Title: An Emergent Area Operator in 2d CFT

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Abstract:

The Ryu-Takayanagi formula implies that the entropy, a non-linear function of the state, is an observable in the semiclassical limit. This is a general phenomenon in a class of quantum error-correcting codes (QECC), but few specific area operators in the boundary CFT. In this work, I define a specific code in a holographic 2d CFT that includes all thermofield double states with arbitrary amounts of time evolution, but no bulk local degrees of freedom. While the naive area operator vanishes, it is possible to write down an area operator for appropriately classical states; the answer matches with previous results obtained from bulk considerations. Interestingly, this area operator involves an explicit coarse-graining. This suggests a construction for a non-isometric holographic map for these states.

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An Emergent Area Operator in 2d CFT

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Motivation: Gravity v/s Discreteness

A recurring theme of QIQG research in the past few years has involved exploring

the fact that semiclassical gravity is a coarse-grained description [Jacobson 1995... Penington-Shenker-Stanford-Yang 2019...].

One aspect of this is that the area of a black hole is a continuous variable, but the dual modular Hamiltonian has a discrete spectrum.

I will explore an example of an area operator in a 2d CFT,

hoping that it will help us think about the emergence of the bulk.

Concretely,

1. I will take a holographic 2d CFT
2. and define a quantum error-correcting code (QECC) as a subspace, taking inspiration from [Casini-Huerta-Magán-Pontello 2019].
3. I will then show that every state in the code subspace has the form anticipated in [Harlow 2016], but that the (information-theoretic) area operator defined there is 0.
4. I will do some explicit coarse-graining to get a non-zero area operator.
5. I will turn the logic around and propose a holographic map.

The Geometric Area Operator

At leading order,

$$S_E(B;\Psi) = -\operatorname{tr}\rho_B \log \rho_B = \operatorname{minext}_{X|X\sim B} \frac{A(X)}{4G_N}$$

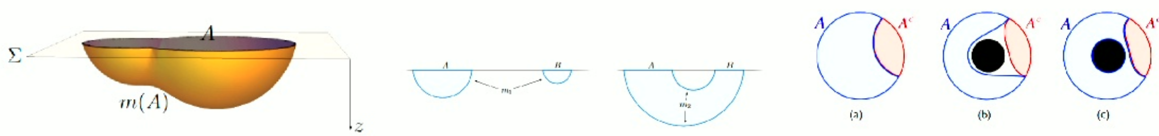


Figure 1: Stolen from [Headrick 1907.08216].

RHS is a gauge-invariant function of a solution

$\xrightarrow{\text{semi-classical}}$ an operator (up to $\mathcal{O}(G_N)$ ambiguities).

So, non-linear function of a state = linear function?

The Information-Theoretic Area Operator

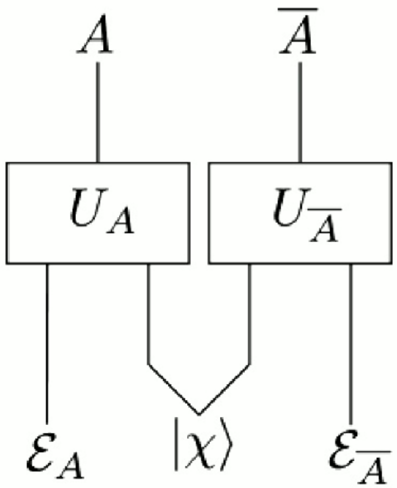


Figure 2: Stolen from [Harlow 2016].

[Harlow 2016, Almheiri-Dong-Swingle 2016] showed that a code subspace can have an information-theoretic area operator.

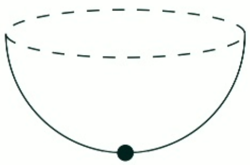
$$\mathscr{H}_{\rm code} = \left\{ \sum_\alpha \sqrt{p_\alpha}\, |\psi_\alpha\rangle_{B_1\overline{B}_1} |\chi_\alpha\rangle_{B_2\overline{B}_2} \middle| \chi_\alpha \text{ fixed} \right\}$$

$$\implies \quad S_E = \Big\langle \underbrace{\oplus_\alpha \log S(\chi_\alpha) \mathbb{1}_\alpha}_{\hat{A}} \Big\rangle_\Psi + \sum_\alpha p_\alpha \left(-\log p_\alpha + S(\rho_\alpha)\right).$$

Hilbert Space of a Generic 2d CFT

The Hilbert space of a 2d CFT on a circle is organised by primaries and descendants,

$$\mathscr{H}_{\text{CFT}} = \bigoplus_{O \in \text{prim}} \mathscr{V}_{h_O} \otimes \bar{\mathscr{V}}_{\bar{h}_O}$$
$$\overset{\text{ICFT}}{=} \mathscr{V}_{\mathbb{1}} \otimes \bar{\mathscr{V}}_{\mathbb{1}} \oplus \mathscr{H}_{\text{prim}} \otimes \mathscr{H}_{\text{desc}}$$



For a generic irrational CFT (ICFT), only identity module has null states so we get RHS.

The primary dimensions $h_O, \bar{h}_O$ can be measured with Virasoro Casimirs $C(L_0), C(\bar{L}_0)$ [Feigin-Fuchs 1983, Fortin-Quintavalle-Skiba 2024].

Holographic 2d CFT

CFTs dual to semiclassical gravity are expected to have the following properties [Hartman-Keller-Stoica 2016. . . Dey, Pal, Qiao 2025]:

- $c = 3\ell/2G_N \gg 1$.
- Unitary and compact (discrete spectrum of primaries).
- The vacuum state is normalisable.
- The light spectrum is sparse

$$\mathrm{tr}\left[\theta\left(L_0 + \bar{L}_0 \leqslant \frac{c}{12} + \epsilon\right) e^{-\beta H}\right] \leqslant A(\beta), \qquad \beta > 2\pi$$

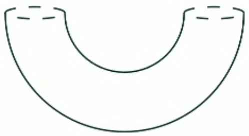
$$\mathrm{tr}\left[\theta\left(\min(L_0, \bar{L}_0) \leqslant \frac{c}{24}\right) e^{-\beta_L L_0 - \beta_R \bar{L}_0}\right] \leqslant B(\beta_L, \beta_R), \quad \beta_L \beta_R \geqslant 4\pi^2.$$

A, B are finite as $c \to \infty$.

The Code

The quantum error-correcting code (QECC) is defined as a subspace of two CFT Hilbert spaces $\mathscr{H}_{\mathrm{CFT}}^{\otimes 2}$,

$$\mathscr{H}_{\mathrm{CFT}}^{\otimes 2} \supset \mathscr{H}_{\mathrm{code}} = \left\{ \left(f(T_L, \bar{T}_L) \otimes \mathbb{1}_R \right) |\epsilon\rangle_{LR} \right\} \approx$$



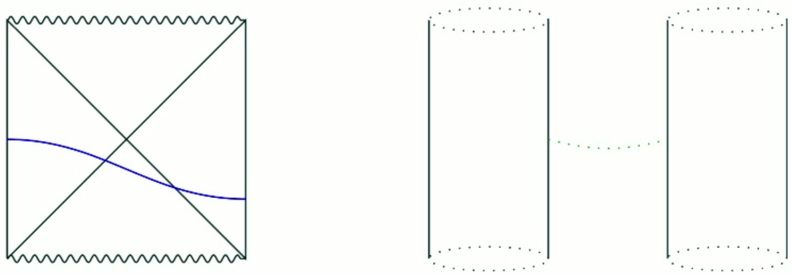
($|\epsilon\rangle$ is a TFD of high temperature.)

Contains all thermofield double states with arbitrary time evolution, and also other states with descendant excitations.

Possible to have 'cleverer' choices of code subspace.

The Bulk Dual

This code contains two-sided black holes with arbitrary time-shifts and arbitrary boundary graviton excitations, entangled pairs of global AdS, etc.



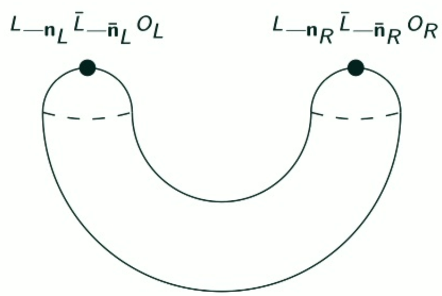
Structure of the Code

Take $|\Psi\rangle \in \mathscr{H}_{\text{code}}$ and expand it in a basis (following [Hung-Wong 2019]),

$$|\Psi\rangle = \sum_{O_L, O_R \in \text{prim}} \sum_{\mathbf{n}_{L,R}, \bar{\mathbf{n}}_{L,R}} \langle O_L, \mathbf{n}_L, \bar{\mathbf{n}}_L; O_R, \mathbf{n}_R, \bar{\mathbf{n}}_R|\Psi\rangle \, |O_L, \mathbf{n}_L, \bar{\mathbf{n}}_L\rangle_L \, |O_R, \mathbf{n}_R, \bar{\mathbf{n}}_R\rangle_R \, .$$

Technically, need Gram matrices.

The wavefunction is



$$\propto \delta_{O_L, O_R} \, .$$

So,

$$\mathscr{H}_{\text{code}} = \bigoplus_{O \in \text{prim}} \mathscr{H}_{O,L} \otimes \mathscr{H}_{O,R} \subsetneq \mathscr{H}_{\text{CFT}}^{\otimes 2} = \oplus_{O,O'} \mathscr{H}_{O,L} \otimes \mathscr{H}_{O',R} \, .$$

Area Operator

A general code state is

$$|\Psi\rangle = \sum_{O\in \text{prim}} \sqrt{p_O} \underbrace{|\psi_O\rangle}_{\in \mathcal{H}_{\text{desc,L}} \otimes \mathcal{H}_{\text{desc},R}} .$$

The EE is

$$S_E = \sum_O p_O \left[-\log p_O + S(\rho_O)\right] .$$

No area type term!

Comparing to [Harlow 2016], the difference is that

there are no $|\chi_O\rangle$ states carrying 'background' EE.

Where's the mistake?

This is because the code is very big and contains factorised states

$$|E\rangle_L |E\rangle_R = \int_{\mathbb{R}} \mathrm{d}t \, e^{i(\hat{H}_L - E)t} |\beta\rangle \in \mathscr{H}_{\mathrm{code}}.$$

Two options:

1. Make a better code.
2. Coarse-Grain or go from exact QEC to approximate QEC.

I will choose the second one, since it leads us to more interesting things (and will likely be required even with option one.)

Coarse-Graining

We coarse-grain by defining a set of bin functions $f_\alpha(O)$,

Demand that they form a partition of unity,

$$\forall O \in \text{prim}, \quad \int \mathrm{d}\mu(\alpha)\, f_\alpha(O) = 1.$$

If you get a state in $\mathscr{H}_O$, put it in a bin between α and $\alpha + \mathrm{d}\mu(\alpha)$ with probability $f_\alpha(O)\,\mathrm{d}\mu(\alpha)$.

So

$$p_O f_\alpha(O) = p_O p(\alpha|O) = p(\alpha, O)$$

is a joint probability distribution for module and bin.

Coarse-Graining the Entropy

The Shannon entropy $H(O) = -\sum p_O \log p_O$ becomes

$$H(O) = \underbrace{-\int \mathrm{d}\mu(\alpha)\, p_\alpha \log p_\alpha}_{H(\alpha)} + \underbrace{\sum_O \int \mathrm{d}\mu(\alpha)\, p(\alpha, O) \log \frac{p_\alpha}{p(\alpha, O)}}_{H(O|\alpha)}$$

$$\underbrace{- \sum_O \int \mathrm{d}\mu(\alpha)\, p(\alpha, O) \log \frac{p_O}{p(\alpha, O)}}_{H(\alpha|O)} .$$

Coarse-Graining the Entropy

The Shannon entropy $H(O) = -\sum p_O \log p_O$ becomes

$$H(O) = H(\alpha) + H(O|\alpha) - H(\alpha|O)$$

Assuming that p_O is slowly varying within each bin,

$$H(O|\alpha) \approx \sum_\alpha p_\alpha \log n_\alpha, \qquad n_\alpha \equiv \sum_O f_\alpha(O)$$
$$= \left\langle \int \mathrm{d}\mu(\alpha) \log n_\alpha \sum_O f_\alpha(O) \left|O\right\rangle \left\langle O\right| \right\rangle.$$

This is our coarse-grained area operator!

$H(\alpha|O) = -\sum_O p_O \int \mathrm{d}\mu(\alpha)\, f_\alpha(O) \log f_\alpha(O) \sim \log \text{width of } f_\alpha(O)$ is a correction term that measures the quality of the coarse-graining.

What is α?

The primaries are indexed by a discrete set of $h, \bar{h}$,

so we can take α to be a continuous set of $h, \bar{h}$.

I will forget about spin and take $\alpha = h + \bar{h}$, $\mathrm{d}\mu(\alpha) = \mathrm{d}\alpha$.

If we take $f_\alpha(O)$ to have width $\mathcal{O}(c^0)$,

$$\log n_\alpha = \begin{cases} 2\pi\sqrt{\frac{(c_{-1})\alpha}{3}} + \mathcal{O}(\log c), & \alpha > \frac{c}{12}, \\ 0 & \alpha < \frac{c}{24}. \end{cases}$$

This is the primary version of Cardy density of states [Hartman-Keller-Stoica 2016, Mukhametzanov-Zhiboedov 2019, Pal-Sun 2019].

Is this the geometric area?

For $\alpha > c/12$ (the black hole regime),

this is the part of the entropy that we can't attribute to boundary gravitons, meaning it should be the area [McGough-Verlinde 2013, Mertens-Simón-Wong 2022, Chua-Jiang 2023].

For $\alpha < c/24$, the bulk dual is two copies of global AdS with entangled boundary gravitons/bulk conical singularities,

so the area does vanish.

The answer seems to be correct, and it is also sensitive to the Hawking-Page transition!

A Note

To get the $\mathcal{O}(c^0)$ piece right, we have to do one more thing:

If the bulk dual has light matter fields ($\Delta \sim c^0$), then each α sector also has bulk entanglement.

But the bulk fields are in a fixed (HH) state, so it's a simple subtraction $\log n_\alpha \to \log n_\alpha - S_{\mathrm{bulk}}$.

However, I expect that this is subleading to $H(\alpha|O)$, which is ambiguous due to the freedom in choosing $f_\alpha(O)$.

Why was this so easy?

Simple answer:

The code has no bulk local degrees of freedom.

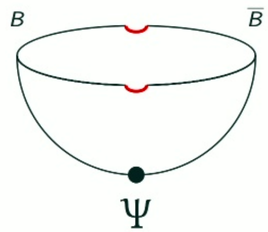
Some Other Cases (Preliminary)

The Vacuum Sector

We can also take [Casini-Huerta-Magán-Pontello 2019]

$$\mathscr{H}_{\text{code}} = \mathscr{V}_{\mathbb{1}} \otimes \bar{\mathscr{V}}_{\mathbb{1}}.$$

and introduce an explicit factorisation map [Hung-Wong 2019]
$J : \mathscr{H}_{S^1} \to \mathscr{H}_I \otimes \mathscr{H}_I$, so that we have

$J\,|\Psi\rangle =$ 

Red line =

Cardy boundary conditions

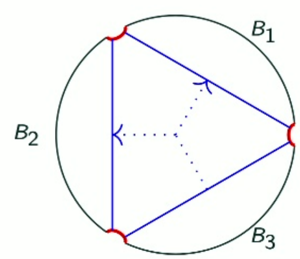
Again, we find a similar story where the primary entropy of the BCFT is the area modulo a potential subtlety.

Three Intervals

In this case, the modules corresponding to B_1, B_2 have to fuse to the one at B_3.

The area operator is the primary entropy of B_3.

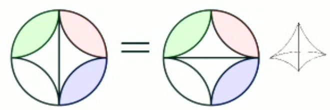
There is a CFT coproduct [Moore-Seiberg 1989, Gaberdiel 1993] using which we can measure the 'total' primary of $B_1 B_2$ (but possible subtleties due to ring-like nature of the fusion [Gaberdiel 1993]).



Non-Commuting Areas

If $S^1 = B_1 \cup \ldots B_4$, A_{12} and A_{23} shouldn't commute [Bao-Penington-Sorce-Wall 2019].

One way to probe that is to look at the inner product of a fixed α_{12} and fixed α_{23} state. Gravitational calculation gives volume of some hyperbolic tetrahedra,



whereas in the boundary it amounts to an st-crossing equation, and the spectrum is given by a Virasoro crossing kernel.

There is a precise equality between the Virasoro crossing kernel and the volume of the tetrahedron [Murakami-Ishijima 2004] (also used in Hung-Jiang et al works.)

Did the Bulk Emerge?

Notice that the emergent (= arising from coarse-graining) area operator has a continuous spectrum even though the modular Hamiltonian doesn't.

Is this somehow a signature of the bulk emerging?

I will assume yes and derive a proposal for the bulk-boundary holographic map from there.

This proposal has been subjected only to preliminary tests.

A Proposed Holographic Map

The bulk Hilbert space in the two-sided black hole case is [McGoush-Verlinde 2013, Mertens-Simón-Wong 2022, Chua-Jiang 2023]

$$\mathscr{H}_{\text{bulk}} = \text{span}\{|\alpha, \bar{\alpha}\rangle \,|\alpha, \bar{\alpha} > c/12\} \otimes \mathscr{H}_{\text{desc}}.$$

My proposed holographic map is

$$V|\alpha, \bar{\alpha}\rangle = \frac{1}{\sqrt{n_{\alpha,\bar{\alpha}}}} \sum_{O} \sqrt{f_{\alpha,\bar{\alpha}}(O)} e^{i\phi_{\alpha,\bar{\alpha}}(O)} |O\rangle_L |O\rangle_R \,,$$

and identity on the descendant factor, with some conditions on the phases.

Notice:

- in $\sum_O$, $\mathscr{O}(e^S)$ phases in $\mathscr{O}(1)$ window of h_O.
- in $\int \mathrm{d}\alpha$ $\mathscr{O}(1)$ phases in $\mathscr{O}(1)$ window of α.

So there's a lot of freedom to get desirable answers.

Checks

Some conditions that the map can satisfy simultaneously:

1. V is approximately isometric: $\langle\phi|\psi\rangle \approx \langle\phi|V^\dagger V|\psi\rangle$.
2. The coarse-graining procedure from before works,

$$V^\dagger |O\rangle \langle O| V \approx \int \mathrm{d}\alpha\, f_\alpha(O) |\alpha\rangle \langle\alpha| .$$

3. Reasonable bulk wavefunctions (e.g. the canonical TFD) map to the expected boundary duals, up to small errors.

$$V \int \mathrm{d}\alpha \sqrt{n_\alpha} e^{-\beta\alpha} |\alpha\rangle \approx \sum_O e^{-\beta\Delta(O)} |O, O\rangle .$$

Comparison with ILM

Can compare with proposal of ILM [Iliesiu-Levine-Lin-Maxfield-Mezei 2024].

$$V_{\mathrm{ILM}}\,|\alpha\rangle = \sum_{O} \delta(\alpha - \Delta(O))\,|O, O\rangle$$

$$V_{\mathrm{S}}\,|\alpha\rangle = \frac{1}{\sqrt{n_\alpha}} \sum_{O} \sqrt{f_\alpha(O)}\, e^{i\phi_\alpha(O)}\,|O\rangle_L\,|O\rangle_R\,,$$

Not necessarily inconsistent, since

$$\sum_{O} e^{-\frac{\beta}{2}\Delta(O)}\,|O, O\rangle = V_{\mathrm{ILM}} \int \mathrm{d}\alpha\, n^0_\alpha e^{-\frac{\beta}{2}\alpha}\,|\alpha\rangle$$

$$\approx V_{\mathrm{S}} \int \mathrm{d}\alpha\, \sqrt{n_\alpha}\, e^{-\frac{\beta}{2}\alpha}\,|\alpha\rangle$$

V_{ILM} works for the energy basis of [Yang 2018] and V_{S} works for the entropy/fixed-area basis.

Summary

- I wrote down a CFT area operator in a very simple setup (no bulk local degrees of freedom).
- Despite the simplicity of the setup (or perhaps due to it), we could think about the coarse-graining required for a bulk to emerge.
- The same idea seems to work with more regions, and shows some expected behaviour like non-commuting areas.
- I proposed a holographic map by taking the coarse-graining procedure very seriously.

Present and Future Directions

1. Can we find the $\mathcal{O}(1)$ ambiguities due to choice of coarse-graining function in gravity? Maybe disprove/refine the story above?
2. Adding multi-boundary wormholes, brane matter and field matter (increasing levels of hardness). Perhaps relevant ideas in [AliAhmad-Klinger 2024-5].

   Expectation: adding field matter in the bulk will allow us to see the emergence of the gravitational crossed product [Witten. . . 2021-].
3. Formulate more rigorous checks of the holographic map and see if they are passed.
4. Can we learn something about observer holography, non-ideal clocks etc?
5. Can we coarse-grain the boundary in different ways to give different Cauchy slices?