theorem is too complicated to explain in a short talk, but here's the statement:

Theorem. *Two quasifree states* φ_S *and* $\varphi_{S'}$ *have quasi-equivalent GNS representations* π_S *and* $\pi_{S'}$ *if and only if the following two conditions hold:*

(1) $\tau_S = \tau_{S'}$ $(\equiv \tau)$.

(2) *Let* $\overline{K}$ *be the completion of* K *by the topology* τ *with an inner product* (x, y) *inducing the topology* $\overline{\tau}$ *and let*

$$S(x, y) = (x, \tilde{S}y), \quad S'(x, y) = (x, \tilde{S}'y).$$

Then $\tilde{S}^{1/2} - \tilde{S}'^{1/2}$ *is in the Hilbert-Schmidt class.*

Part Three

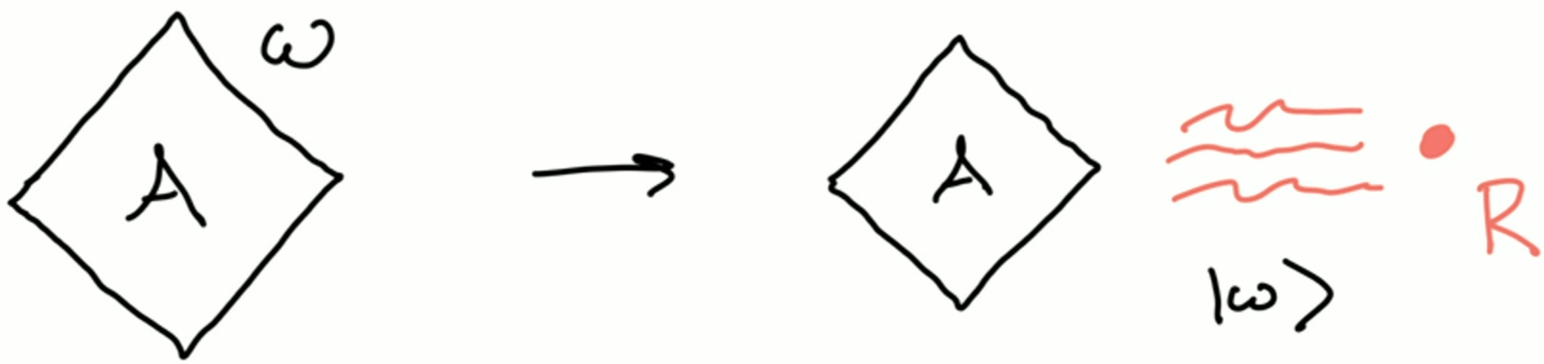
GNS, canonical, and CRT sewing

Back to GNS

- A quantum state is a list of correlation functions on an algebra:

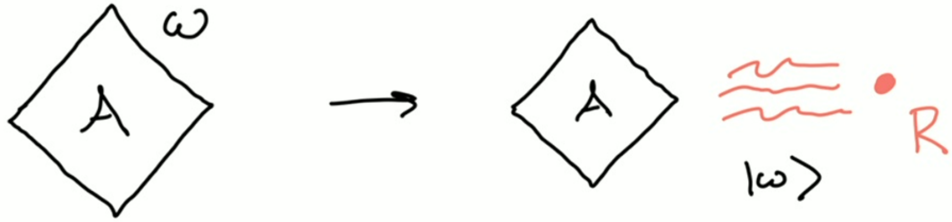
$$\omega : \mathcal{A} \to \mathbb{C}$$

- GNS realizes this on a Hilbert space with a purifying system:

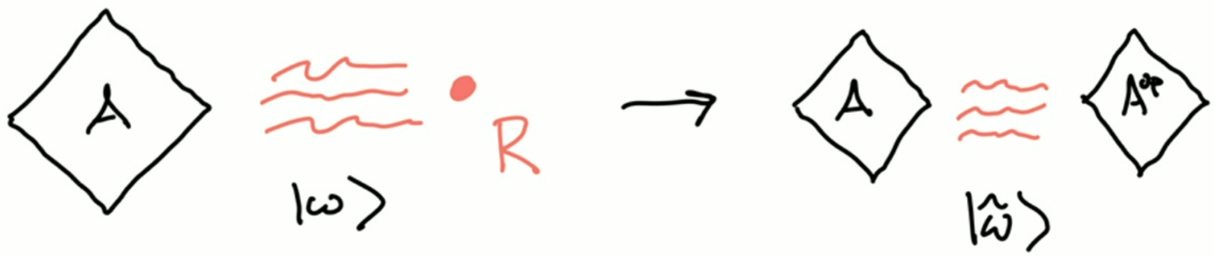


The purifying system in GNS

- A priori, there's no structure in the purifying system:



- But (in certain settings) there's a natural way to identify it with a time-reversed copy of the original d.o.f.:



Purifying systems and modular conjugation

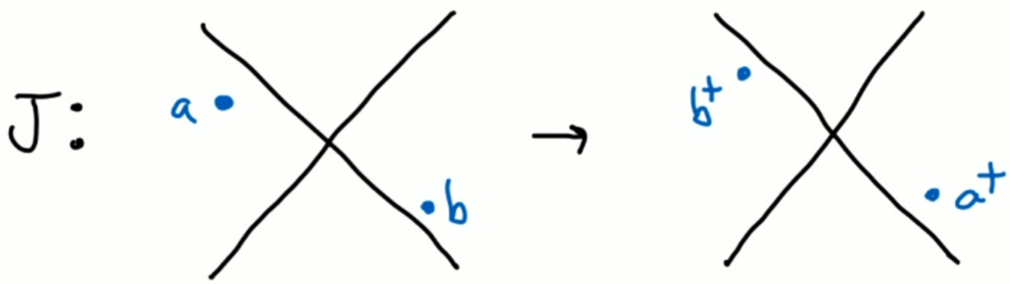
- The purifying system is described algebraically by the "commutant" algebra $\mathcal{A}'$.

- In the "separating" case, modular theory gives you an operator that maps

$$J_\omega \mathcal{A} J_\omega = \mathcal{A}'$$

A familiar example

- This map generalizes the CRT reflection in the Minkowski vacuum sector:



But it’s a mathematical structure that always exists.

Correlation functions on a doubled space

- Define the “canonical purification” to be a state on two copies of the original d.o.f., plus time reversal:

$$\hat{\omega} : \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \to \mathbb{C}$$

- Just use the modular conjugation on the GNS space:

$$\hat{\omega}(a \otimes b) = \langle \omega | a J_\omega b^\dagger J_\omega | \omega \rangle$$

- (Note for experts: this does something interesting in the non-separating case; ask me offline.)

Part Four

Simplifying sector equivalence

An application to sector equivalence

- Sector equivalence in pure states is pretty easy: you just hunt for a unitary.

$$\mathcal{H}_{\omega_1} \xrightarrow{U?} \mathcal{H}_{\omega_2}$$

- In practice, you write down an ansatz for $U|\omega_1\rangle$ and solve it. [Shale 1962, Wald 1975]

- Solvability is controlled by whether the ansatz is normalizable. This is the condition you check in practice.

In equations:

$$U|\omega_1\rangle = \oplus_{n \text{ particles}} |\psi^{(n)}\rangle$$

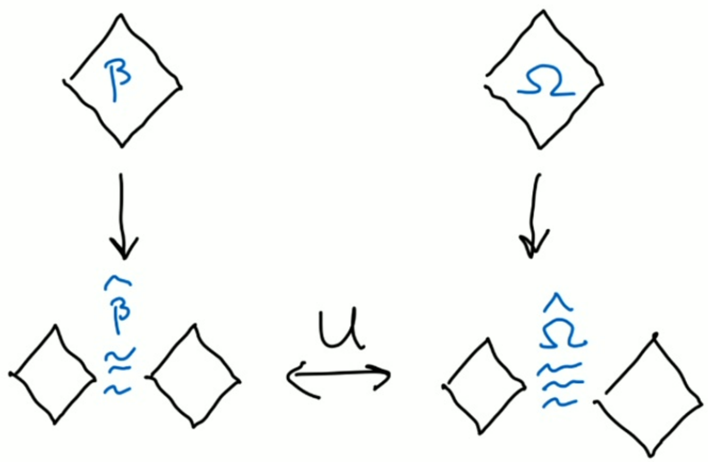
constrain amplitudes with $\langle U\omega_1 | \ldots | U\omega_1 \rangle = \langle \ldots \rangle_{\omega_1}$

demand $\sum_n \langle \psi^{(n)} | \psi^{(n)} \rangle < \infty$

Then $\tilde{S}^{1/2} - \tilde{S}'^{1/2}$ *is in the Hilbert-Schmidt class.*

Canonical purifications help!

- OK, so let's just purify away all the IR junk!



- This is the essence of the proof — the Araki-Yamagami theorem really just tells you when canonical purifications are unitarily related.

Summary

- Quantum information has been very successful at turning hard problems into easy ones.
- Turning a mixed-state problem into a pure-state one is a classic way to do this.
- We have the technology, building on [Dutta & Faulkner '19], to do this in a controlled way in very general settings.
- Araki-Yamagami simplification is a proof of concept: we should be thinking about where else we can use this tool.

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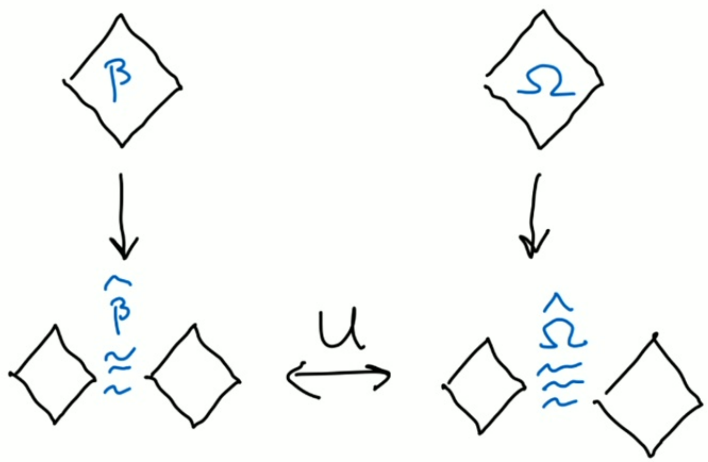

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