Title: Towards an information theory of scrambling

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Abstract:

A scrambling unitary never destroys information according to quantum information/Shannon theory. However, this framework alone doesn’t capture the fact that scrambled information can be effectively inaccessible. This limitation points to the need for a new kind of information theory—one that quantifies how much information is scrambled, rather than how much is lost to noise. To address this, we propose introducing a new family of entropies into physics: free entropy. Unlike conventional quantum entropies, which are extensive under tensor independence, free entropy has the defining feature of extensivity under freeness—the appropriate notion of independence pertaining to quantum scrambling.

I will present a preliminary result showing how free entropy naturally arises in a variant of Schumacher compression, providing it with an operational interpretation as the quantum minimum description length of quantum states. I will sketch how this interpretation extends to observables and unitaries, allowing free entropy to capture an operational aspect of quantum scrambling. Finally, I will highlight striking parallels between free entropy and von Neumann entropy, suggesting that free entropy may form the foundation of a new, complementary information theory.

Classical independence

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$$

Classical independence

i.e. $$\mathbb{E}(\tilde{X}\tilde{Y}) = 0, \quad \tilde{X} := X - \mathbb{E}(X)$$

↑ Connected 2pt function

Quantum correlations in two ways

$$\varphi(\tilde{O}_A \tilde{O}_B) \neq 0, \quad \tilde{O}_{A,B} := O_{A,B} - \varphi(O_{A,B})1$$

1. $[O_A, O_B] = 0,$
   φ entangled (while observables separate in space)

2. $[O_A, O_B] \neq 0,$
   φ unentangled (while observables separate in time)

Two notions of independence

if $\forall O_A, O_B \text{ s.t. } [O_A, O_B] = 0$,

$$\varphi(\tilde{O}_A \tilde{O}_B) = 0$$

(Tensor) Independence : $\varphi \cong \varphi_A \otimes \varphi_B$

when regions A and B are spatially distant.

Two notions of independence

Given O_A, O_B s.t. $[O_A, O_B] \neq 0$,

$$\varphi(\tilde{O}_A \tilde{O}_B \tilde{O}_A \tilde{O}_B \cdots) = 0, \text{ etc } , \forall \text{ words}(O_A, O_B) .$$

Freeness : O_A and O_B are free w.r.t. φ

when O_A and $O_B = U_t O_A U_t^\dagger$ are temporally distant (and at large N).

[Voiculescu '91]

Intuitively,

Independence		Freeness
Spatially distant	vs	Scrambled in time
"Locality: what happens here doesn't influence something in a distance"		"Butterfly effect: Knowing the past doesn't help you predict the future."

~~Tensor Independence~~ ⟶ Freeness

~~Quantum information theory~~ ⟶ ?

$$H(X)_p = \sum_x - p_x \log p_x$$

$$p_X \rightarrow \rho_X$$

commuting

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

non-commuting

Another formula for Shannon entropy

$$H(X)_p = \sum_x - p_x \log p_x$$

"Gibbs"

Counting Measure

Typical strings

$$H(X)_p = \lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \#\{x_N \in \mathcal{X}^N \,|\, |P_{x_N} - p| \leq \varepsilon\}$$

"Boltzmann"

Entropy counts the number of typical strings => Compression vN Entropy

"Gibbs" formula

$p_X \to \rho_X$

Shannon Entropy

?

"Boltzmann" formula

Counting typical ~~strings~~ matrices

Matricial "microstates"

$$x_N \rightarrow X_{N\times N}$$

commuting

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

non-commuting

Lebesgue measure

N-by-N Hermitian matrix

$$\chi(a) := \lim_{\varepsilon\to 0}\lim_{N\to\infty}\frac{1}{N^2}\log \Lambda\left\{X = X^\dagger \in \mathbb{R}^{N^2} \,\middle|\, \forall k, |\,\mathrm{tr_N} X^k - \tau(a^k)\,| \leq \varepsilon\right\}$$

$a = a^* \in (\mathscr{A}, \tau)$

Free Entropy !

[Voiculescu '93]

*It was invented as a tool to study operator algebra.

Free Entropy also has two formulas:

$$\chi(a) := \lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{1}{N^2} \log \Lambda \left\{ X = X^\dagger \in \mathbb{R}^{N^2} \,\middle|\, \forall k, |\,\mathrm{tr}_N X^k - \tau(a^k)\,| \le \varepsilon \right\}$$

$(a = a^*)$

"Boltzmann"

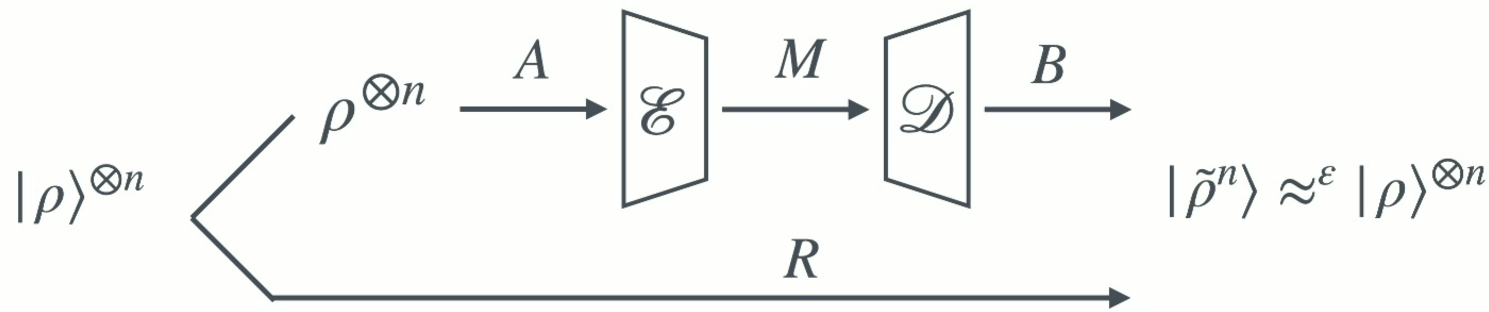
$$\chi(a) = \int\int \log|x - y|\, \mathrm{d}\mu_a(x)\mathrm{d}\mu_a(y) + \mathrm{Const}$$

"Gibbs"

Spectral density of a

Does free entropy have an operational meaning? If so, it's gotta be compression of some sort in quantum theory.

Revisiting Schumacher compression



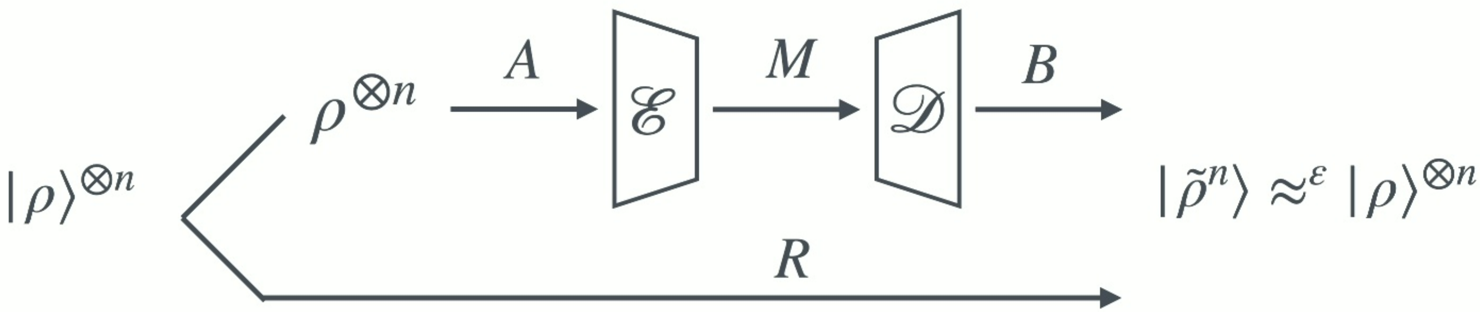
$$|\rho\rangle_{AR} \equiv \sum \sqrt{p_x}\,|\psi_x\rangle_A |\psi_x\rangle_R$$

Task: Minimize $|M|$ over $(\mathscr{E}, \mathscr{D})$.

Result: $|M| = nS(\rho) + O(\sqrt{n})$ [Schumacher '95]

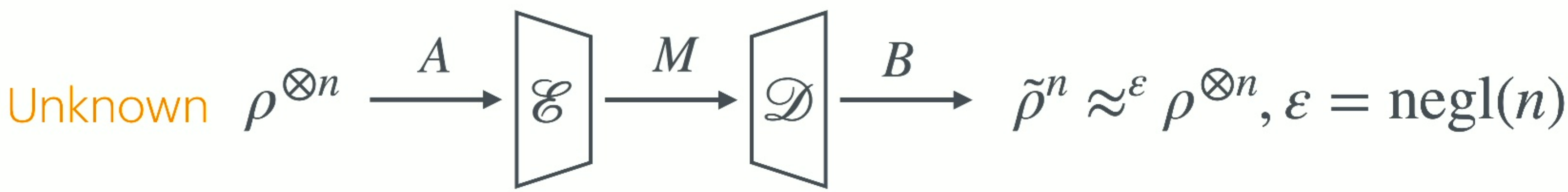
Idea: Let's consider a variant of Schumacher compression.

Minimal quantum description



$|\rho\rangle_{AR} \equiv \sum \sqrt{p_x}\,|\psi_x\rangle_A |\psi_x\rangle_R$

$\Downarrow$ Remove the purification R



What's the minimal quantum description $M(\rho^{\otimes n})$ of some unknown $\rho^{\otimes n}$?

We show

Free entropy

$$|M(\rho^{\otimes n})| = \frac{1}{2}\chi_{\text{phy}}(\rho; 1/n) := \frac{1}{2}[d^2 \log n + \chi(\rho)] + \text{negl}(n)$$

$$= \frac{1}{2}(d^2 - \sum_i g_i^2)\log n + \sum_{i<j} g_i g_j \log|p_i - p_j| + \text{negl}(n)$$

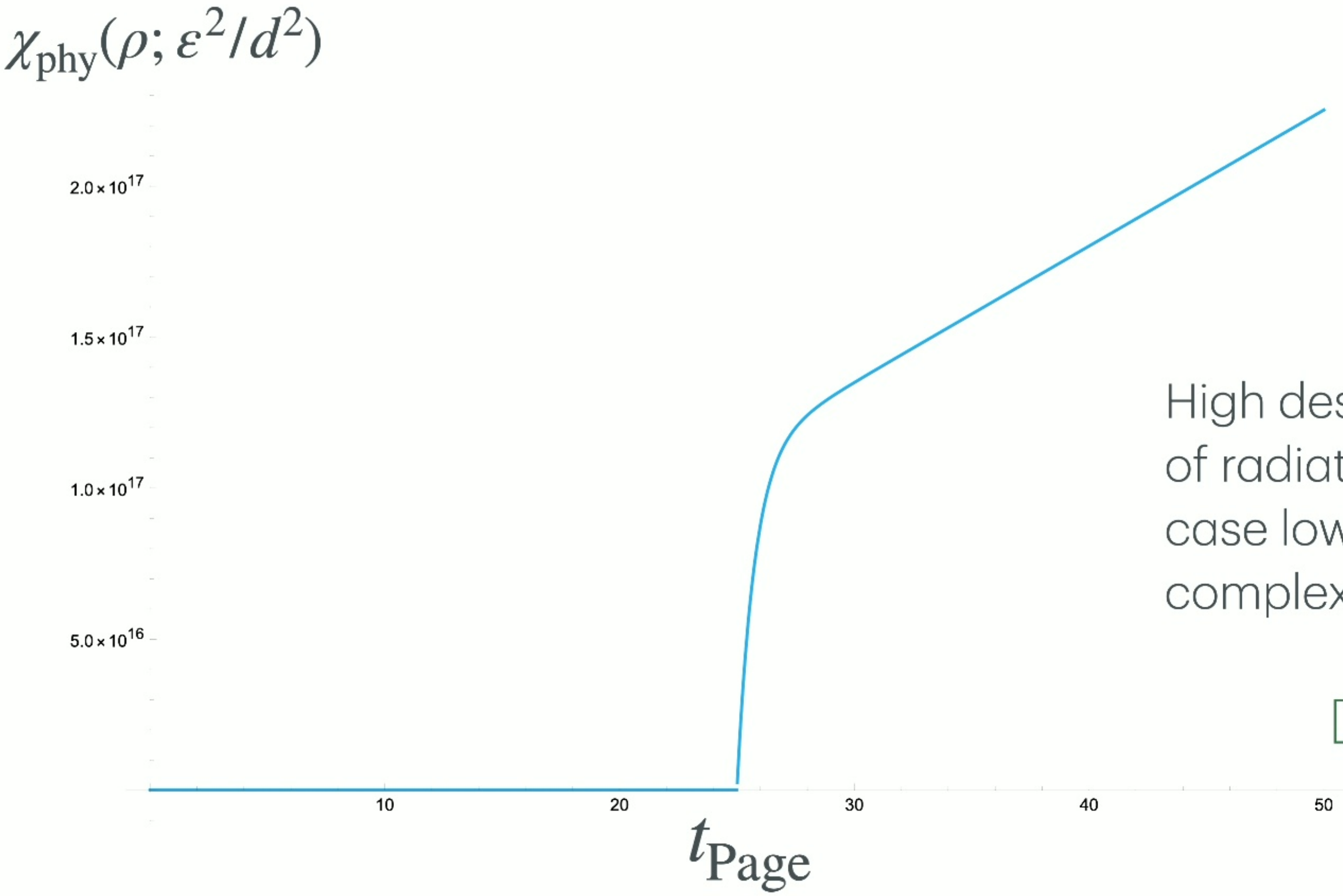
Degeneracy

Spectrum

[Hayden, Maloney, JW, Yang, wip]

Free entropy characterizes the Quantum Minimum Description Length (QMDL) of $\rho^{\otimes n}$, i.e. its descriptive complexity.

A Page curve of free entropy



High descriptive complexity of radiation, i.e. a worst-case lower bound on comp. complexity.

[Harlow, Hayden '13]

Free Entropy of two variables

Free entropy is defined for any operators, such as unitary evolutions and Hermitian observables, not only density operators.

Monomial of degree m.

$$\chi(u,v) := \inf_{\varepsilon>0,\, m\in\mathbb{N}} \lim_{N\to\infty} \frac{1}{N^2} \log \gamma_{\text{Haar}}\left\{(U,V)\in U(N)^{\otimes 2} \,\middle|\, |\operatorname{tr}[P_m(U,V)] - \tau[P_m(u,v)]| \le \varepsilon\right\}$$

$u, v \in (\mathscr{A}, \tau)$

Free entropy is defined on the non-commutative distribution $\{\tau(P(u,v))\}_P$

$\chi(u,v) \le \chi(u) + \chi(v)$ (subadditivity), and it saturates for free $u, v \in (\mathscr{A}, \tau)$.

Free entropy and Universal programming

We believe free entropy of unitaries measures the QMDL of their universal programs, which encode the unitaries into quantum states.

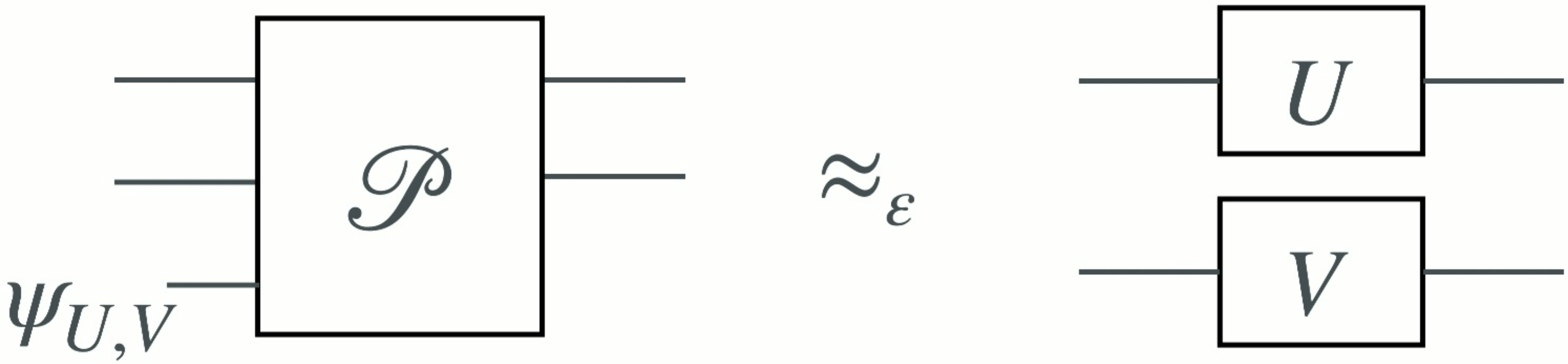
QMDL of U is determined by $\chi_{\text{phy}}(U;\varepsilon) = d^2 \log \varepsilon^{-1} + \chi(U)$

[wip]

No programming theorem: $\varepsilon \to 0$, QMDL $\to \infty$

[Nielson, Chuang '97]

Universal programming of (U, V)



QMDL of (U, V) is determined by $\chi_{\mathrm{phy}}(U, V; \varepsilon) = 2d^2 \log \varepsilon^{-1} + \chi(U, V)$

[wip]

$\chi_{\mathrm{phy}}(U, V; \varepsilon) \leq \chi_{\mathrm{phy}}(U; \varepsilon) + \chi_{\mathrm{phy}}(V; \varepsilon)$ follows from concatenation.

Free mutual information

The subadditivity gap of QMDL is the free mutual information!

$$\lim_{\varepsilon \to 0} \left(\chi_{\text{phy}}(U; \varepsilon) \qquad - \chi_{\text{phy}}(U \,|\, V; \varepsilon) \qquad \right)$$
$$= \lim_{\varepsilon \to 0} \left(\chi(U) \quad - \chi(U \,|\, V) \quad \right) =: I_{\text{free}}(U : V)$$

$I_{\text{free}}(U : V)$ measures the gain in QMDL of U if we know how it relates to some V. Suppose $U = V(t), V = V(0),\ I_{\text{free}}(U : V)$ quantifies how much knowing the past can help us describe the future, and hence a good probe of how scrambling the dynamics is. (cf. Shreya's talk)

Unscrambling Capacity

Given a unitary dynamics U_t, we can then use I_{free} to probe its scrambling. We conjecture that the following should be thought of as some kind of unscrambling capacity of U_t in a hypothetical communication task.

[wip]

$$C(U_t) = \sup_O I_{\text{free}}(O : U_t O U_t^\dagger)$$

More scrambling U_t is, smaller its unscrambling capacity.

*There is no good entropic measure in quantum Shannon theory that can do this job.

Laws of Information Theory

[Pippenger, '86]

$H \geq 0$ (Positivity)

$H(X) \leq H(X,Y)$ (Monotonicity)

$H(X,Y) \leq H(X) + H(Y)$ (Subadditivity)

$H(X,Y) = H(X) + H(Y)$ (X, Y Independent)

$H(X,Y,Z) + H(Z) \leq H(X,Z) + H(Y,Z)$
(Strong Subadditivity)

...

$S \geq 0$ (Positivity)

$S(A,B) \leq S(A) + S(B)$ (Subadditivity)

$S(A,B) = S(A) + S(B)$ (A, B Independent)

$S(A,B,C) + S(C) \leq S(A,C) + S(B,C)$
(Strong Subadditivity)

Intriguingly, It turns out that

$\chi(a) \leq \chi(a,b)$ (Monotonicity)

$\chi(a,b) \leq \chi(a) + \chi(b)$ (Subadditivity)

$\chi(a,b) = \chi(a) + \chi(b)$ (a, b free)

$\chi(a,b,c) + \chi(c) \leq \chi(a,c) + \chi(b,c)$
(Strong Subadditivity [Jung, '03])

An information theory
of scrambling?

Thank you!