Title: Von Neumann algebras in gravity (Vision Talk)

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Von Neumann algebras in gravity

Geoff Penington

Von Neumann algebras in gravity

- Pre-modern era

  Algebraic QFT (1960s -), Quantum null energy condition (Faulkner and collaborators), holographic QEC (Harlow?, Kang + Kolchmeyer) …

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2021

Von Neumann algebras in gravity

- Pre-modern era — ? - 2021

  Algebraic QFT (1960s -), Qu[illegible] (Faulkner and collaborators), holographic QEC (H[illegible]meyer)

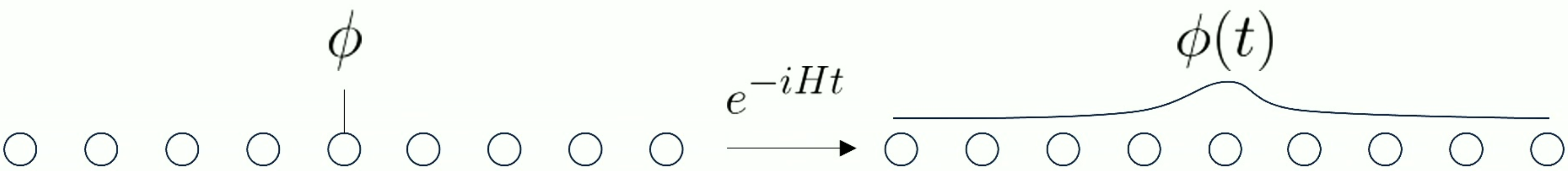
- Modern era — 2021 - present

  Large N algebras and modular theory (Leutheusser + Liu), crossed products and entropy (Witten), observers and de Sitter (CLPW) …

- Future?

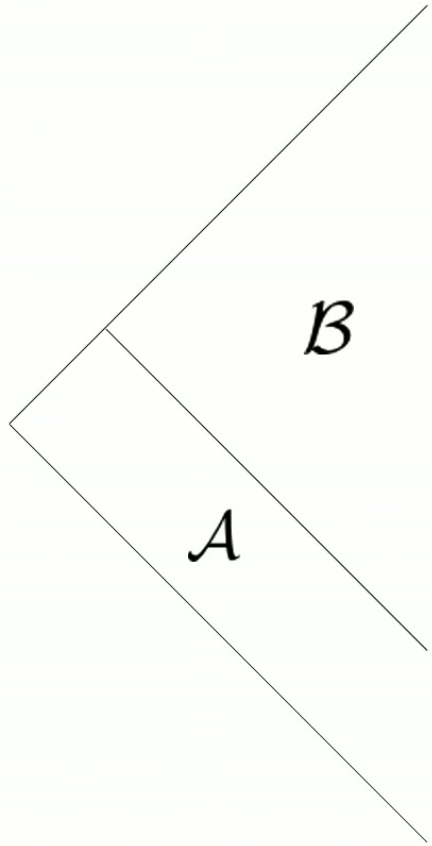
Why use von Neumann algebras?

- We often pretend that quantum field theories can be written as a tensor product over subregions and include formal divergences. Why bother with von Neumann algebras?
- We want to understand the structure of relativistic causality in QFT:
  - Hamiltonian evolution of local operators in a lattice theory always leads to infinite tails
  - Two options: take limits and worry about epsilons or just work directly with the right, natural mathematical structure (von Neumann algebras)
  - Second option is a bit fancier but much cleaner and nicer to deal with! Structure turns out to be both very powerful but also constraining
  - Use them any time you want to describe observables in a local region and care about having sharp lightcones (pretty often!)



Half-sided modular inclusions

- We are often interested in subregions that are null separated (e.g. Rindler wedges)
- Morally, the (larger) Rindler wedge has density matrix is $\rho = e^{-2\pi H_{boost}}$
- Modular flow (evolution using $\mathrm{H} = -\log \rho$) preserves the smaller wedge towards the future but not the past
- This is impossible for tensor product subsystems!



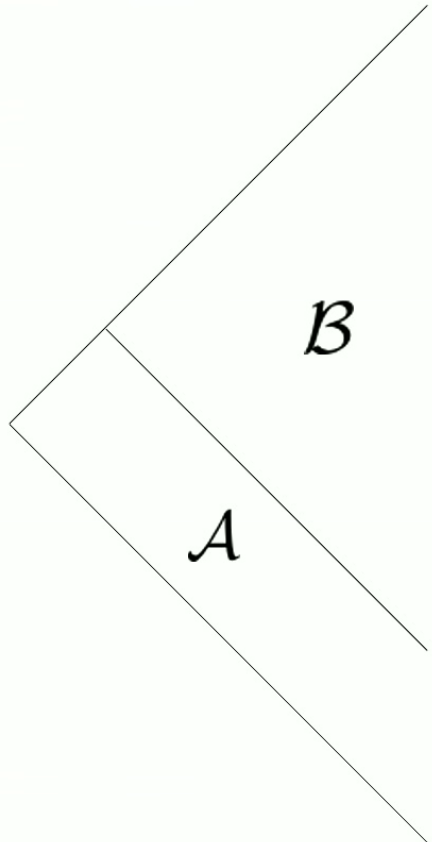
Half-sided modular inclusions

- If we remember that these are Type III von Neumann algebras, the same structure is possible but tightly constrained
- Density matrices don't exist, but modular flow does: now generated by the modular operator Δ " $= \rho \otimes \rho'$"
- Must have:

$$\mathcal{B} = e^{iP} \mathcal{A} e^{-iP} \qquad P = \frac{1}{2\pi} (\log \Delta_B - \log \Delta_A)$$

$$P \geq 0 \qquad [\log \Delta_A, P] = 2\pi i P$$

- Only possible for Type III algebras
- Obtain exactly the expected structure with P the null Hamiltonian
- Also works if null translation depends on transverse coordinates => **average null energy condition**



Relativistic causality in quantum gravity

- Gravitational spacetimes also feature relativistic causal structure, although in quantum gravity this structure will fluctuate
- Some features of this causal structure remain sharp (even nonperturbatively):
  - Disconnected and noninteracting boundaries in AdS/CFT can never be in causal contact
  - Left and right boundaries in the thermofield double state are exactly null separated
- But the Hilbert space of disconnected boundary regions is a genuine tensor product!

*For the moment we exclude all conserved charges (e.g. the Hamiltonian)

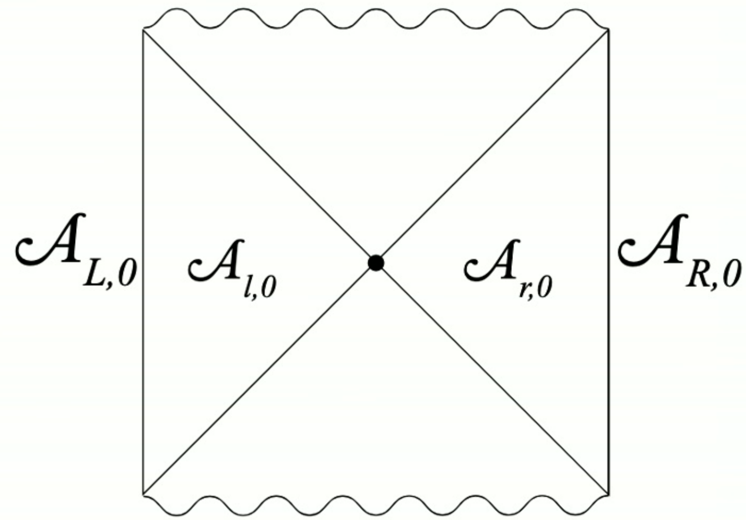
The large N limit and bulk QFT

- At leading order as $N \to \infty$, gravity decouples and correlation functions reduce to bulk QFT (including free gravitons)
- Specifically, after subtracting $\langle a \rangle = O(N)$, appropriately normalised single-trace operators a have $O(1)$ correlation functions that can be computed using free QFT
- **Leutheusser, Liu**: describe this structure in algebraic language
- Formally define a large N GNS Hilbert space $a_1 a_2 \dots |\Psi\rangle$ with inner products given by thermal correlation functions $\langle \Psi | b^\dagger a | \Psi \rangle = \langle b^\dagger a \rangle_\beta$ (+ completion)
- Algebra of single-trace operators* acts on this Hilbert space
- We can complete this algebra in the strong/weak operator topology to get a von Neumann algebra

[Leutheusser, Liu (arXiv:2110.05497, arXiv:2112.12156)]

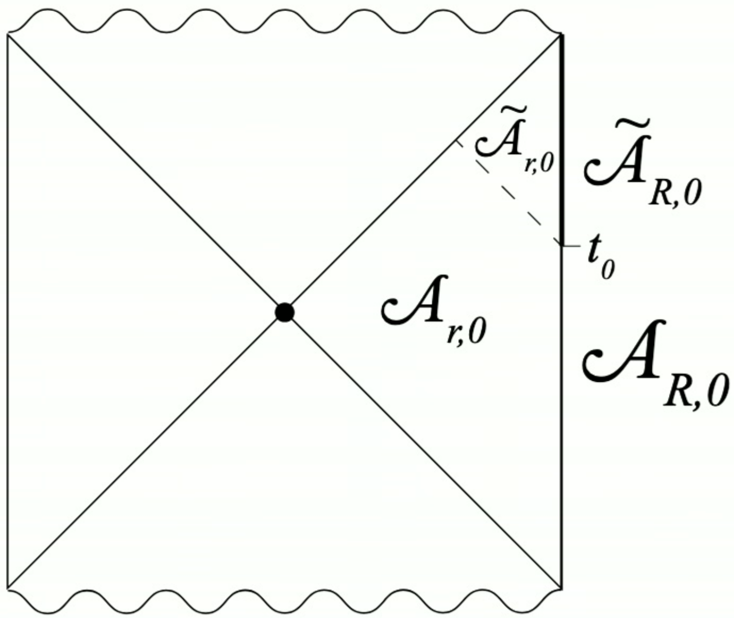
The bulk picture and commutant algebras

- This large N algebra is Type III and can be identified with the bulk algebra associated to the black hole right exterior
- The commutant algebra on the large N Hilbert space can be identified (using modular conjugation) with the large N limit of left boundary operators acting on the thermofield double state
- In the bulk picture, this becomes the left exterior algebra
- What about operators in the black/white hole interiors?



Null translations on the horizon

- Want to understand from a boundary perspective how to evolve exterior operators into the interior
- Schwarzschild time evolution <=> modular flow (but preserves the exteriors)
- Can define subalgebras of operators at time $t > t_0$ (or $t < t_0{}'$)
- These are half-sided modular inclusions => define operators $P_\pm$ that act as null translations along black/white hole horizons
- Act nonlocally elsewhere (as determined by bulk equations of motion)



OTOCs and chaos

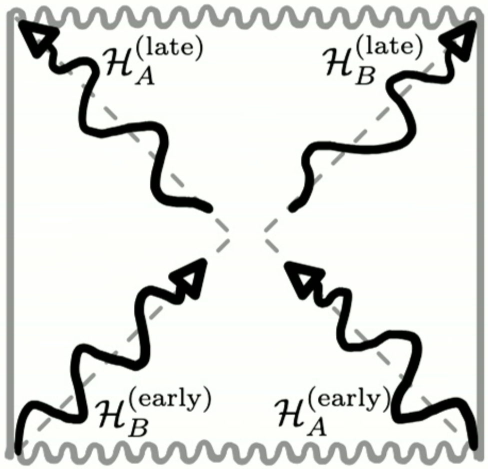
- Important probe of quantum chaos: out-of-time-ordered correlators e.g $\langle a(0)b(t)ab(t)\rangle$
- $O(1)$ corrections to free-field-theory answer when t is roughly the scrambling time $t_{scr} = \frac{\beta}{2\pi}\log N^2$
- Cannot be seen in Leutheusser-Liu algebra where all operators satisfy $t/\beta \sim O(1)$
- We can construct bigger large N Hilbert spaces/algebra by including not just operators $a(t)$ but also $b(T+t')$ with t, t' arbitrary but finite, while T is a fixed but diverging function of N
- When $\beta \ll T \ll t_{scr}$, the Hilbert space and algebras are just tensor products of early + late time modes
- When $T \gg t_{scr}$, OTOCs vanish and the algebra is a free product

[Chandrasekaran, GP, Witten (arXiv:2209.10454)]

An algebra for scrambling

- When $T = t_{scr}$, the large N algebra needs to capture the entire behaviour of all (n-point) OTOCs from **early Lyapunov growth** of commutators to **late-time decay**
- However, the nontrivial relevant physics all comes from gravitational eikonal scattering
- Phase proportional center of mass energy (=product of null momenta of scattering modes)
- Hilbert space is just a tensor product, but

$$\mathcal{H}_A^{(\text{late})} \otimes \mathcal{H}_B^{(\text{late})} = e^{iP_A P_B} \mathcal{H}_A^{(early)} \otimes \mathcal{H}_B^{(early)}$$



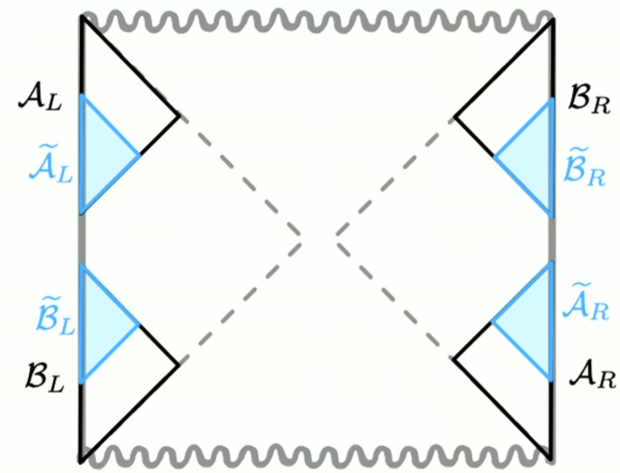
[GP, Tabor (arXiv:2506.?????)]

An algebra for scrambling

- As before, we can define P_A, P_B from a boundary perspective via half-sided modular inclusions
- Early- and late-operators act on different modes, but also at different times:
- So A_R acts on H_A^{early} while B_R acts on H_B^{late} and

$$[\mathcal{A}_R, e^{+\mathrm{i}P_A P_B}\mathcal{B}_R e^{-\mathrm{i}P_A P_B}] = 0.$$

- Very nontrivial (but true!) that these generate a proper subalgebra (Type III_1 factor) that we call a **modular-twisted product**
- Commutant algebra can again be identified with left boundary operators (complementary reconstruction)



The boundary Hamiltonian and Type II algebras

- So far we have excluded conserved charges from our algebras
- In particular, in gravity/holography, the Hamiltonian is a boundary operator
- If we include the Hamiltonian in our algebra, time evolution (= modular flow) is an inner automorphism (generated by H)
- The state $|\Psi\rangle$ has a "density matrix" $\rho = e^{-\beta H}$ => the algebra is not Type III.
- Since it also isn't a tensor product subsystem (Type I), it must be Type II. In fact it is a **modular crossed product**
- We can define a **trace** (linear functional satisfying $\mathrm{Tr}[ab] = \mathrm{Tr}[ba]$) on this algebra by $\mathrm{Tr}(a) = \left\langle\Psi\right| e^{\frac{\beta H}{2}} a e^{\frac{\beta H}{2}} \left|\Psi\right\rangle$.
- This trace is semifinite and unique up to rescaling (algebra is a Type II_∞ factor)

[Witten (arXiv:2112.12828); Chandrasekaran, GP, Witten (arXiv:2209.10454)]

Semiclassical states and generalised entropy

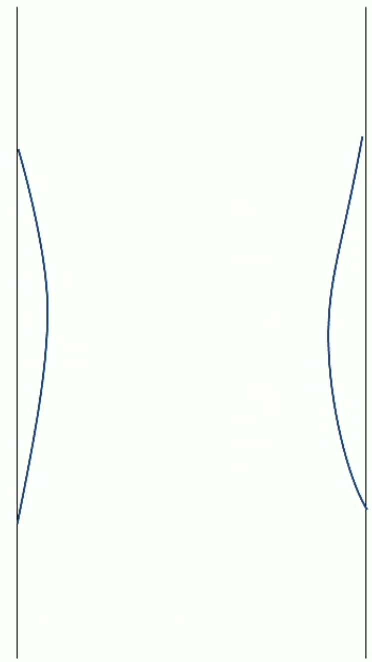
- The trace uniquely defines density matrices by $\langle a \rangle = \mathrm{Tr}(\rho a)$ and hence entropies by $S = -\mathrm{Tr}[\rho \log \rho]$
- We expect that for semiclassical states entanglement entropies should be equal to bulk generalised entropy
- However, states with finite fluctuations in H as $N \to \infty$ (**microcanonical ensemble**) must also have finite fluctuations in the timeshift between the left and right boundaries
- To make semiclassical states, we need to make the timeshift fluctuations small => energy fluctuations large
- In those state, we find that entropy = generalised entropy

Euclidean path integrals and traces

- What have we learned here? We already knew about the relationship between entropy and generalised entropy from Euclidean path integrals…
- Main lesson: better understanding of **why** Euclidean path integrals compute entropies
- AdS/CFT: Euclidean path integrals are directly related to boundary partition functions <- not true in general (e.g. canonically quantised JT gravity)
- But Euclidean path integrals are cyclic by construction => always define an algebraic trace
- So long as we have a sufficiently rich algebra of boundary observables this trace will be unique up to normalisation

Example: canonically quantised JT gravity

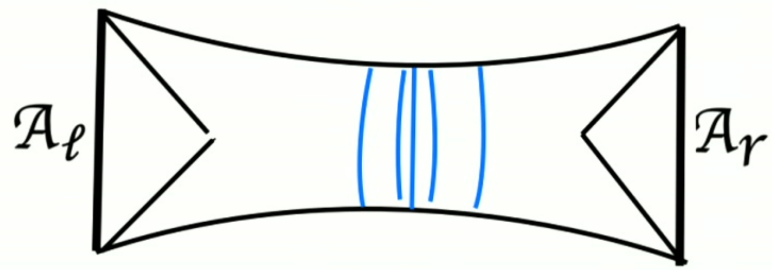
- In JT gravity without matter, the only boundary observables are functions of the Hamiltonian (equal at left/right boundary)
- Operators that don't commute with Hamiltonian (e.g. length of wormhole) can't be written using boundary operators (=> not holographic)
- Since the boundary algebra is commuting, any positive linear functional defines a trace



[Harlow, Jafferis (arXiv:1804.01081); GP, Witten (arXiv:2209.10454), Kolchmeyer (arXiv:2303.04701)]

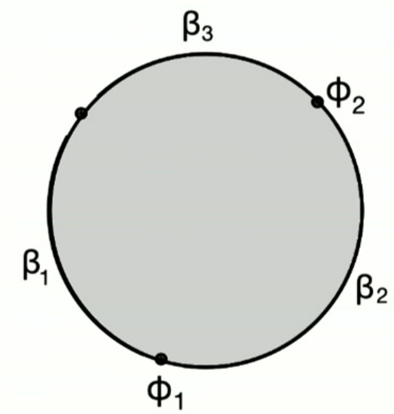
JT gravity plus matter

- If we add matter, the boundary Hamiltonians are no longer equal
- Matter boundary operators don't commute with the Hamiltonians
- In fact, one can prove that the boundary algebras are Type II_∞ factors and commutants
- No "gap" between between the boundary algebras even though there may be long wormholes with large backreaction
- Any operator can be written using products of boundary operators (theory is holographic)



Euclidean path integrals again

- Because the boundary algebras are Type II_∞ factors, they have a unique trace
- This trace can be found and described by purely Lorentzian methods, but it is easiest to describe in terms of Euclidean disc partition functions
- Entropies given by Euclidean replica trick calculations



$$\mathrm{Tr}\left[e^{-\beta_1 H}\Phi_1 e^{-\beta_2 H}\Phi_2 e^{-\beta_3 H}\Phi_3\right]$$

Canonically quantised super-JT gravity

- A particularly interesting phenomenon occurs in N=2 super-JT gravity:
  - For any bulk matter configuration, we can find a state where both boundaries have exactly zero energy
  - Time evolution is trivial, but can find nontrivial boundary algebras by projecting matter primaries onto the space of ground states
  - These algebras are Type II_1 factors and commutants
  - Can be understood concretely as (very complicated) algebras on the matter QFT, for which the matter vacuum is maximally entangled
  - All information is accessible from the boundary even though there is no boundary time

[Lin, Maldacena, Rozenberg, Shan (arXiv:2207.00407/8); GP, Witten (arXiv:2412.15549)]

De sitter, observers and clocks

- In the static patch of de Sitter space, there is not just no boundary Hamiltonian but in fact no boundary at all
- Instead, the boost Hamiltonian is a gauge constraint that physical operators should commute with this constraint
- But this means modular flow of physical observables should be trivial => Type I/II algebra with Bunch-Davies state maximally mixed
- Only one problem: there are no physical observables except for c-numbers!

[Chandrasekaran, Longo, GP, Witten (arXiv:2206.10780)]

A clock is just a way to tell the time

- Solution: add a (quantum) clock that can be used to distinguish different times
- CLPW model: clock has $H = x$ with x the position operator on $L^2(R^+)$
- More generally, any system will work so long as it never reaches equilibrium in the semiclassical limit
- This allows the integral over boost times to converge so that boost-invariant operators can exist
- Examples: slow-rolling inflation, black holes in de Sitter
- Entropy = generalised entropy for states where the clock is semiclassical (evolves quickly in time)

[Chen, GP (arXiv:2406.02116)]

Where next?

- Canonically quantised gravity still has a lot to teach us
- Can we precisely understand the algebra of a boundary (or observer) in a background-independent (but semiclassical) spacetime?
  - Witten's work + talks this week are interesting progress, but work to be done
  - Right structure seems to a large N/bulk GR Poisson algebra
- Can it help understand holography beyond AdS?
  - What information is encoded in a neighbourhood of (flat space) null infinity?
  - What does this tell us about the microscopic degrees of freedom?
  - Connections to e.g. Sabrina's talk
- What happens to the algebraic structure nonperturbatively?
  - How do Type I algebras appear?
  - How important is the integer quantisation of N?
  - Beyond asymptotic infinity, do nonperturbative algebras even exist?

Thank you!