Title: Chaos in deformed SYK models

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Abstract:

The SYK model has attracted significant interest for its maximal chaos, connections to two-dimensional holography, and relevance to strange metals.
Two signatures of chaos in this model are out-of-time-ordered correlators (OTOCs) and Krylov complexity, both exhibiting early-time exponential behaviours characterized by the Lyapunov and Krylov exponents, respectively. In this talk, I explore these quantities in a class of relevant deformations of the SYK model, including flows which interpolate between two regions of near-maximal chaos and flows that lead to a nearly-integrable behaviour at low temperatures. I present both analytic and numerical results showing that the Krylov exponent consistently upper-bounds the Lyapunov exponent. Notably, while the Lyapunov exponent varies non-monotonically with temperature, the Krylov exponent remains smooth and monotonic, showing no clear signatures across chaotic transitions. This challenges the effectiveness of Krylov complexity as a diagnostic of chaos in quantum mechanical systems. I will also comment on the potential relevance of SYK flows for de Sitter holography and on connections to other recent works.

Chaos in deformed SYK models

Shira Chapman, Ben-Gurion University

QIQG 2025: Quantum Information in Quantum Gravity

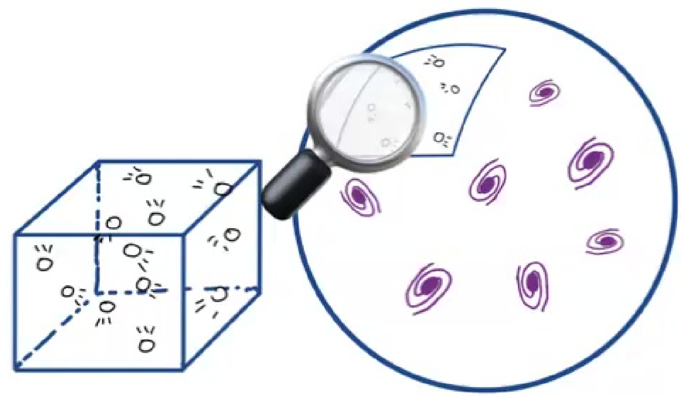
26-06-2025

SC, Saskia Demulder, Damian Galante, Sam Sheorey, Osher Shoval [PRB 2025']

Motivation

- Better understanding many-body systems.
- Signatures of quantum chaos.
- OTOC and Krylov Complexity

- Microscopic models of spacetime?
- The role of chaos in holography.
- Potential for dS holography?



Outline

- Motivation [V]
- SYK model and its deformations
- Signatures of Chaos
- Results – signatures of chaos in SYK deformations
- Summary and Outlook: hints for dS holography?

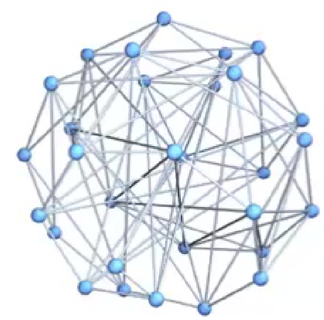
SYK model: ensemble of 1D many-body QM

$$H_q = i^{\frac{q}{2}} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} J_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q} \ , \qquad \{\psi_i, \psi_j\} = \delta_{ij}$$

random q-body interactions

N Majorana fermions ($i \in 1 \ldots N$)

$$\langle J_{i_1 i_2 \cdots i_q} \rangle = 0 \ , \qquad \langle J^2_{i_1 i_2 \cdots i_q} \rangle = \frac{2^{q-1}}{q} \frac{\mathcal{J}^2 (q-1)!}{N^{q-1}}$$



[Sachdev and Ye (1993); Maldacena and Stanford (2016); Kitaev and Suh (2018)]
[Review: Chowdhury, Georges, Parcollet, Sachdev]

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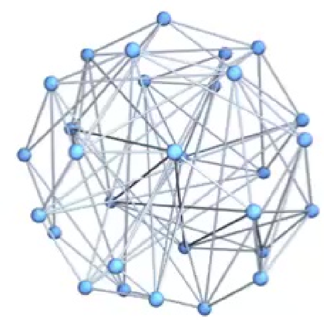
Appealing properties:

- Exactly solvable at large N, in terms of effective action for fermion bilinears, whose equations of motion are the Schwinger-Dyson equations.
- Further simplification when q also large (ratio $\lambda \equiv 2q^2/N$ fixed: DSSYK, we take $\lambda \to 0$).
- Exhibits chaos for $q \geq 4$; Special case: $q = 2$ is integrable.
- Used to model strange metals (many-body quantum states without quasiparticles).



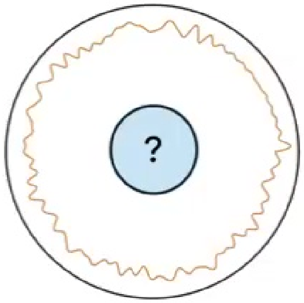
[Sachdev and Ye (1993); Maldacena and Stanford (2016); Kitaev and Suh (2018)]
[Review: Chowdhury, Georges, Parcollet, Sachdev]

The low temperature limit and gravity

- At low energies, the SYK model is approximated by a conformal field theory where the fermions obtain a conformal dimension of $\Delta_\psi = 1/q$.
- The effective action: $S_{\text{eff}} = S_{\text{CFT}} + S_{\text{Sch}} + \ldots$

with a leading Schwarzian correction making it nearly conformal.

- Maximally chaotic at low temperatures (and yet tractable!)



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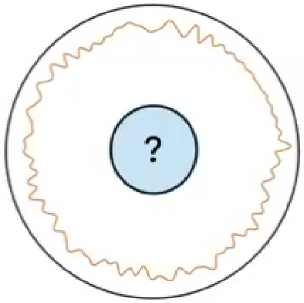
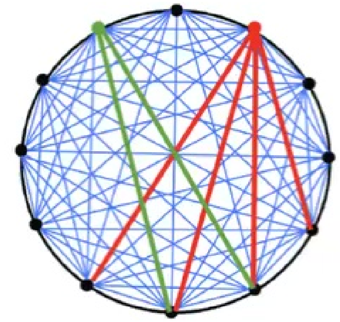
Dual gravitational description JT gravity (gravity + dilaton)

- Bulk is rigid AdS_2 and dynamics reduces to that of the dilaton boundary mode [Maldacena, Stanford, Yang (2016)]
- Can we obtain more general gravity backgrounds from SYK deformations? with different IR?



Deformations of the SYK model



- Relevant deformation of SYK – hard to find!
- Why? In the IR the fermion acquired a dimension

$$H_q = i^{\frac{q}{2}} \sum J_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q} \to \Delta_\psi = 1/q$$

- Naive additions to the Hamiltonian - vanishing $\mathcal{O} = \sum_i \psi_i \psi_i$ or irrelevant: $\mathcal{O} = \Sigma_i \psi_i \partial_\tau^{2n+1} \psi_i$
- Consider disordered operators, like the Hamiltonian itself

$$H_{\text{def}} = H_q + s H_{\tilde{q}} \ , \qquad s > 0 \ , \quad q \geq \tilde{q}$$

*Naive scaling of deformation around IR fixed point $\Delta = \tilde{q}/q < 1$.
*Becomes important at very low energies (deep IR)

[Jiang and Yang (2019); Anninos, Galante (2020), Anninos, Galante, Sheorey (2022)]

[Garcia Garcia et al. (2018); Lunkin et al. (2020); Nandy et al. (2022)]

q = # of fermions in each interaction term

$$H_{\rm def} = H_q + sH_{\tilde{q}} \ , \qquad s > 0 \ , \quad q \geq \tilde{q} \ .$$

- Large N & large q techniques can be generalized to this case.
- Effective theory in terms of bilinear fields (fermion bi-linear G and self-energy Σ):

$$G(\tau_1, \tau_2) = \frac{1}{N}\sum_{i=1}^{N} \langle T\psi_i(\tau_1)\psi_i(\tau_2)\rangle, \qquad \Sigma(\tau_1, \tau_2) = \text{self energy}$$

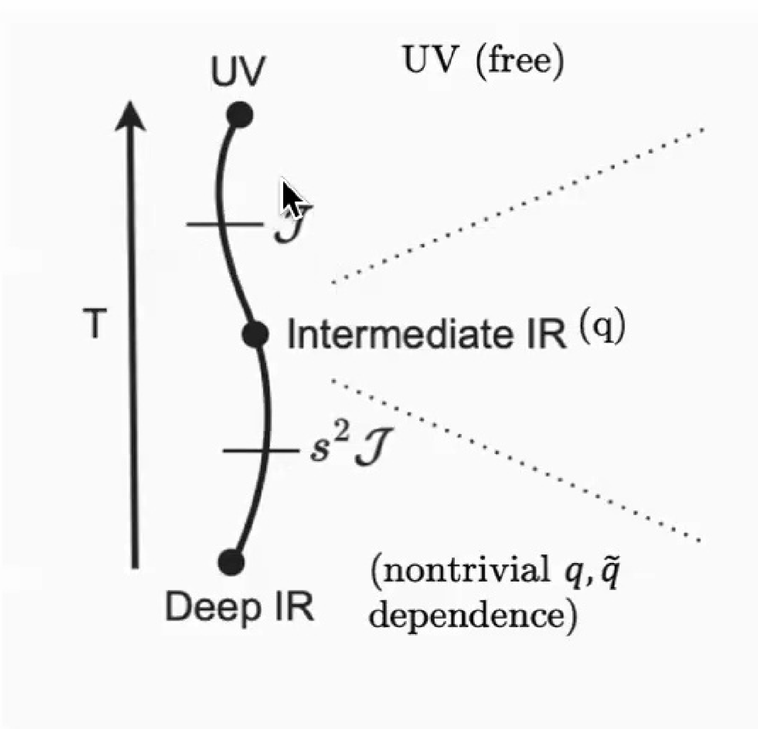
- Deformed Schwinger-Dyson equations (EOM):

$$G^{-1}(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2)\partial_{\tau_2} - \Sigma(\tau_1, \tau_2) \ ,$$

$$\Sigma(\tau_1, \tau_2) = \mathcal{J}^2 \left(\frac{2^{q-1}}{q} G(\tau_1, \tau_2)^{q-1} + s^2 \frac{2^{\tilde{q}-1}}{\tilde{q}} G(\tau_1, \tau_2)^{\tilde{q}-1} \right) \ .$$

- Large N and large q ($q, \tilde{q} \to \infty$ with $n \equiv q/\tilde{q}$ fixed), have analytic solution

Regimes along the RG flows



*If not $s \ll 1$ go directly to deep IR

$$H_{\text{def}} = H_q + sH_{\tilde{q}} \ , \qquad s > 0 \ , \quad q \geq \tilde{q}$$

- Renormalization group flows away from the near-fixed point of the Sachdev-Ye-Kitaev (SYK) model

- **Expect two different regimes of nearly maximal chaos, can we see them?**

[Jiang, Yang]
[Anninos, Galante, Sheorey, ...]

Quantum Chaos

(and how to recognize it)

Signature 1 - Out of time ordered correlation functions (OTOCs)

- Quantum version of hypersensitivity to initial conditions.

$$\mathrm{OTOC}(t) = \frac{1}{N^2}\sum_{i,j=1}^{N} \mathrm{Tr}\left(\rho^{\frac{1}{4}}\psi_i(t)\rho^{\frac{1}{4}}\psi_j(0)\rho^{\frac{1}{4}}\psi_i(t)\rho^{\frac{1}{4}}\psi_j(0)\right), \qquad \rho = \frac{1}{Z(\beta)}e^{-\beta H}$$

$$\boxed{\mathrm{OTOC} = f_0 - \frac{f_1}{N}\exp(\lambda_{\mathrm{OTOC}}t) + \ldots}$$

← **OTOC Lyapunov exponent**

- A bound on chaos [Maldacena, Shenker, Stanford (2016)]: $\lambda_{\mathrm{OTOC}} \leq \frac{2\pi}{\beta}$
- Need semiclassical limit to have separation between the dissipation and scrambling time.
- In SYK – relevant signature when N is large.

Signature 2 –Krylov Complexity

- No need for large N in this approach
- How fast does an operator spread with time $\mathcal{O}(t) = e^{iHt}(\mathcal{O}_{local})e^{-iHt}$?
- More precisely – how complex does it become with time?
- Construct a basis for the time evolution:

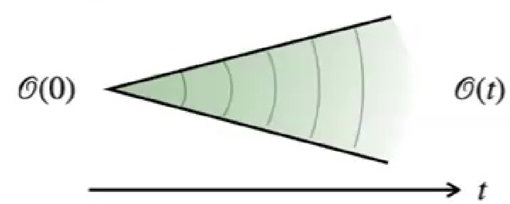
$$\mathcal{H}_{\mathcal{O}} = \text{span}\{\mathcal{O}, [H,\mathcal{O}], [H,[H,\mathcal{O}]], \ldots\} \longrightarrow \mathcal{H}_{\text{Krylov}} = \text{span}\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \ldots\}$$

Simple ↔ complicated

- Measure of complexity of the time-evolved operator:

Krylov exponent

$$C_K(t) = \sum_n n \times \begin{pmatrix} \text{amplitude} \\ \text{to be at } \mathcal{O}_n \end{pmatrix} = \exp(\lambda_K t)$$



- Conjecture: exponential spreading in Chaotic systems
- Computable from the thermal two point function - easier.

[Parker, Cao, Avdoshkin, Scaffidi, Altman (2019)]

Relation between the two exponents

- Bounds on Krylov complexity
  [Parker, Cao, Avdoshkin, Scaffidi, Altman (2019); Avdoshkin, Dymarsky (2020); Gu, Kitaev, Zhang (2022)]

$$\lambda_{\mathrm{OTOC}} \leq \lambda_K \leq \frac{2\pi}{\beta}$$

- The bound turned out to be tight $\lambda_{OTOC} = \lambda_K$ for SYK at large N, large q, at any value of the temperature.
- Conjecture: the two are always equal. Easier way to calculate OTOC exponent?

Questions

- What are λ_{OTOC} and λ_K in simple and in deformed SYK models?
- Are they generally equal?
- Can they both diagnose chaos transitions along RG flows?

Answers:

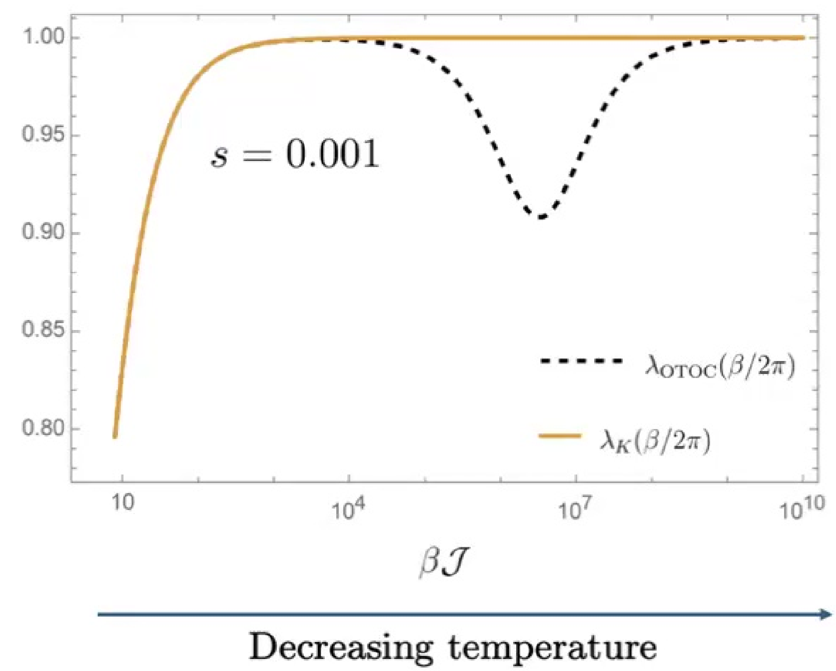
- The two exponents are not equal! Can already see at next order in 1/q expansion for a single SYK:

$$\frac{\beta}{2\pi}(\lambda_K - \lambda_{\rm OTOC}) = \frac{4\pi^2}{3q\beta\mathcal{J}} + \dots .$$

- Only λ_{OTOC} can diagnose chaos, λ_K cannot!

Flows - Chaos to Chaos $H_{\text{def}} = H_q + sH_{q/2}, \quad q \to \infty$

- **Two regimes of nearly maximal chaos.**
- Chaos transition at $\beta \mathcal{J} \sim 1/s^2$
- OTOC feels the transitions
- Krylov does **not** diagnose chaos transitions!
- Krylov complexity correctly (but poorly!) bounds (above) the Lyapunov exponent.



Large N and finite q – Numeric strategy

Outline of the numerical procedure:

1) Solve the the Schwinger-Dyson equation iteratively in Fourier space to extract the fermion 2pt function.
2) Evaluate the Krylov exponent from the moments of the 2pt functions
3) Solve a Kernel equation for the OTOC, as a matrix diagonalization in discretized time.
4) Extract the OTOC Lyapunov exponent.
5) Numerical difficulties associated with low temperatures (many coding hours of our talented students & postdocs)

$$K(t_1,t_2,t_3,t_4) = G^R(t_{13})G^R(t_{24})\mathcal{J}^2\left(\frac{2^{q-1}}{q}(q-1)G^W(t_{34})^{q-2} + s^2\frac{2^{\tilde{q}-1}}{\tilde{q}}(\tilde{q}-1)G^W(t_{34})^{\tilde{q}-2}\right)$$

$$F(t_1,t_2) = \int dt_3 dt_4\, K(t_1,t_2,t_3,t_4)\, F(t_3,t_4)$$

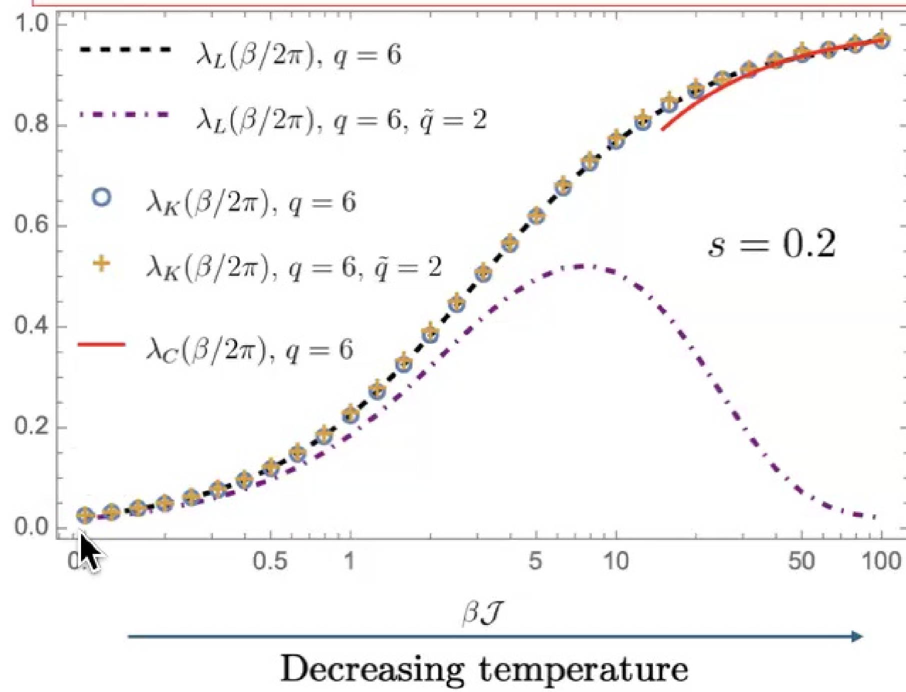
$$\mathrm{OTOC}(t_1,t_2) = F_0(t_1,t_2) + \frac{1}{N}F(t_1,t_2) + \cdots\ .$$

Flows - Chaos to integrable

$$H_{\text{def}} = H_6 + sH_2$$

- OTOC exponent sees the transition
- Krylov seems unable to diagnose the chaotic properties of the flows

4+2/6+2 of fermions in each interaction term



Other studies of chaos-integrable transitions: [Berkooz, Brukner, Jia, Mamroud, 2024; Garcia-Garcia, Loureiro, Romero-Bermudez, Tezuka, 2018; Kim, Cao, 2021; Lunkin, Kitaev, Feigel'man, 2020; Nandy, Cadez, Dietz, Andreanov, Rosa 2022; Menzler, Jha 2024...]

Summary

- We saw deviations between the two exponents (Krylov vs. OTOC) already for the single SYK.
- For flow SYK, Krylov did not provide a good tool for diagnosing chaos transitions along RG flows
- In all examples, the Krylov exponent grew monotonically (when it could be defined)

❖Some authors have suggested that it is the covariances of the Lanczos coefficients, that should be used to diagnose chaos [Balasubramanian, Magan, Wu, 2023']

Outlook: SYK Flows and dS Holography

- SYK at low energies dual to near–AdS_2
- Deformations of SYK: IR deformations of nearly–AdS_2 [Anninos, Galante (2020); Anr Galante, Sheorey (2022)]
- In 2D gravity, dS_2 bubbles can be embedded inside AdS_2 $\rightarrow$ enables dS holography with timelike boundary [Anninos, Hofman (2017)]
- Open question: Can SYK flows capture these dS–AdS configurations?

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- Hints from gravity: Thermodynamics matches SYK flows upon analytic continuation [Anninos, Galante, Sheorey (2022)] $H_{\text{def}} = H_q + sH_{\tilde{q}} \Rightarrow H_q + isH_{\tilde{q}}$
  - Non-Hermitian Hamiltonian interpretation?
  - dS dual to open quantum system? (dS has open boundary at future infinity!)

- Chaotic signatures differ from the usual AdS_2. [Anninos, Galante, Hofman 2019; SC, Galante, Kramer 2022] For example, the OTOC oscillates. Compare to signatures of chaos in open systems? [Sá Et al. 2021; Bhattacharya et al. 2022; Liu, et al. 2023; Bhattacharjee et al. 2023; Srivatsa et al. 2024…]

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Lots to explore!