Title: Visions of RealTime: The Lorentz-signature gravitational path integral for fun and profit (Vision Talk)

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Abstract:

The Euclidean gravitational action is unbounded below. As a result, even at the effective field theory level, the gravitational path integral cannot be formulated as an integral over real Euclidean geometries. I therefore review recent efforts to formulate the path integral directly in Lorentz-signature in a manner that allows general topology-changing transitions. I will also describe how this formulation resolves certain puzzles associated with computing the density of states for nearly-extremal black holes.

(VIRTUAL TALK)

Visions of RealTime:

The Lorentz-signature gravitational path integral for fun and profit

Don Marolf

UCSB

6/24/25

Recent results from Euclidean gravitational path integrals:

- Page curve from replica wormholes
- Schwarzian-mode contributions to near-extreme BH density of states
- Phases in dS partition functions?

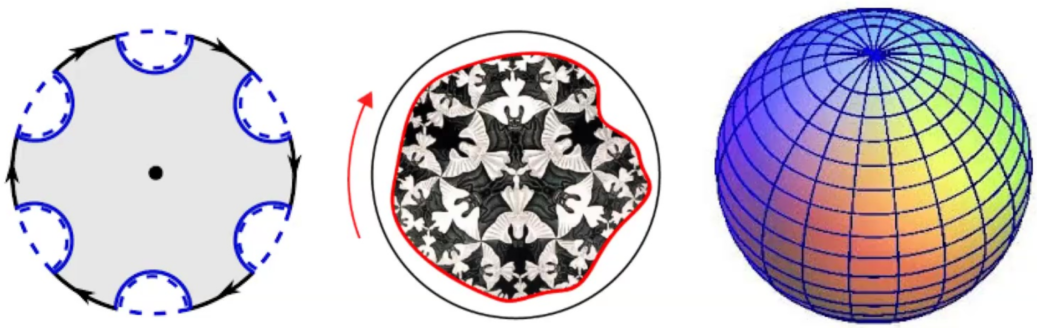


Image credits: G. Pennington, T. Mertens & J. Turiaci, mathcurve.com

GPI an Oracle?

Oracles are known for being dangerously enigmatic!

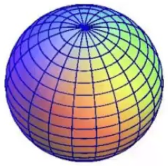
We wish to be Themistocles* and not Croesus†!!

*Who saved Athens by realizing Oracle's advice to build "wooden walls" meant to build the navy that defeated the Persians at Salamis.

† Who was told that, by attacking Persia, he would destroy a great empire – and who thus destroyed his own.

Real Euclidean metrics can have arbitrarily negative S_E.

Recall: spheres have negative action.

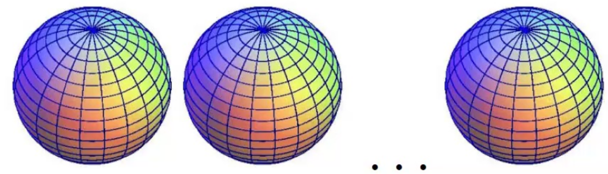


$$S_E(S^d) < 0.$$

(For $\Lambda > 0$, this requires $r < \sqrt{\frac{d(d-1)}{2\Lambda}}$).

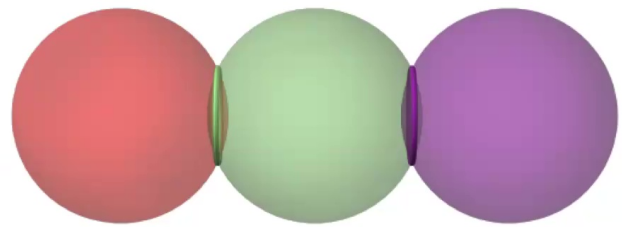
But real Euclidean metrics can have arbitrarily negative S_E.

With many spheres, S_E is very negative.



$$S_E(S^d \sqcup S^d \sqcup \cdots \sqcup S^d) = nS_E(S^d) < 0.$$

If we join neighboring spheres using necks of size ϵ, the contribution to $\int \sqrt{g}R$ scales like ϵ^{d-2}. This vanishes at small ϵ for $d > 2$.



Chains of n spheres with small necks have large negative action at large n.

$$S_E \sim nS(S^d) \to -\infty.$$

Curvatures need not be large!

A problem at the EFT level. Does not appear to be UV sensitive. [Problem remains when grav. constraints are imposed. arxiv:2505.13600 w/ Horowitz & Santos. And also on-shell for "necklaces" from Jonah's talk?]

Prevailing Wisdom: Choose a different contour

But which one?

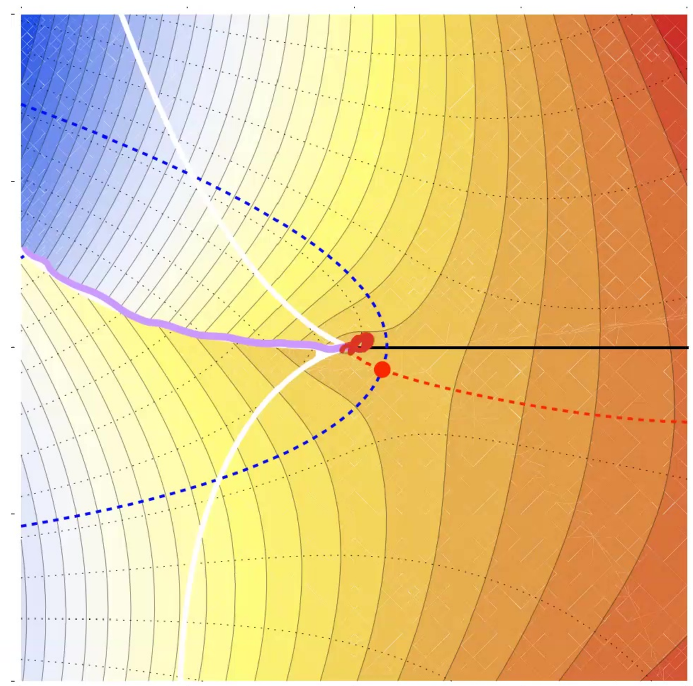
Cartoon example: Red shows large e^{-S_E}.

Euclidean metrics are positive real axis since signature $++\cdots+$.

Lavender curve is a branch cut.

Dot is a saddle w/ descent/ascent curves in blue/red.

Two possible choices of convergent contour.



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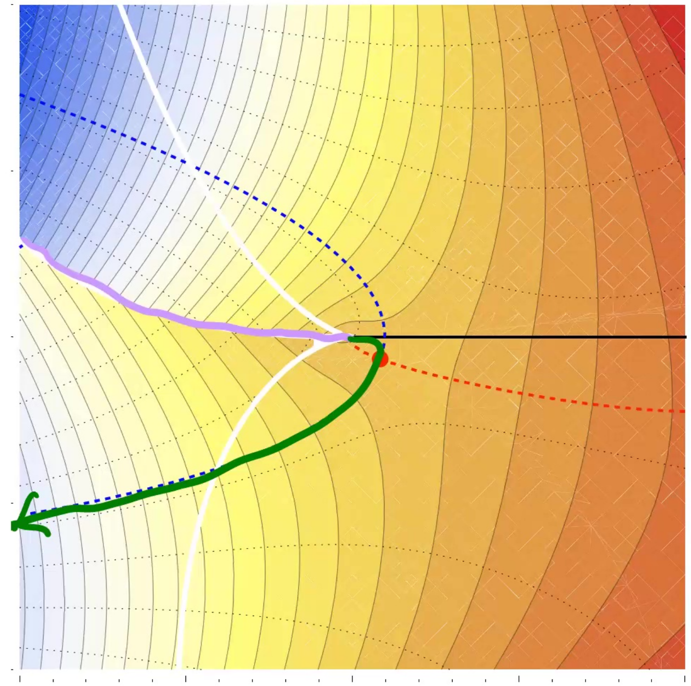

Green contour follows descent contour over saddle

Saddle contributes!

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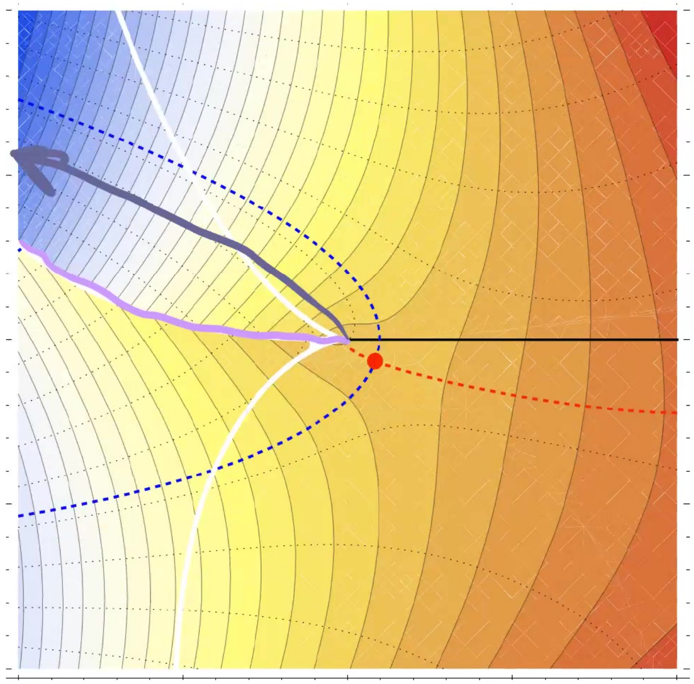


Silver contour runs downhill away from saddle

Saddle contribution would exceed integrand anywhere along contour!
So saddle cannot contribute!

Some Options

Gibbons-Hawking-Perry: Wick rotate the conformal factor!

- Gives physically sensible results for perturbations around real black hole saddles.
- Less clear what it means around complex saddles.
- Originally defined for asympt flat metrics $g = \Omega^2 \tilde{g}$ with $\tilde{R} = 0$.
- Deemed to fail since not every asympt flat metric is of this form.

Old Alternative Proposal (many authors):

Define path integral using the real *Lorentz*-signature contour!
(Integrand e^{iS} oscillates, but that's better than diverging.)

Topology change in the Lorentzian path integral: Some Initial References

1. J. Louko & R. Sorkin, *Complex actions in two-dimensional topology change*, [arXiv:gr-qc/9511023.
2. Y. Neiman, *The imaginary part of the gravity action and black hole entropy*, arXiv:1301.7041.
3. X. Dong, A. Lewkowycz and M. Rangamani, *Deriving covariant holographic entanglement*, arXiv:1607.07506.
4. D. Marolf and H. Maxfield, *Observations of Hawking radiation: the Page curve and baby universes*, arXiv:2010.06602.
5. S. Colin-Ellerin, X. Dong, D. Marolf, M. Rangamani and Z. Wang, *Real-time gravitational replicas: Formalism and a variational principle*, arXiv:2012.00828.
6. S. Colin-Ellerin, X. Dong, D. Marolf, M. Rangamani and Z. Wang, *Real-time gravitational replicas: low dimensional examples, arXiv:2105.07002.*
7. D. Marolf, *Gravitational thermodynamics without the conformal factor problem: partition functions and Euclidean saddles from Lorentzian path integrals*, arXiv:2203.07421.
8. B. Dittrich, T. Jacobson and J. Padua-Argüelles, *de Sitter horizon entropy from a simplicial Lorentzian path integral*, arXiv:2403.02119.

Example of method: BH partition functions from Lorentzian Path Integrals [arxiv:2203.07421]

Strategy: For an operator $H \geq E_0$, we can write

$$e^{-\beta H} = \int_{\mathbb{R}} dT e^{-iHT} f_\beta(T)$$

where $f_\beta(T) = \frac{1}{2\pi i} \frac{e^{E_0(-\beta+iT)}}{T+i\beta}$.

We therefore write

$$Z = \mathrm{Tr} e^{-\beta H} = \int_{\mathbb{R}} dT \left(\mathrm{Tr} e^{-iHT}\right) f_\beta(T).$$

We then interpret the right-hand side as an integral over Lorentzian path integrals with periodic Lorentz-signature time of period T.

Of course, $\mathrm{Tr} e^{-iHT}$ is ill-defined.

However, the right-hand-side should make sense if the integral over T is performed before the trace is fully evaluated.

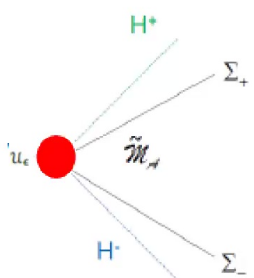
BH partition functions from Lorentzian Path Integrals

$$Z = \mathrm{Tr} e^{-\beta H} = \int_{\mathbb{R}} dT \left(\mathrm{Tr} e^{-iHT}\right) f_\beta(T) \quad \text{for} \quad \mathrm{Tr} e^{-iHT} = \int_{\substack{\text{real}\\ \text{Lorentzian}^*\\ \text{spacetimes}\\ \text{with period } T}} \mathcal{D}g\, e^{iS}.$$

We will integrate over BCS!

*: We wish to allow general toplogies.

$\Longrightarrow$ Allow spacetimes with codim-2 Lorentzian conical singularities where Lorentz structure breaks down; e.g., time-periodic quotients of BH exteriors.



Identifying Σ_+ with Σ_- creates a spacetime with a conical singularity at γ. Note that no null geodesics reach γ; i.e., it has no light cone ($\mathcal{N} = 0$ instead of the usual $\mathcal{N} = 4$).

$$S_{EH} = \frac{1}{16\pi G_N} \int_{\tilde{\mathcal{M}}} \sqrt{-g} R \quad \text{Louko \& Sorkin arxiv : gr-qc/9511023}$$

$$:= \lim_{\epsilon \to 0} \left(\frac{1}{16\pi G_N} \int_{\tilde{\mathcal{M}} \setminus \mathcal{U}_\epsilon} \sqrt{-g} R - \frac{1}{8\pi G_N} \mathcal{P} \int_{\partial \mathcal{U}_\epsilon} \sqrt{|h|} K \right) + i\left(\frac{\mathcal{N}}{4} - 1\right) \frac{A\gamma}{4G_N}$$

$$S_{Total} = -ET + \cancel{\Omega J T} - iA_\gamma/4G_N \quad \text{since } \Omega = 0 \text{ for our problem.}$$

Save A, T integrals for last

Evaluate others integrals semiclassically.

Saddles are quotients $\mathcal{M}_{A,T}$ of static black holes with area $A_\gamma = A$ under a time translation with T.

$$Z \approx \int_{\mathbb{R}^+} dA \int_{\mathbb{R}} dT \; f_\beta(T) e^{iS(\mathcal{M}_{A,T})} = \int dA dT \; f_\beta(T) e^{A/4G} e^{-iET}$$

$$= \int dA dT \; \frac{1}{2\pi i} \frac{e^{E_0(-\beta + iT)}}{T + i\beta} e^{A/4G} e^{-iET}$$

$$= \int dA \; e^{A/4G} e^{-\beta E}.$$

$E = E(A)$

$A = A(E)$

So the density of states is $A/4G$ as desired.

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So the density of states is $A/4G$ as desired.

Generalizations

- Double cones w/ higher topology 2411.16922 by Blommaert et al
- Charged (and rotating) singularities 2501.08409 by Hong Zhe [Vincent] Chen

$$Z \approx \int dA \, dQ \; e^{A/4G} e^{-\beta(E + \mu Q)}.$$

Application: Near-extremal density of states

w/ Maciej Kolanowski

Puzzle raised by L. Iliesiu and J. Turiaci, arxiv:2003.02860 (non-SUSY)

Studied density of states by using partition function for JT gravity w/ Maxwell field.

Compact U(1) gauge group, so $\mu_n = \mu + \frac{2\pi n i}{\beta}$ gives same holonomy $e^{-q\int_{S^1} A} = e^{-q\beta\mu_n}$ around Euclidean time circle for all $q, n \in \mathbb{Z}$.

$\Longrightarrow$ Sum over these 'shifted' μ_n in the path integral. (Complex BCS!)

Sum converges in Maxwell-JT!

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Sum converges in Maxwell-JT!

But corresponding sum diverges for a (large) AdS-RN black hole! (unpublished)

(Even though JT should be a dim reduction of (nearly-extreme) Einstein-Maxwell.)

Resolution

Note: Iliesiu and Turiaci emphasized one-loop contributions, but puzzle arises already at leading semiclassical level. So let us just focus on the saddles.

The Lorentzian prescription gives a well-defined starting point.

$$(*) \quad Z \approx Z_{TAdS} + \sum_n \int dA\, dQ\; e^{A/4G} e^{-\beta(E + [\mu + \frac{2\pi n i}{\beta}]Q)} \qquad \text{[Chen 2501.08409]}$$

Since the path integral sums over topologies, I have included a separate contribution Z_{TAdS} from thermal AdS.

We could also just take (*) as our starting point on physical grounds.

Details:

We wish to study the partition function

$$Z \approx Z_{TAdS} + \sum_n Z_n, \quad \text{with } Z_n := \int dA\, dQ\, e^{A/4G} e^{-\beta(E+[\mu+\frac{2\pi n i}{\beta}]Q)} = \int dA\, dQ\, e^{-S_E},$$

where (say, in AdS_4) for $n = 0$ we have

$$-S_E = \frac{A}{4G} - \beta(E + \mu Q) = \left(\frac{1}{2}\beta - \frac{r_+^3}{L^2} - \frac{Q^2}{r_+} + 2\mu Q - r_+\right) + \pi r_+^2,$$

and $r_+ := \sqrt{A/4\pi} \geq 0$.

Above, $Z_{TAdS} = 1$ is a thermal AdS contribution with $A = 0, Q = 0, E = 0$.

Since $r_+ \geq 0$, the Q-integral is a convergent Gaussian for all $\mu \in \mathbb{C}$.

Since taking $n \neq 0$ just inserts a phase, we must have $|Z_n| < Z_0$.

The r_+ integral

$dA \sim r_+ dr_+$

Performing the Gaussian Q integral gives

$$Z_0 = \frac{8\sqrt{2}\pi^{3/2}}{\sqrt{\beta}} \int_0^\infty dr_+ r_+^{3/2} e^{\frac{1}{2} r_+ \left(\beta\left(\mu^2 - \frac{r_+^2}{L^2} - 1\right) + 2\pi r_+\right)} = \frac{8\sqrt{2}\pi^{3/2}}{\sqrt{\beta}} \int_0^\infty dr_+ r_+^{3/2} e^{-S_E},$$

and similarly for Z_n.

Saddles for $n = 0$ are large and small AdS-RN BHs with

$$r_+ = \frac{L}{3\beta}\left(2L\pi \pm \sqrt{4L^2\pi^2 + 3\beta^2(\mu^2 - 1)}\right).$$

Results for $n \neq 0$ are obtained by replacing $\mu \to \mu + \frac{2\pi n i}{\beta}$.

For large $n > 0$ the saddles satisfy

$$r_+ = \pm\frac{2\pi niL}{\sqrt{3}\beta} + \frac{L\left(2L \pm \sqrt{3}\beta\mu\right)}{3\beta} + O(n^{-1}),$$

$$S_E = \pm\frac{8i\pi^3 Ln^3}{3\sqrt{3}\beta^2} + \frac{4n^2\pi^2 L\left(\pi L \pm \sqrt{3}\beta\mu\right)}{3\beta^2} + O(n).$$

For $\sqrt{3}\beta\mu > \pi L$, the saddle with least $\mathrm{Re} S_E$ has $\mathrm{Re} S_E \propto -n^2$ and gives a divergent sum!

But this (-) saddle cannot actually contribute, as it would give $|Z_n| > Z_0$.

Recall : $Z_n := \int dA\, dQ\; e^{A/4G} e^{-\beta(E+[\mu+\frac{2\pi ni}{\beta}]Q)} = \frac{8\sqrt{2}\pi^{3/2}}{\sqrt{\beta}} \int_0^\infty dr_+ r_+^{3/2} e^{-S_E}.$

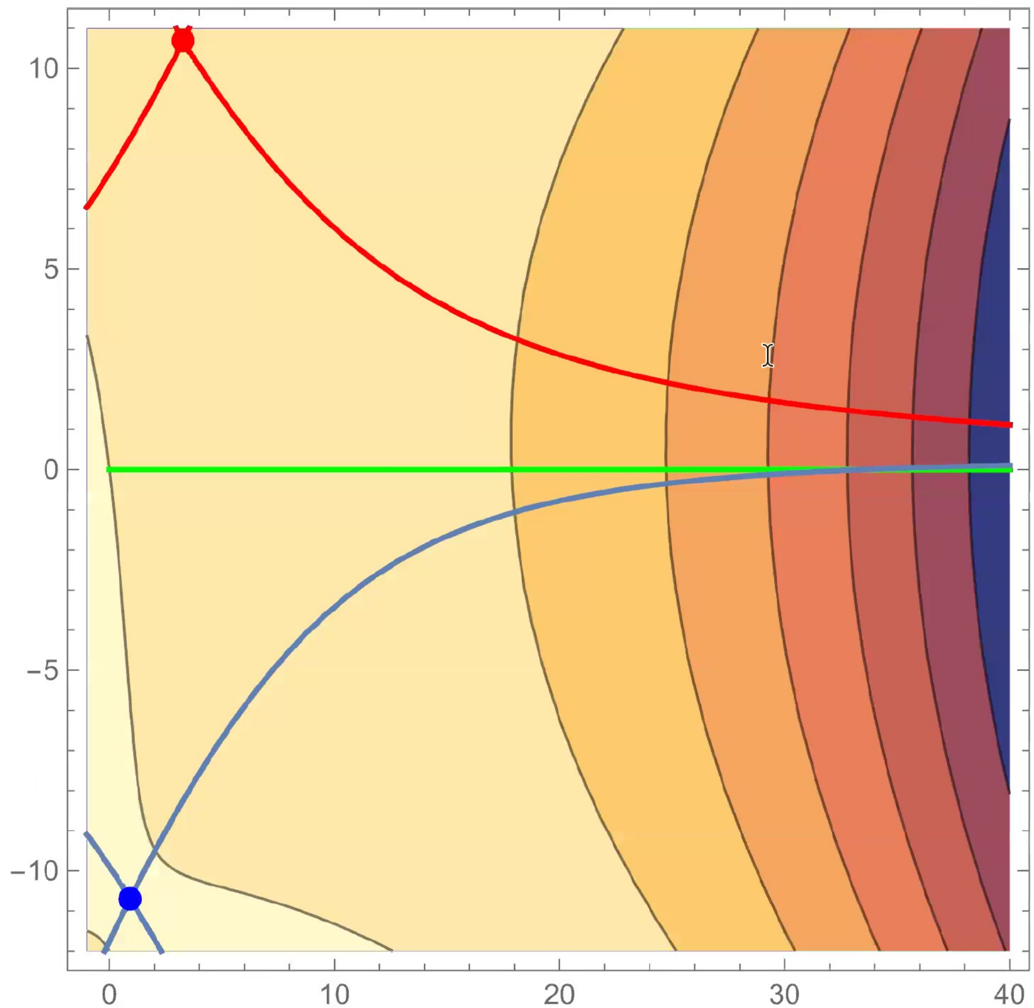
with

$$S_E = -\frac{1}{2} r_+ \left(\beta\left([\mu + \frac{2\pi ni}{\beta}]^2 - \frac{r_+^2}{L^2} - 1\right) + 2\pi r_+\right).$$

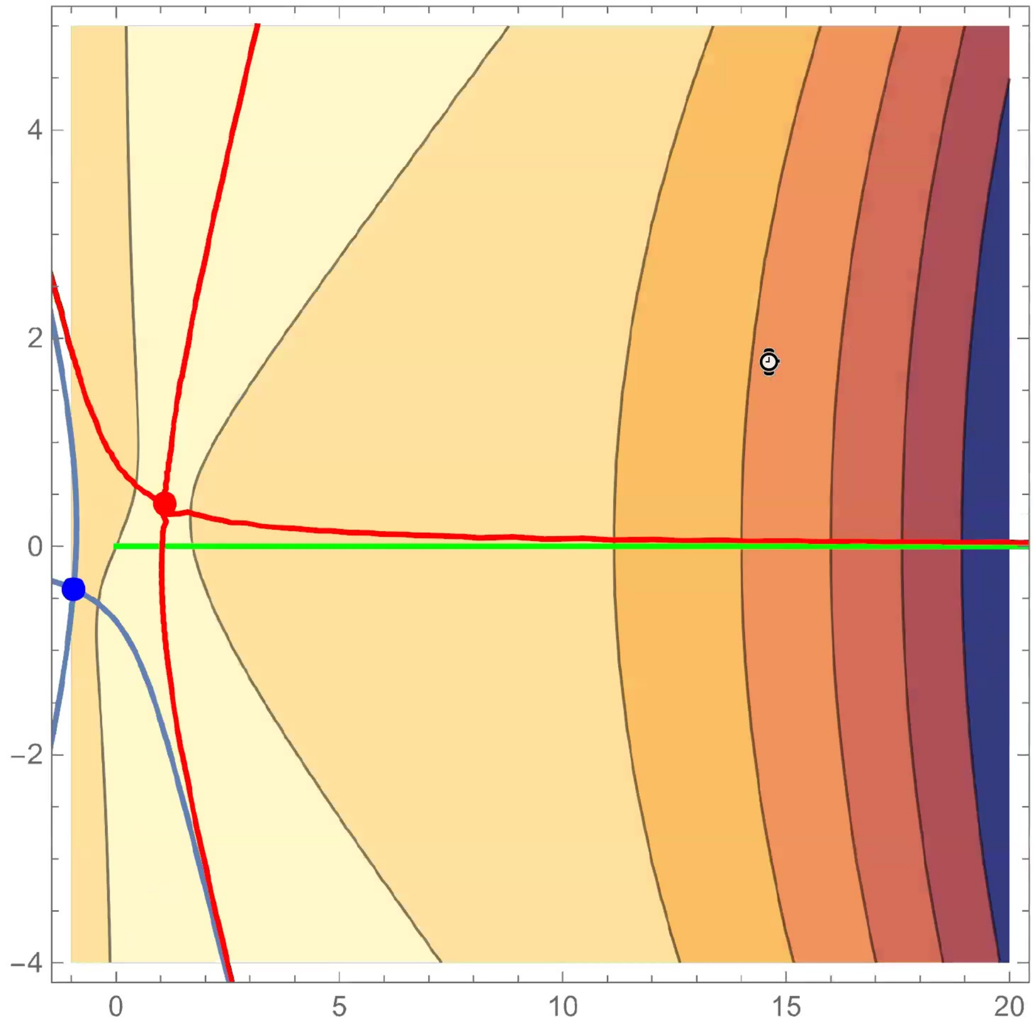
Also note: For $\mu, \beta > 0$ the RHS integrand has has $\mathrm{Im}\, S_E < 0$.

But the $(+)$ $\mathrm{Im} S_E < 0$ saddle has $\mathrm{Im}\, S_E > 0$. So it cannot contribute either. Thus for fixed β, μ, only thermal AdS contributes at large $|n|$.

$\left|e^{-S_E(r_+)}\right|$ for $n = 3, \beta = 1, \quad Z_3 \propto \int_0^\infty dr_+ r_+^{3/2} e^{-S_E}$.



$\left|e^{-S_E(r_+)}\right|$ for $n = 3, \beta = 30, \quad Z_3 \propto \int_0^\infty dr_+ r_+^{3/2} e^{-S_E}.$

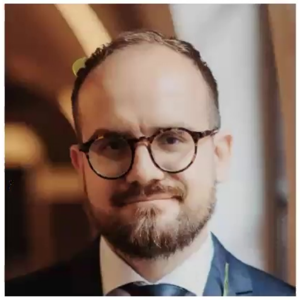
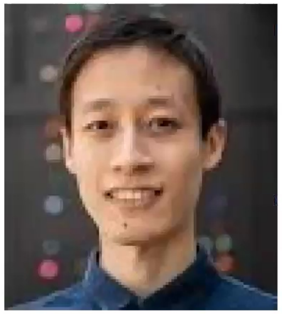


Summary

- The gravitational path integral has taught us a lot. But a full definition requires a choice of contour.
- Proposal: Use the real Lorentzian contour with singularities that allow topology change and the Louko-Sorkin prescription for the associated (complex!) action.
- Reproduces standard Euclidean calculations of BH thermodynamics with real potentials.
- Gives sensible results for sums over shifted complex potentials associated with compact gauge groups and charge quantization.
- An especially useful tool when saddles are far from the real axis and physical intuition from real black holes is not obviously reliable.

Further comments:

- Reproduces Euclidean computations of gravitational Reyni's.
  Colin-Ellerin et al, arxiv:2012.00828,2105.07002
  J. Held, X. Liu, DM, Z. Wang arxiv:2409.17428
- Excludes AdS axion wormholes, despite previous Euclidean analyses finding them to be "stable" in the asymptotically flat context.
  [to appear w/ J. Held, M. Kaplan, and Z. Wang]
- Nevertheless, appears to allow Garcia-Garcia-Godet wormholes (JT + imaginary scalar) and Marolf-Santos wormholes.
- Much more to do! E.g., Vincent and Maciej are looking at one-loop effects near extremality in this framework.
- Better understanding of story with higher derivative terms?

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