Title: Area and time

Speakers: Joshua Kirklin

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Subject: Quantum Gravity, Quantum Information

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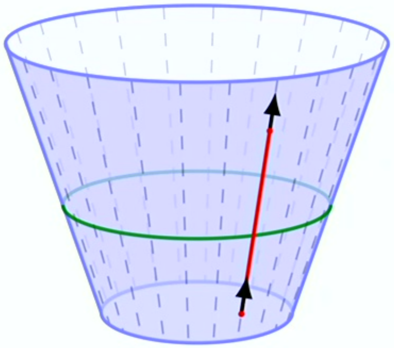
Abstract:

A diffeomorphism-invariant definition of physical subsystems is crucial for understanding local quantum information in gravity. Recent toy models have made progress by adding observers with clocks and imposing boost invariance, thereby enabling a rigorous von Neumann algebraic definition of generalized entropy. But they depend on auxiliary degrees of freedom and leave infinitely many diffeomorphisms unaddressed. It is natural to ask what happens when one tries to surmount these limitations.

I'll show that including null translations lets us prove a version of the generalized second law beyond the semiclassical regime, with potential direct implications for black hole information. Then I'll outline how the area degrees of freedom on a null surface define a quantum null time coordinate, and show how to use it to construct dressed operators invariant under all null diffeomorphisms (work in progress with Laurent Freidel.)



Area and **Time**

Josh Kirklin

QIQG 2025

[Based on: 2405.00114 & 2412.15502 with Julian De Vuyst, Stefan Eccles & Philipp Höhn;
2412.01903; and 2507.????? with Laurent Freidel]




PERIMETER INSTITUTE

- **QRFs** (quantum reference frames) are key to understanding **QI in QG**.
- E.g. *boost clocks* regularise entropy *(via crossed product)*.

- **QRFs** (quantum reference frames) are key to understanding **QI in QG**.
- E.g. *boost clocks* regularise entropy *(via crossed product)*. But:
  - Only one diffeomorphism $\mathbb{R}_{\text{boost}} \subset \text{Diff}(\mathcal{M})$ accounted for? What about the others?
  - Boost clock is typically *auxiliary*. Can we use an *intrinsic* QRF?
- *This talk:* how to surmount these limitations — and what we stand to gain.

1. **Generalised second law** beyond the semiclassical regime.
   - Two diffeomorphisms $\text{Aff}(\mathbb{R}) \subset \text{Diff}(\mathcal{M})$ are accounted for, using *dynamical cuts*.
   - Leads to a modification: $S_{\text{gen}}(v_2) \geq S_{\text{gen}}(v_1) +$ free energy of QRF.
2. The **area element** on a null surface as a QRF.
   - Completely intrinsic to the gravitational system.
   - Accounts for $\text{Diff}(\mathbb{R}) \subset \text{Diff}(\mathcal{M})$. *Rich mathematical and physical structure.*

General Covariance

Reference Frames

(coordinate systems)

Background Independence

General Covariance

Dynamical Reference Frames

"clocks and rods"

Background Independence

General Covariance

Quantum ~~Dynamical~~ Reference Frames (QRFs)

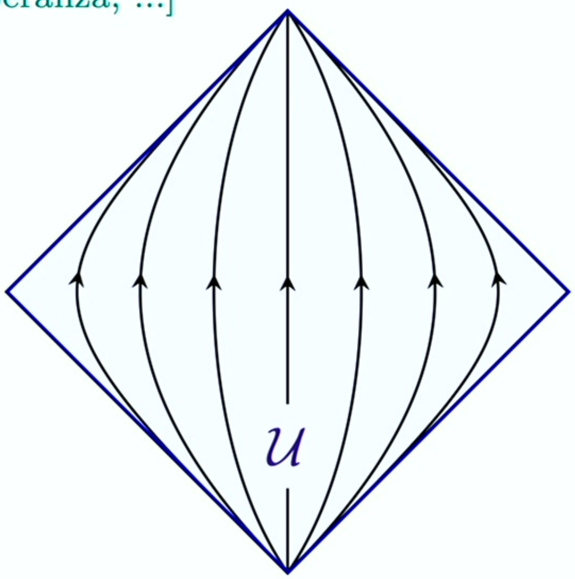
"quantum clocks and rods"

Quantum Theory

Quantum clocks **regulate entropies**.

[Chandrasekaran, Longo, Penington, Witten, Jensen, Sorce, Speranza, ...]

[De Vuyst, Eccles, Höhn, **JK**, 2405.00114 & 2412.15502]



Let $\mathcal{A}_{\mathcal{U}} \subset \mathcal{B}(\mathcal{H}_{\mathrm{QFT}})$ be the algebra of field operators with support in some subregion $\mathcal{U}$.

$$(\mathcal{A}_{\mathcal{U}} \otimes \mathcal{B}(\mathcal{H}_{\mathrm{clock}}))^{C} = \underbrace{\mathcal{A}_{\mathcal{U}} \rtimes \mathbb{R}}_{\text{'crossed product'}}$$

$C = H_{\mathrm{QFT}} + H_{\mathrm{clock}}, \quad \mathcal{H}_{\mathrm{clock}} = L^2(\mathbb{R}).$

Fields are thermal in $\mathcal{U}$. Choose a clock that measures the associated time.

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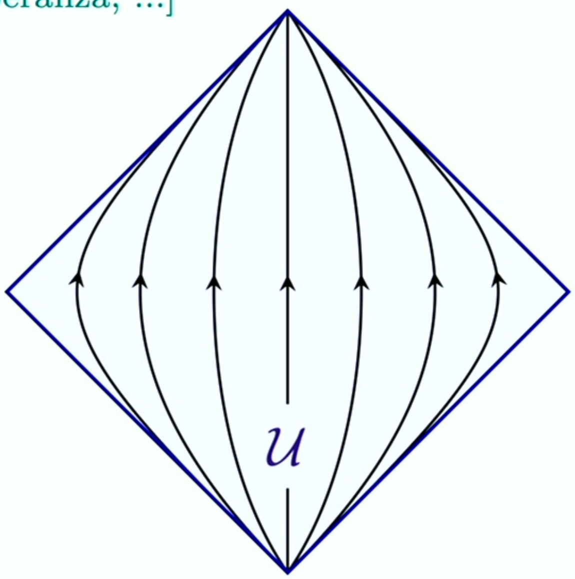

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Fields are thermal in $\mathcal{U}$. Choose a clock that measures the associated time.

Entropy of fields is UV-divergent: $S(\mathcal{A}_\mathcal{U}) \to \infty$

But entropy of fields and clock is finite! $S(\mathcal{A}_\mathcal{U} \rtimes \mathbb{R}) < \infty$

Semiclassically, $S(\mathcal{A}_{\mathcal{U}} \rtimes \mathbb{R})$ *is generalised entropy*.

'Semiclassically' means small $\hat{t}$ fluctuations, i.e. small $(\Delta\hat{t})^2 = \langle(\hat{t} - \langle\hat{t}\rangle)^2\rangle$.

Alternatively: large H_{clock} fluctuations. More precisely:

$$\Delta H_{\text{clock}} \gg \Delta H_{\text{QFT}}$$

In this regime one finds:

$$S(\mathcal{A}_{\mathcal{U}} \rtimes \mathbb{R}) = \underbrace{S_0}_{\text{some constant}} - \langle H_{\text{clock}}\rangle - \underbrace{S^{\text{rel}}_{\text{QFT}}}_{\text{relative entropy}}(\psi||\underbrace{\Omega}_{\text{vacuum}}) + \dots$$

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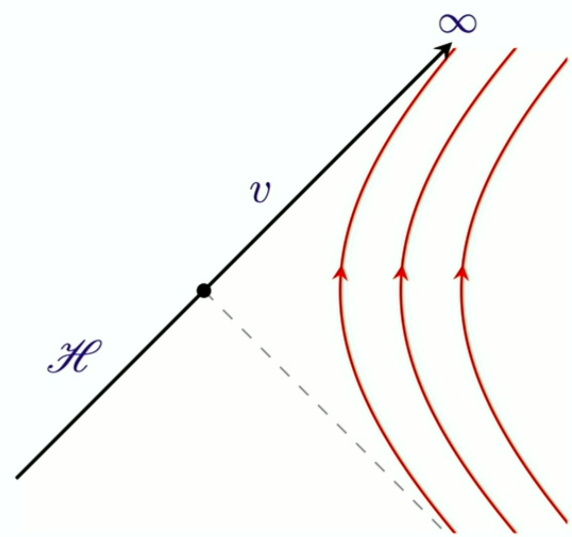
Introducing a UV regulator, and using Einstein's equations $G_{\mu\nu} = 8\pi G_{\text{N}} T_{\mu\nu}$ (with $T_{\mu\nu} = T^{\text{fields}}_{\mu\nu} + T^{\text{clock}}_{\mu\nu}$) leads to:

$$S(\mathcal{A}_{\mathcal{U}} \rtimes \mathbb{R}) = \frac{\langle \text{Area}(\partial\mathcal{U})\rangle}{4G_{\text{N}}} + S_{\text{QFT}}(\mathcal{U}) + \dots = S_{\text{gen}}\,!$$

Consider a Killing horizon $\mathscr{H}$.

The fields outside a given *cut* are thermal with respect to a *boost*.

The constraint generating the boost comes from Raychaudhuri's equation.



$$C = \underbrace{K_{\text{boost}}}_{H_{\text{QFT}}} - \underbrace{\frac{\text{Area}(\infty)}{4G_{\text{N}}}}_{H_{\text{clock}}} \qquad K_{\text{boost}} = \int_{\mathscr{H}} vT_{vv}.$$

Semiclassical regime $\Delta H_{\text{clock}} \gg \Delta H_{\text{QFT}}$ is $\Delta\,\text{Area}(\infty)/4G_{\text{N}} \gg \Delta K_{\text{boost}}$

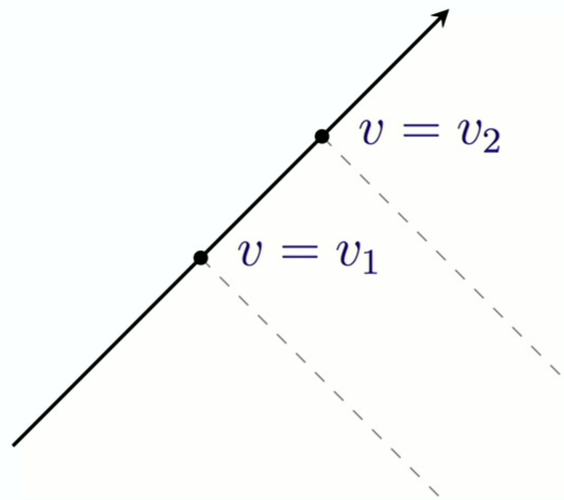
large area fluctuations *in Planck units*

For a cut of the horizon located at v, get an algebra $\mathcal{A}(v) \rtimes \mathbb{R}$.

Semiclassical regime: $(\Delta\,\mathrm{Area}(\infty)/4G_{\mathrm{N}} \gg \Delta K_{\mathrm{boost}})$

$$S(\mathcal{A}(v_2) \rtimes \mathbb{R}) \gtrsim S(\mathcal{A}(v_1) \rtimes \mathbb{R}).$$

[Wall 2011; Faulkner, Speranza 2024]



This is the *generalised second law*: [Bekenstein]

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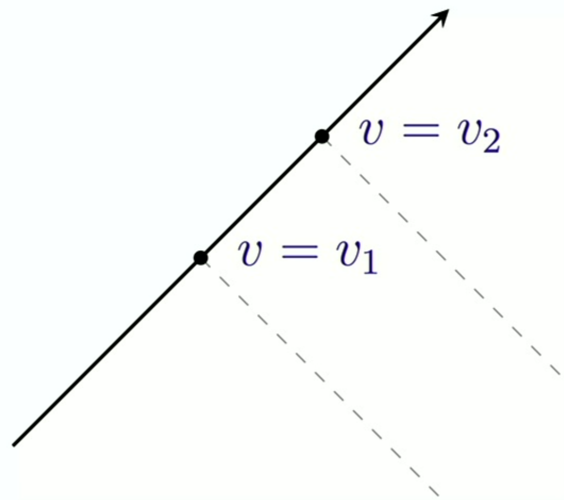

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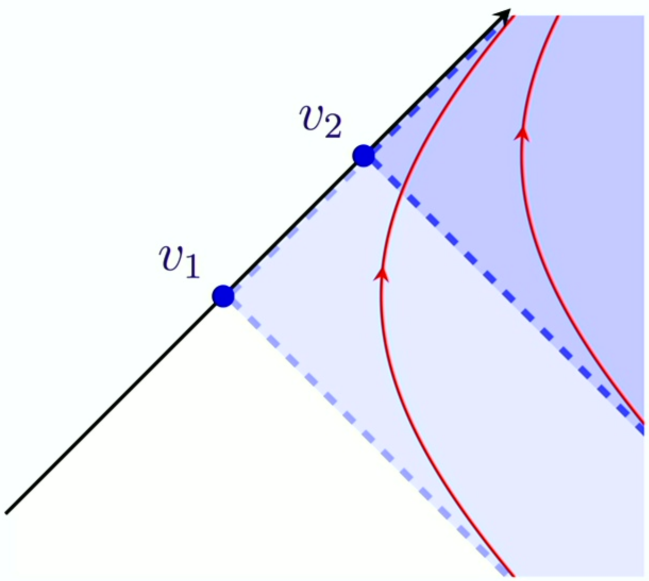
How do we go **beyond** the semiclassical regime?



Entropy inequalities typically come from subalgebras $\mathcal{A}_2 \subseteq \mathcal{A}_1$.

But here:

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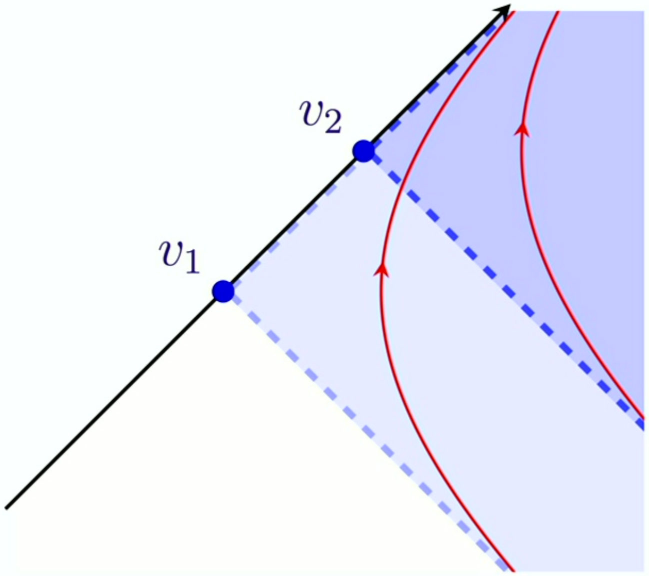
Algebra of later cut not contained in algebra of earlier cut.

('isotony' is violated)

This happened because a different boost invariance is imposed in each region.



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This is the reason why the previous approach does not give a GSL beyond the semiclassical regime.

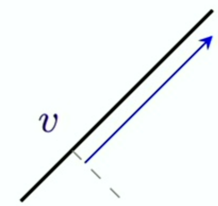
Solution: impose invariance under both boosts simultaneously!

Two constraints:

$$\underset{\text{boost around } v = v_1}{C_1} \qquad \underset{\text{boost around } v = v_2}{C_2}$$

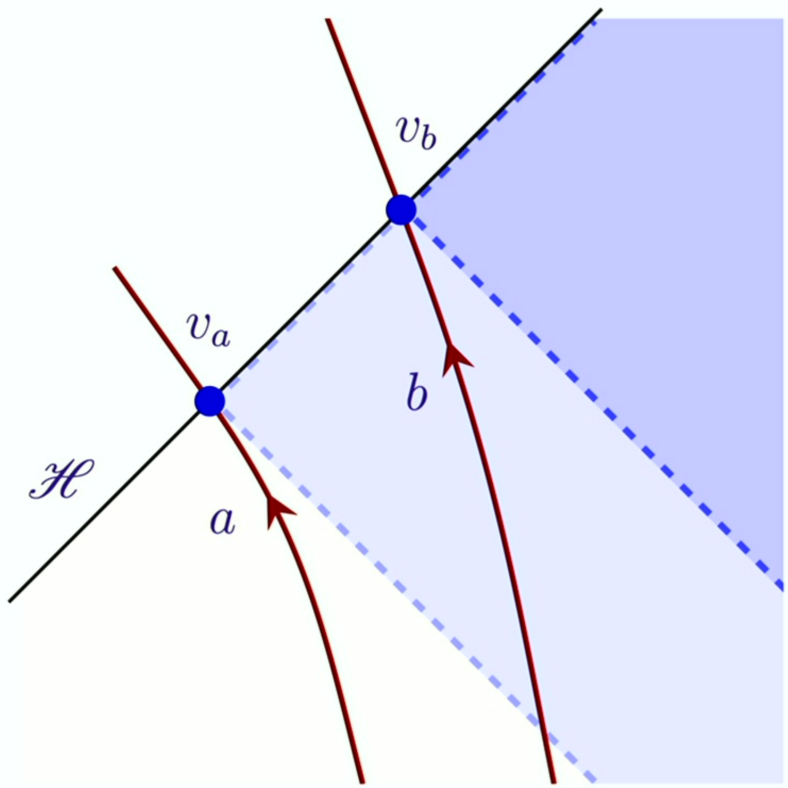
$C_2 - C_1$ generates a null translation $v \to v + s$ – moves the cuts.

$\implies$ *fixed cuts are not compatible with gauge symmetry*



Local subsystems must be defined relative to QRFs.

Toy Model: *dynamical cuts*



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For example: cuts v_a, v_b may be defined as the locations at which infalling particles a, b cross the horizon.

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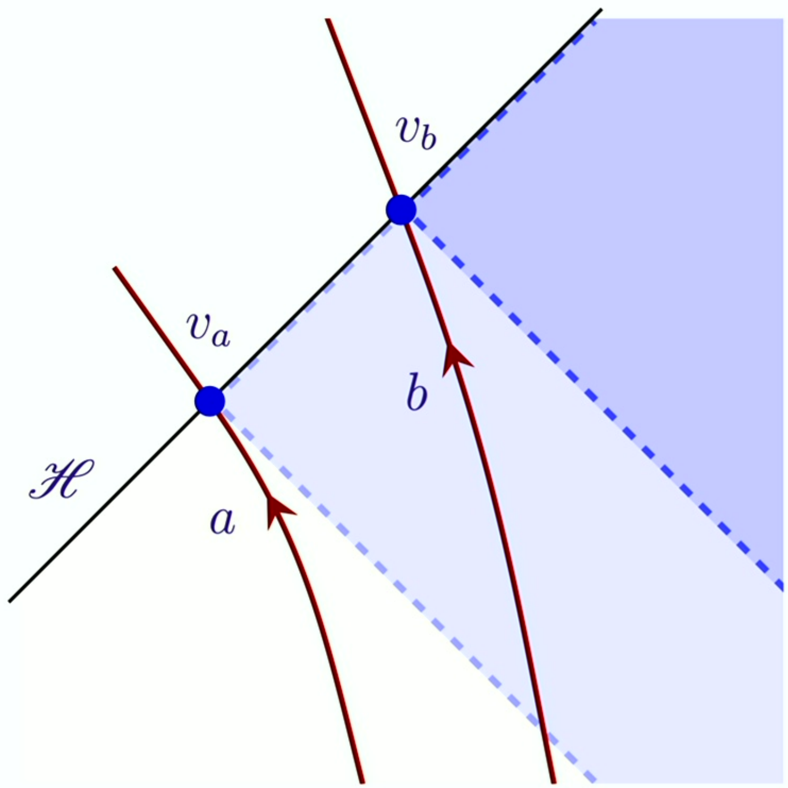

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For example: cuts v_a, v_b may be defined as the locations at which infalling particles a, b cross the horizon.

These are moved around by gauge transformations in the appropriate way, and so give rise to well-defined physical subsystems.

There are many other ways to construct such *dynamical cuts*.

In [JK 2412.01903]: dynamical cuts labelled by $a = 1, 2, \ldots$.

Each has Hilbert space $\mathcal{H}_a = L^2(\mathbb{R})$: wavefunctions $f(v_a)$ of the cut's location.

Algebra of observables outside of a dynamical cut: a crossed product *dressed to the location of the cut*: $\mathcal{A}(\hat{v}_a) \rtimes \mathbb{R}$.

Now have subalgebra structure: (RHS holds if we condition on the LHS)

$$\hat{v}_2 \geq \hat{v}_1 \implies \mathcal{A}(\hat{v}_2) \rtimes \mathbb{R} \subseteq \mathcal{A}(\hat{v}_1) \rtimes \mathbb{R}$$

Using ordinary QI theory (*monotonicity of relative entropy*), one may then show:

$$\hat{v}_2 \geq \hat{v}_1 \implies S(\mathcal{A}(\hat{v}_2) \rtimes \mathbb{R}) \geq S(\mathcal{A}(\hat{v}_1) \rtimes \mathbb{R}) + \text{free energy of cuts}$$

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The **GSL** must account for the *heat of quantum clocks*.

Taking QRFs seriously sheds light on **black hole information**.

$$S_{\text{gen}}(\hat{v}_2) \geq S_{\text{gen}}(\hat{v}_1) + \text{free energy of cuts.}$$

Free energy of cuts can be arbitrarily negative.

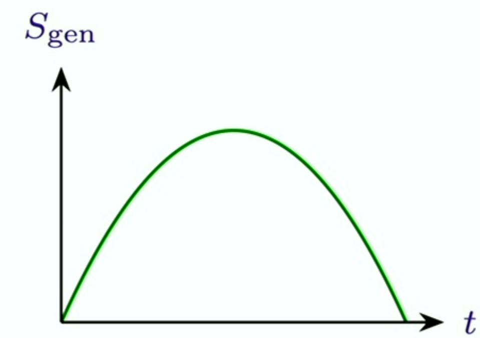
$\implies S_{\text{gen}}$ can *decrease*!

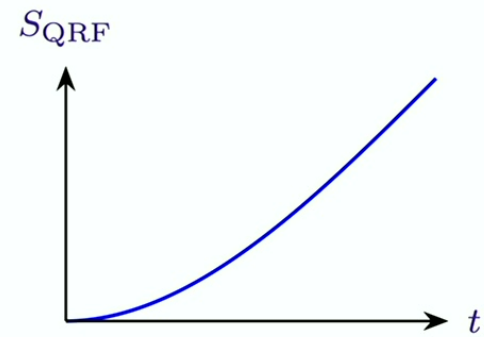
For black hole unitarity, we need something like this to happen, to recover a *Page curve*.

Is this a loophole for black hole unitarity?

Speculative argument:

> *"At late times, QRF becomes entangled with a large amount of Hawking radiation, bla bla bla..."*





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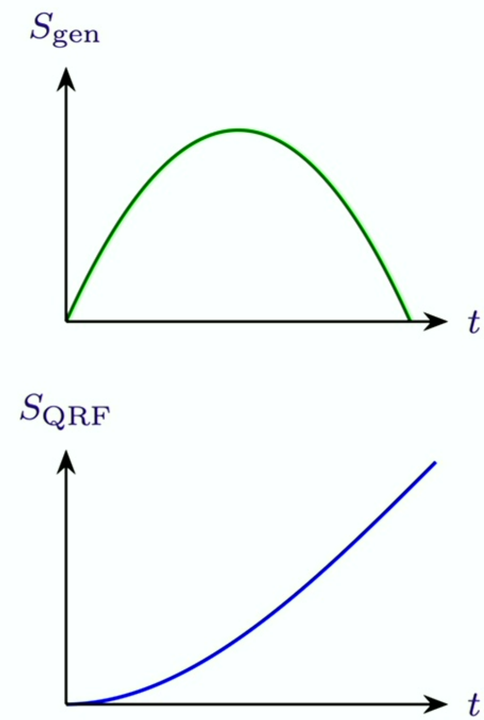
It's possible this is just an artifact of the toy model.



The *gravitational field itself* provides natural QRFs.

[Freidel, **JK** W.I.P.]

On a null surface (more general than just horizon), the fields decompose:

$$\Omega, \mu \qquad + \qquad \text{radiative degrees of freedom}$$

area element   canonical partner

('surface tension')

[Ciambelli, Freidel, Leigh]

Ω, μ give null time coordinates. E.g. 'affine time' V_{aff} satisfies $\partial_{V_{\text{aff}}}\Omega + 2\Omega\mu = 0$.

Upon quantisation, Ω, μ give a QRF for null times. Gauge group:

null diffeomorphisms: $\quad \text{Diff}^+(\mathbb{R}) \quad \ni$ boost, null translation, ...

Constraint is *Raychaudhuri equation*:

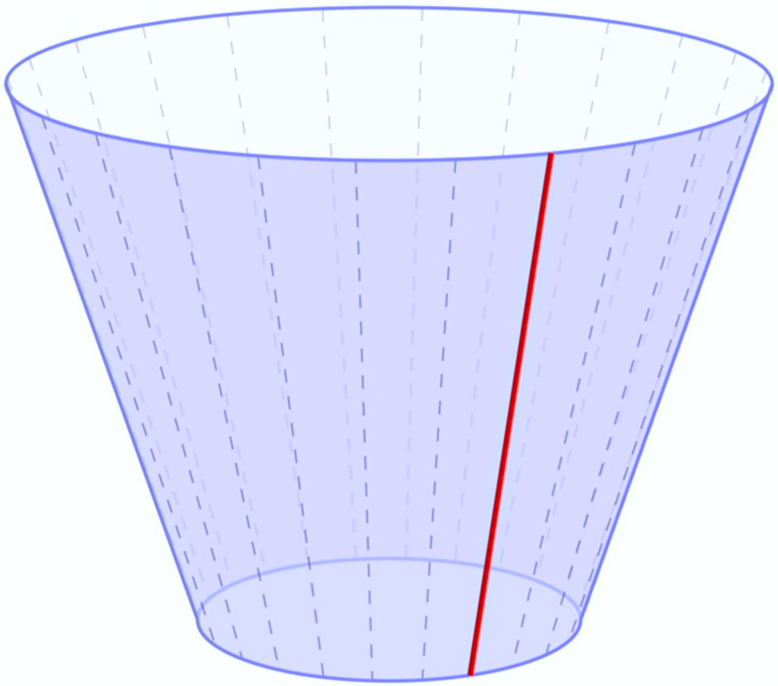
$$C(v) = \partial_v^2\Omega - \mu\partial_v\Omega + 8\pi G_{\text{N}}\Omega T_{vv}.$$

⋆ The many challenges of an Ω, μ QRF for $\mathrm{Diff}^+(\mathbb{R})$ ⋆

- Ω, μ interact with the rest of the fields via a cubic interaction term ΩT_{vv}. Non-interacting QRFs (like the boost clock and dynamical cuts) are easier.
- Quantum mechanical QRFs are typically Schrödinger-quantised (wavefunctions on configuration space). But Ω, μ are Kähler-quantised: *positive/negative modes*.
- Well-defined field operators involve normal-ordering. But normal-ordering is not preserved by diffeomorphisms: the modes get mixed (Bogoliubov transformation).
- Normal-ordering the constraint $:C(v):$ leads to anomalies, $\mathrm{Diff}^+(\mathbb{R}) \to$ Virasoro.
- Integrating over the gauge group is very useful for constructing observables dressed to QRFs. But this is subtle on $\mathrm{Diff}^+(\mathbb{R})$.
- Ω, μ states do not have a positive-definite inner product. Hilbert space?

In [Freidel, **JK** W.I.P.] we address all these problems.

Dressing time: a decoupled QRF on a gravitational null ray



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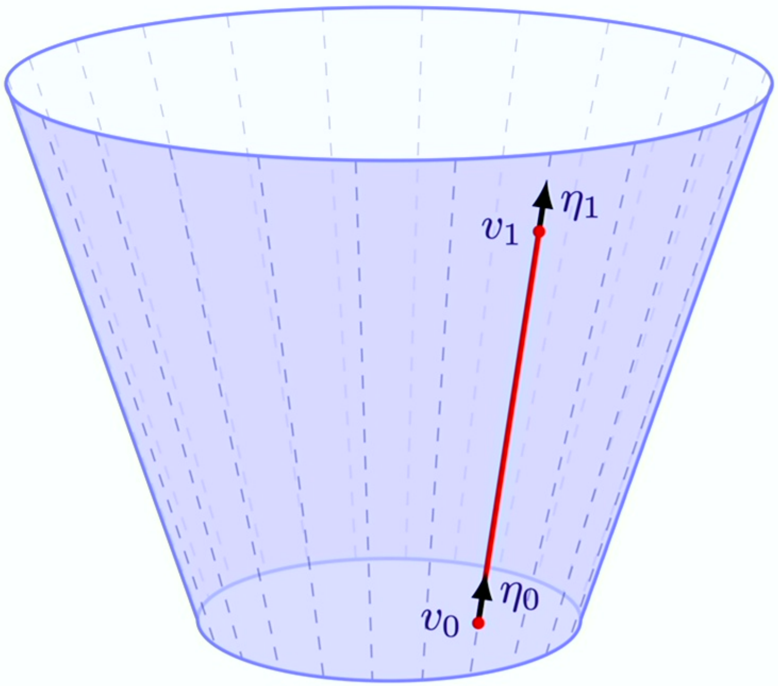

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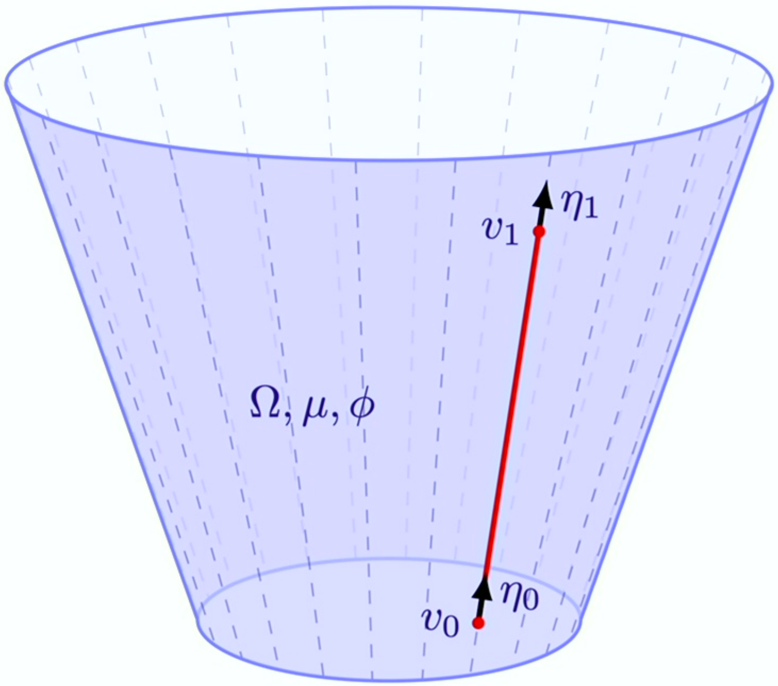

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- Actually, a segment. Edge modes at endpoints.
- Go to a perturbative regime for matter and radiation (but non-perturbative for Ω, μ). Treat it as a bunch of scalar fields ϕ.

$$C = \partial_v^2 \Omega - \mu \partial_v \Omega + 8\pi G_{\mathrm{N}} \Omega \partial_v \phi \partial_v \phi.$$

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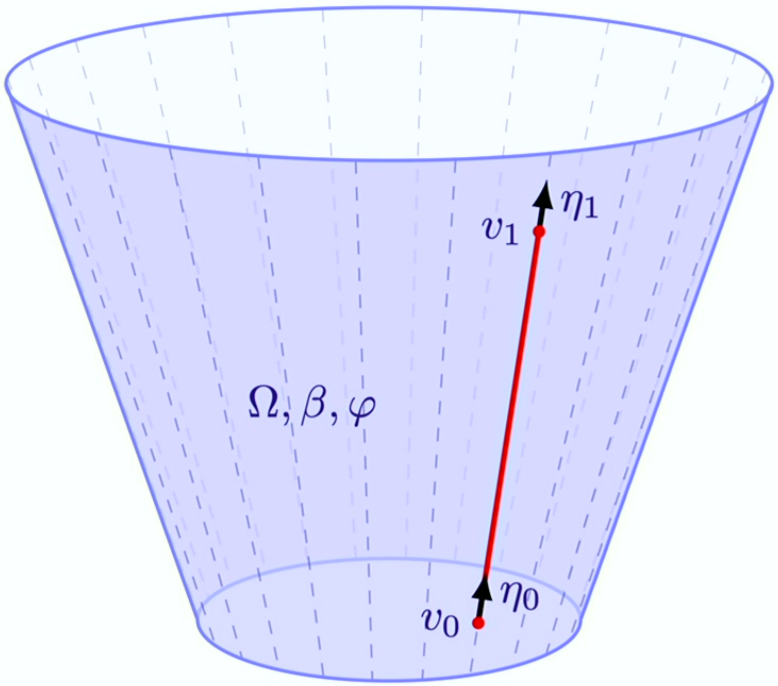

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Define 'half-densitised' matter $\varphi = \sqrt{\Omega}\phi$. β is conjugate momentum to Ω.

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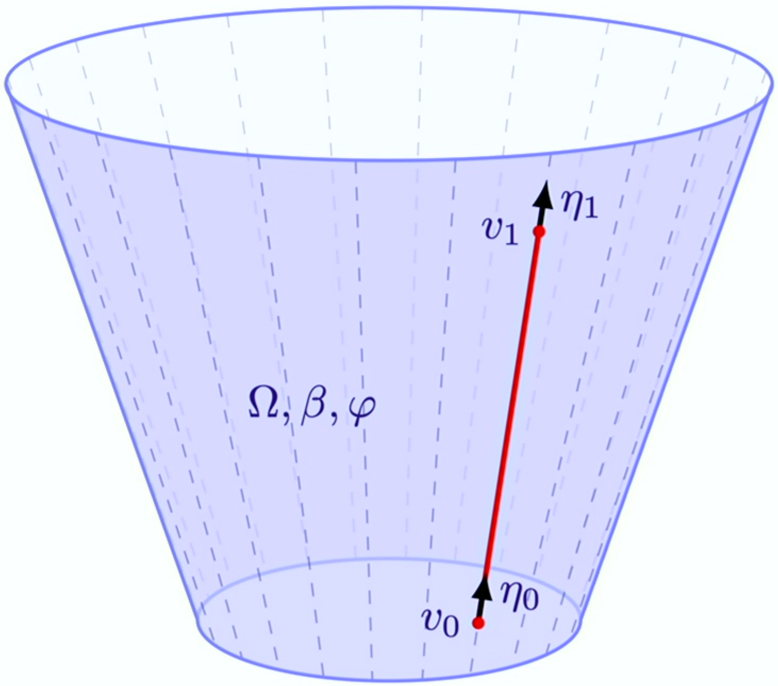

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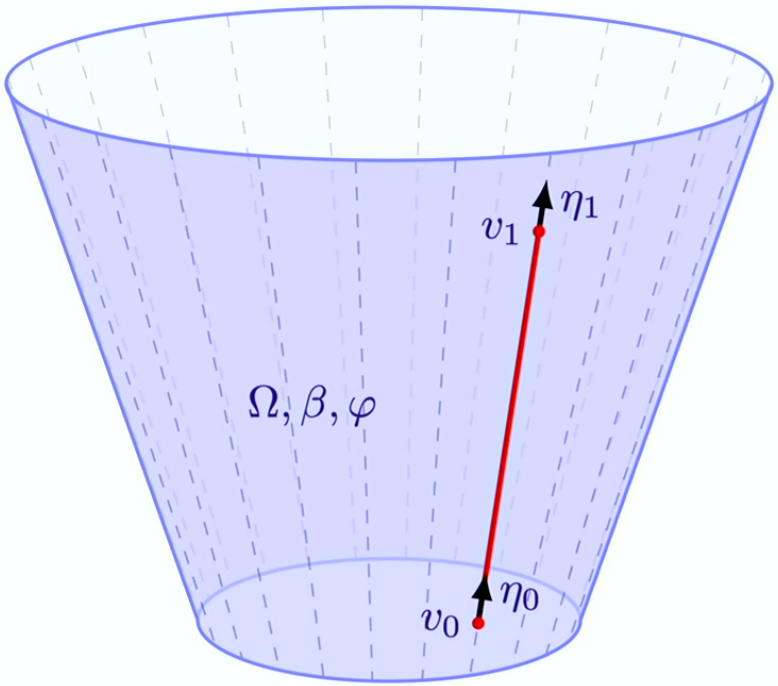

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V is non-interacting with φ. Under $F \in \mathrm{Diff}^+(\mathbb{R})$, have $V \mapsto V \circ F$. Use V as QRF.

⋆ The many challenges of an Ω, μ QRF for $\mathrm{Diff}^+(\mathbb{R})$ ⋆

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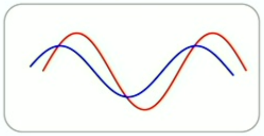
This deals with the first challenge. For the next ones let's look at some Toy Models.

Toy Model: *Anomalous Kähler QRF*

frame

system



Gauge group: $\mathbb{R}^2$, *2d* translations.

$\mathcal{H}_S$: Hilbert space of a *system* with gauge generators P_0, P_1. $\quad [P_0, P_1] = 0.$

$\mathcal{H}_F$: Hilbert space of the *frame*: a *1d* harmonic oscillator.

The position $\hat{x}$ and momentum $\hat{p}$ of the harmonic oscillator generate a projective (i.e. anomalous) representation of $\mathbb{R}^2$. $\quad [\hat{x}, \hat{p}] = i.$

Gauge constraints:

$$C = \zeta + a, \qquad C^\dagger = \zeta^\dagger + a^\dagger,$$

where $\zeta = P_0 + iP_1$, $a = \hat{x} + i\hat{p}$.

There is a preferred vacuum $|0\rangle$ for the frame, satisfying $a\,|0\rangle = 0$.

What are the observables?

$$D(A) \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_F)^{C,C^\dagger} \qquad \longleftarrow \text{ gauge-invariant}$$

1-to-1

$$A \in \mathcal{B}(\mathcal{H}_S) \qquad \longleftarrow \text{ gauge-fixed}$$

Dressing map: $D(A) = \colon e^{\zeta^\dagger a - \zeta a^\dagger} A e^{\zeta a^\dagger - \zeta^\dagger a} \colon \qquad D^{-1}(\cdot) = \langle 0| \cdot |0\rangle.$

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1-to-1 $\qquad \longleftarrow$ *not* an isomorphism

$$A \in \mathcal{B}(\mathcal{H}_S) \qquad \longleftarrow \text{ gauge-fixed}$$

Dressing map: $\quad D(A) = \mathbf{:} e^{\zeta^\dagger a - \zeta a^\dagger} A e^{\zeta a^\dagger - \zeta^\dagger a} \mathbf{:} \qquad D^{-1}(\cdot) = \langle 0| \cdot |0\rangle.$

Deformation: $\quad D(A)D(B) = D(A \star B)$, where

$$\begin{aligned} A \star B &= \int_{\mathbb{C}} \frac{\mathrm{d}^2 z}{\pi} e^{-|z|^2} e^{\bar{z}\zeta^\dagger} A e^{-\bar{z}\zeta^\dagger} e^{-z\zeta} B e^{z\zeta} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \operatorname{ad}^n_{\zeta^\dagger}(A) \operatorname{ad}^n_{\zeta}(B). \end{aligned}$$

(comes from moving $a, a^\dagger$ past each other)

Gauge-fixed algebra is $(\mathcal{B}(\mathcal{H}_S), \star)$. $A \star B = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \operatorname{ad}^n_{\zeta^\dagger}(A) \operatorname{ad}^n_{\zeta}(B)$.

Rich structure, and *physical consequences*.

For example, it affects **von Neumann entropies**:

$$\exp(a) \to \exp^\star(a) = \sum_{m=0}^{\infty} \frac{1}{m!} \underbrace{a \star a \star \cdots \star a}_{m}, \qquad \log \to \log^\star = (\exp^\star)^{-1},$$

$$S = -\operatorname{tr}(\rho \log \rho) \to S^* = -\operatorname{tr}(\rho \star \log^\star \rho).$$

Things simplify in a semiclassical regime.

$$a \star b \approx \underbrace{ab - [\zeta^\dagger, a][\zeta, b]}_{(\sim \textit{ Dirac bracket})}. \qquad S^* \approx S + \underbrace{(\cdots)}_{(\sim \textit{ area term})}$$

(In this regime $D : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_F)^G$ can probably be thought as approximate **OAQEC**)

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A *non-Abelian* Toy Model: SL$(2,\mathbb{R})$ highest weight QRF

Gauge group SL$(2,\mathbb{R})$, frame $\mathcal{H}_F =$ highest weight h irrep, i.e.

$$L_0 \left|0\right\rangle = h \left|0\right\rangle , \qquad L_- \left|0\right\rangle = 0, \qquad \mathcal{H}_F = \mathrm{span}\{L_+^n \left|0\right\rangle \mid n \geq 0\}.$$

Then gauge-invariant operators are given by a *dressing map*:

$$D(A) = {}^\star_\star U_S(\hat{z},\hat{z}^\dagger) A U_S(\hat{z},\hat{z}^\dagger)^{-1} {}^\star_\star$$

'covariant normal ordering'

$\hat{z}$ to the right of $\hat{z}^\dagger$

where U_S is SL$(2,\mathbb{R})$ representation on $\mathcal{H}_S$, and $\hat{z} = (L_0 + h)^{-1} L_+$.

Deformed product: $\quad D(A)D(B) = D(A \star B)$ where

$$A \star B = \left\langle 0\right| e^{\hat{z}M_-} A e^{-\hat{z}M_-} e^{-\hat{z}^\dagger M_+} B e^{\hat{z}^\dagger M_+} \left|0\right\rangle .$$

(M_+, M_0, M_- the generators of SL$(2,\mathbb{R})$ on $\mathcal{H}_S$)

Dressing time as a *quantum reference frame*

Quantise dressing time: $V \longrightarrow \hat{V}$, an operator acting on Ω, β states.

Get the same structure!

Gauge-invariant operators are given by a *dressing map*:

$$D : \{\Omega\}'' \otimes \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_F \otimes \mathcal{H}_S)^{\mathrm{Diff}^+(\mathbb{R})}$$

which can be written with a *covariant normal ordering* as:

$$D(A) = {}^\star_\star U[\hat{V}^{-1}] A U[\hat{V}]^\star_\star,$$

where U is the Virasoro representation.

Loosely speaking: this *covariant normal ordering* is normal ordering of modes of positive/negative frequency *in dressing time*.

(whereas ordinary normal ordering is normal ordering with respect to background time frequency)

For $F \in \mathrm{Diff}^+(\mathbb{R})$:

$$:U[F]AU[F^{-1}]: \neq U[F]:A:U[F^{-1}],$$

$$\text{but} \quad {}^\star_\star U[F]AU[F^{-1}]{}^\star_\star = U[F]{}^\star_\star A{}^\star_\star U[F^{-1}].$$

This is what makes the dressing map work.

An interesting special case:

$${}^\star_\star C(v){}^\star_\star - :C(v): = \tfrac{2}{3} G_{\mathrm{N}} \{\hat{V}(v); v\}.$$

Covariant normal ordering changes the Raychauduri constraint by the Schwarzian derivative of the dressing time.

(This is the covariant manifestation of the anomaly)

The *deformed product* $D(A)D(B) = D(A \star B)$ may be explicitly written as

$$A \star B = \mathrm{Ad}_{U\left[V[\overrightarrow{\mathrm{ad}}_{\beta_+}]^{-1}\right]}(A)\ \mathrm{Ad}_{U\left[V[-\overleftarrow{\mathrm{ad}}_{\beta_-}]^{-1}\right]}(B).$$

where $\beta_\pm$ are positive/negative modes, and $V[X] \in \mathrm{Diff}^+(\mathbb{R})$ is defined by $X = \frac{\partial_v^2 V[X]}{\partial_v V[X]}$, $V[X](\hat{v}_{0,1}) = 0, 1$.

Some examples: (recall $\phi = \Omega^{-1/2}\varphi$)

$$\Omega(u) \star \varphi(v) = \Omega(u)\varphi(v), \quad \text{but} \quad \Omega(u) \star \phi(v) \neq \Omega(u)\phi(v).$$

Also:

$$\partial_u^2\Omega(u) \star \partial_v^2\Omega(v) = \frac{1}{(u - v - i\epsilon)^4} + \frac{2\partial_v^2\Omega(v)}{(u - v - i\epsilon)^2} + \frac{\partial_v^3\Omega(v)}{u - v - i\epsilon} + \partial_v^2\Omega(u)\partial_v^2\Omega(v).$$

This is a stress tensor OPE.

⋆ The many challenges of an Ω, μ QRF for $\mathrm{Diff}^+(\mathbb{R})$ ⋆

- Ω, μ interact with the rest of the fields via a cubic interaction term ΩT_{vv}. Non-interacting QRFs (like the boost clock and dynamical cuts) are easier. ✓
- Quantum mechanical QRFs are typically Schrödinger-quantised (wavefunctions on configuration space). But Ω, μ are Kähler-quantised: *positive/negative modes*. ✓
- Well-defined field operators involve normal-ordering. But normal-ordering is not preserved by diffeomorphisms: the modes get mixed (Bogoliubov transformation). ✓
- Normal-ordering the constraint $:C(v):$ leads to anomalies, $\mathrm{Diff}^+(\mathbb{R}) \to$ Virasoro. ✓
- Integrating over the gauge group is very useful for constructing observables dressed to QRFs. But this is subtle on $\mathrm{Diff}^+(\mathbb{R})$. ✓
- Ω, μ states do not have a positive-definite inner product. Hilbert space?

What is $\mathcal{H}_{\text{phys}}$?

States of Ω, β have indefinite inner product – they do not form a Hilbert space. For example:

$$\left\| \frac{\Omega(v) + \beta(v')}{\sqrt{2}} |0\rangle \right\|^2 = \frac{1}{4G_{\text{N}}} \, \text{p. v.} \, \frac{1}{v - v'}.$$

But the space of states obtained by acting on the vacuum $|0\rangle$ with dressed operators $D(A)$ has a positive definite inner product. Therefore, the physical Hilbert space can be identified with the GNS Hilbert space of the vacuum:

$$\mathcal{H}_{\text{phys}} = \{D(A) |0\rangle \mid A \in \mathcal{B}(\mathcal{H}_S)\}.$$

Intuition: at the gauge-fixed level $\partial_v^2 \Omega$ is a stress tensor at central charge $c \geq 1$, which is straightforward to represent on a Hilbert space.

(c.f. also what happens with worldsheet conformal invariance in string theory)

Algebras and entropies of null ray segments

Each interval of dressing time has an algebra $\mathcal{A}_{[v_0,v_1]}$ of dressed operators.

Vacuum modular flow is a diffeomorphism – so these algebras have **traces**!
Each interval of dressing time has a well-defined **von Neumann entropy**.

Moreover, isotony is obeyed:

$$[v_0, v_1] \subset [v_0', v_1'] \implies \mathcal{A}_{[v_0,v_1]} \subset \mathcal{A}_{[v_0',v_1']}.$$

$\implies$ diffeomorphism-invariant **generalised second law**
without auxiliary degrees of freedom.

Valid on *any* null surface:
horizon, lightcone, ∂(causal diamond), non-extremal, ...

Thank you for listening!