Title: von Neumann algebraic quantum information theory and entanglement in infinite quantum systems

Speakers: Lauritz van Luijk

Collection/Series: Quantum Information

Subject: Quantum Information

Date: May 29, 2025 - 3:00 PM

URL: https://pirsa.org/25050049

Abstract:

In quantum systems with infinitely many degrees of freedom, states can be infinitely entangled across a pair of subsystems. But are there different forms of infinite entanglement?

In the first part of my talk, I will present a von Neumann algebraic framework for studying information-theoretic properties of infinite systems. Using this framework, we find operational tasks that distinguish different forms of infinite entanglement, and, by analysing these tasks, we show that the type classification of von Neumann algebras (types I, II, III, and their respective subtypes) is in 1-to-1 correspondence with operational entanglement properties. Our findings promote the type classification from mere algebraic bookkeeping to a classification of infinite quantum systems based on their operational entanglement properties.

In the second part, I will discuss what is known about the type classification of the von Neumann algebras arising in quantum many-body systems. Together with our results, this identifies new operational properties, e.g., embezzlement of entanglement, of well-known physical models, e.g., the critical transverse-field Ising chain or suitable Levin-Wen models.

Joint work with: Alexander Stottmeister, Reinhard F. Werner, and Henrik Wilming

von Neumann algebraic Quantum Information Theory

Extend Definitions & Theorems

von Neumann algebraic Quantum Information Theory

Extend Definitions & Theorems

Relative entropy, f-divergences, Petz recovery, Nielsen's theorem, smooth entropy formalism, asymptotic equipartition theorem, …

Correspondence between **operational** and **algebraic** properties

Bell's inequalities, Embezzlement of entanglement, secret sharing, Tsirelson's problem, …

Understand vN-algebraic properties of concrete models/systems.

rel. QFT, CFT, semiclassical QG, gapped spin systems in 2D (Toric code, Levin-Wen), critical spin chains, quasi free fermions & bosons, …

→ **Information-theoretic properties in concrete models/systems.**

Example: No information without disturbance

→ Measurements cannot reveal information about the state of a system without altering it.

Operational property

The principle applies to the agent's subsystem

the agent's subsystem is "purely quantum"

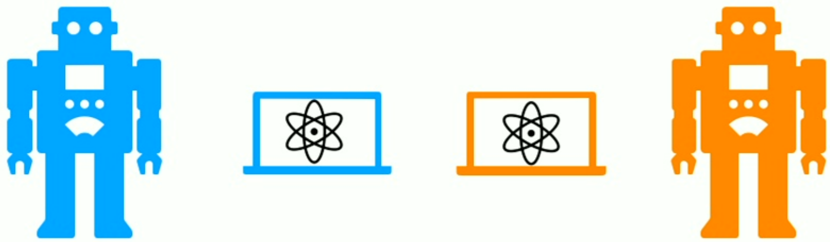
$$\Updownarrow$$

Algebraic property

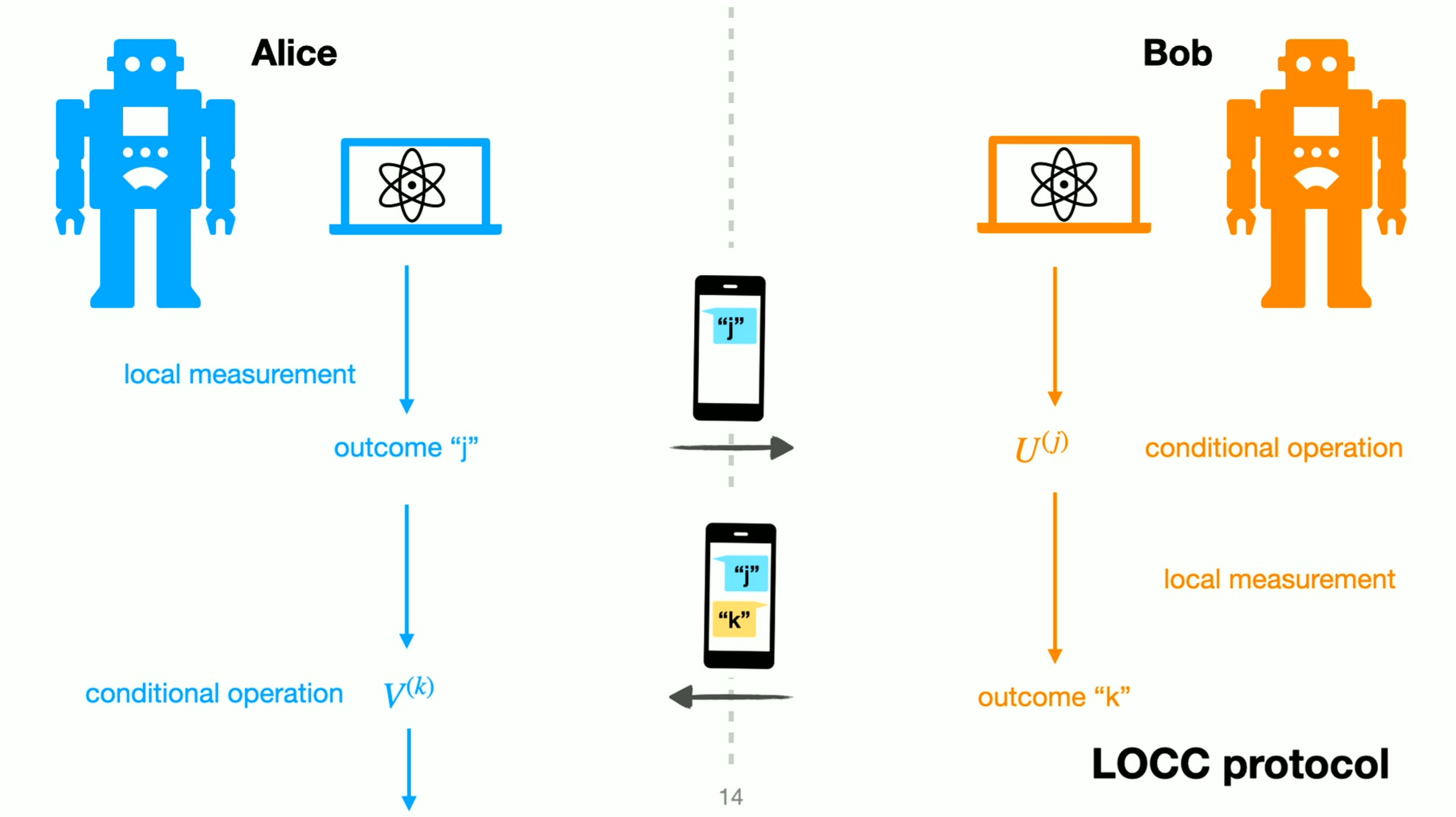
The agents observable algebra $\mathscr{M}$ is a **factor**:

$$\mathscr{M} \cap \mathscr{M}' = \mathbb{C}1_{\mathscr{H}}$$

Entanglement



The resource that cannot be created by **local (quantum) operations** and **classical communication** (LOCC).



Alice
Bob
local measurement
outcome “j”
“j”
$U^{(j)}$
conditional operation
local measurement
“j”
“k”
conditional operation
$V^{(k)}$
outcome “k”
LOCC protocol
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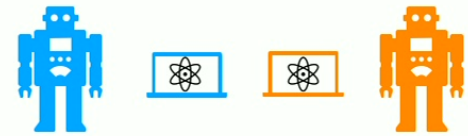
Pure state entanglement theory



When is $|\Psi\rangle_{AB}$ more entangled than $|\Phi\rangle_{AB}$?

more entangled $|\Psi\rangle_{AB} \longrightarrow$ **LOCC protocol** $\longrightarrow |\Phi\rangle_{AB}$ less entangled

(We allow for errors if they can be made arbitrarily small)

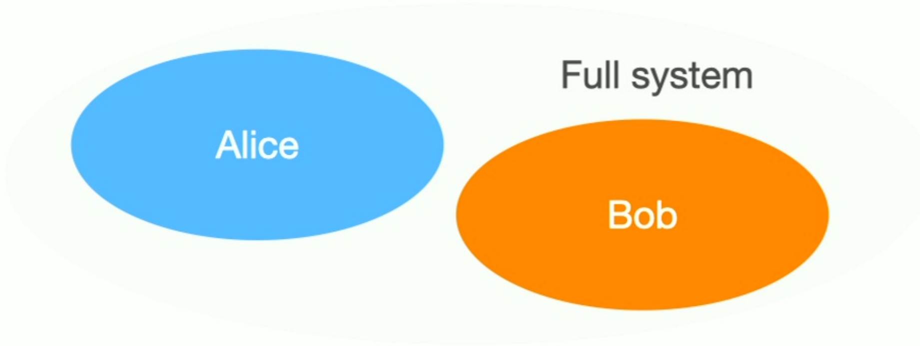
To define comparative entanglement theory for, we only need to understand:

1) bipartite systems

2) local operations

Bipartite systems

Minimal assumption:
commuting von Neumann algebras $(\mathscr{M}_A, \mathscr{M}_B)$ on $\mathscr{H}$



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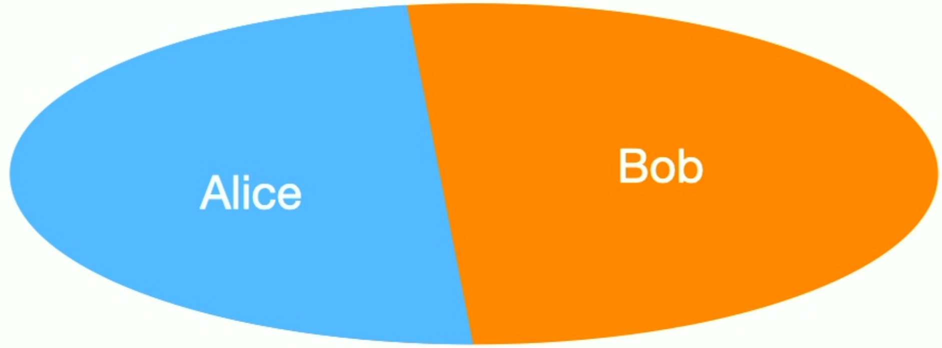
Bipartite systems



Minimal assumption:
commuting von Neumann algebras $(\mathscr{M}_A, \mathscr{M}_B)$ on $\mathscr{H}$

Tomography: States can be uniquely determined in correlation experiments.

$\Updownarrow$

$\mathscr{M}_{A/B}$ are factors & $\mathscr{M}_A \vee \mathscr{M}_B = B(\mathscr{H})$

Uniqueness of purifications

$|\Psi\rangle, |\Phi\rangle$ in $\mathscr{H}$ induce the same states on $\mathscr{M}_{A/B} \iff |\Psi\rangle \in \overline{\mathscr{U}(\mathscr{M}_B)|\Phi\rangle}$

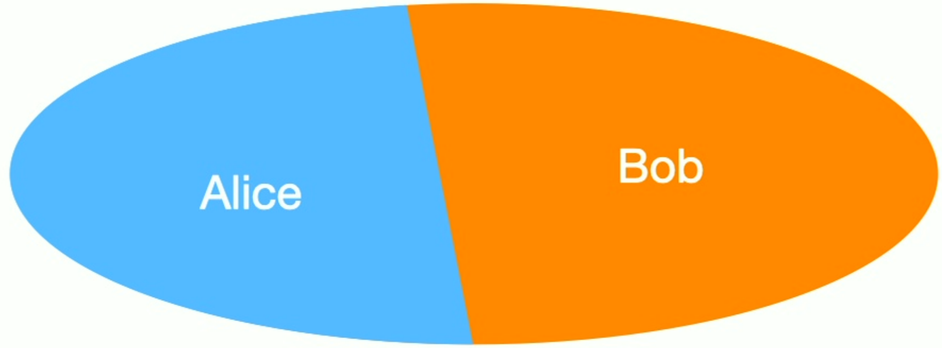
$\Updownarrow$

Haag duality:

$$\mathscr{M}_A = \mathscr{M}_B'$$

(unpublished)

Bipartite systems



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$$\not\Longleftarrow$$

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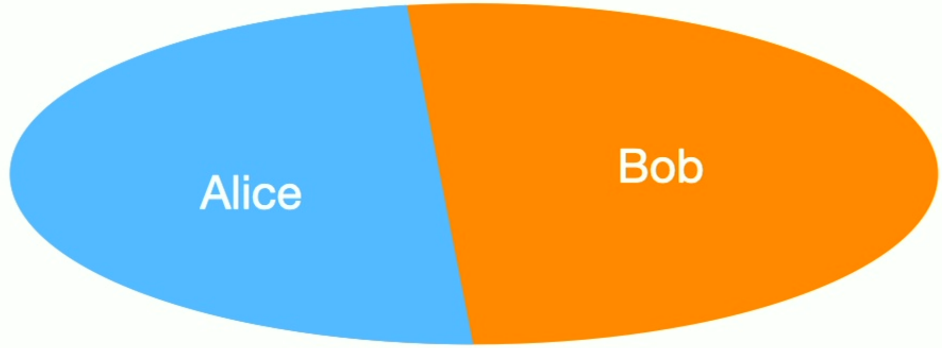
Bipartite systems



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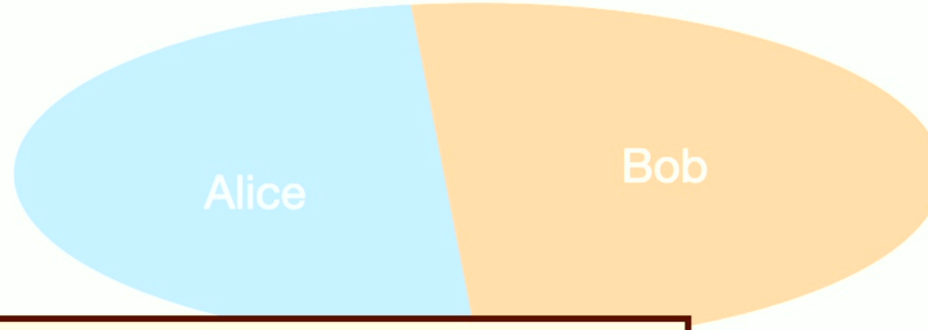
Bipartite systems



Minimal assumption:

commuting von Neumann

algebras $(\mathscr{M}_A, \mathscr{}$

A bipartite system is a pair $(\mathscr{M}_A, \mathscr{M}_B)$ of commuting factors on a Hilbert space $\mathscr{H}$ with

$$\mathscr{M}_A = \mathscr{M}_B' \,.$$

Tomography: S

determined in c

fications

same states on

$(\mathscr{M}_B) \,|\, \Phi\rangle$

$\Updownarrow$ $\qquad$ $\not\Longrightarrow$ $\qquad$ $\Updownarrow$

$\not\Longleftarrow$

$\mathscr{M}_{A/B}$ are factors & $\mathscr{M}_A \vee \mathscr{M}_B = B(\mathscr{H})$

Haag duality:

$\mathscr{M}_A = \mathscr{M}_B'$

(unpublished)

Nielsen's theorem

Given: $|\Psi\rangle, |\Phi\rangle \in \mathscr{H}$ with A-marginal states ψ_A, ϕ_A.

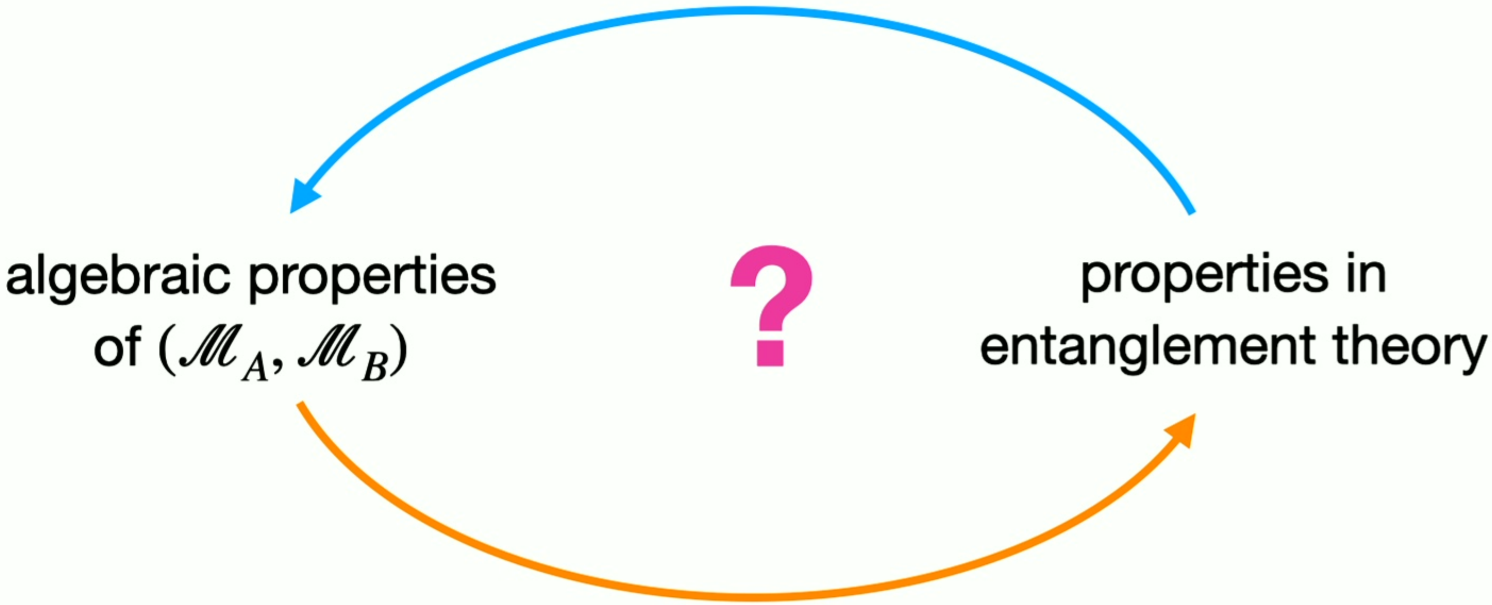
Theorem: $|\Psi\rangle \xrightarrow{\overline{LOCC}} |\Phi\rangle \iff \psi_A \preceq \phi_A$

$$\psi_A \preceq \phi_A \quad :\iff \quad \forall \epsilon > 0 \; \exists \{p_x, u_x\} \quad \text{s.t.} \quad \psi_A \approx_\epsilon \sum p_x u_x \phi_A u_x^*$$

Proof sketch: Go through proof in Nielsen & Chuang and replace objects such as $\rho^{1/2}\sigma^{-1/2}$ with corresponding objects in "Tomita-Takesaki modular theory".

J. Crann et al., *"State Convertibility in the von Neumann Algebra Framework"*, Comm. Math. Phys. 378, no. 2 (2020): 1123–56,

L. v. L., A. Stottmeister, R.F. Werner. H. Wilming, *"Pure state entanglement and von Neumann algebras"*, arXiv:2409.17739


algebraic properties of $(\mathscr{M}_A, \mathscr{M}_B)$
?
properties in entanglement theory

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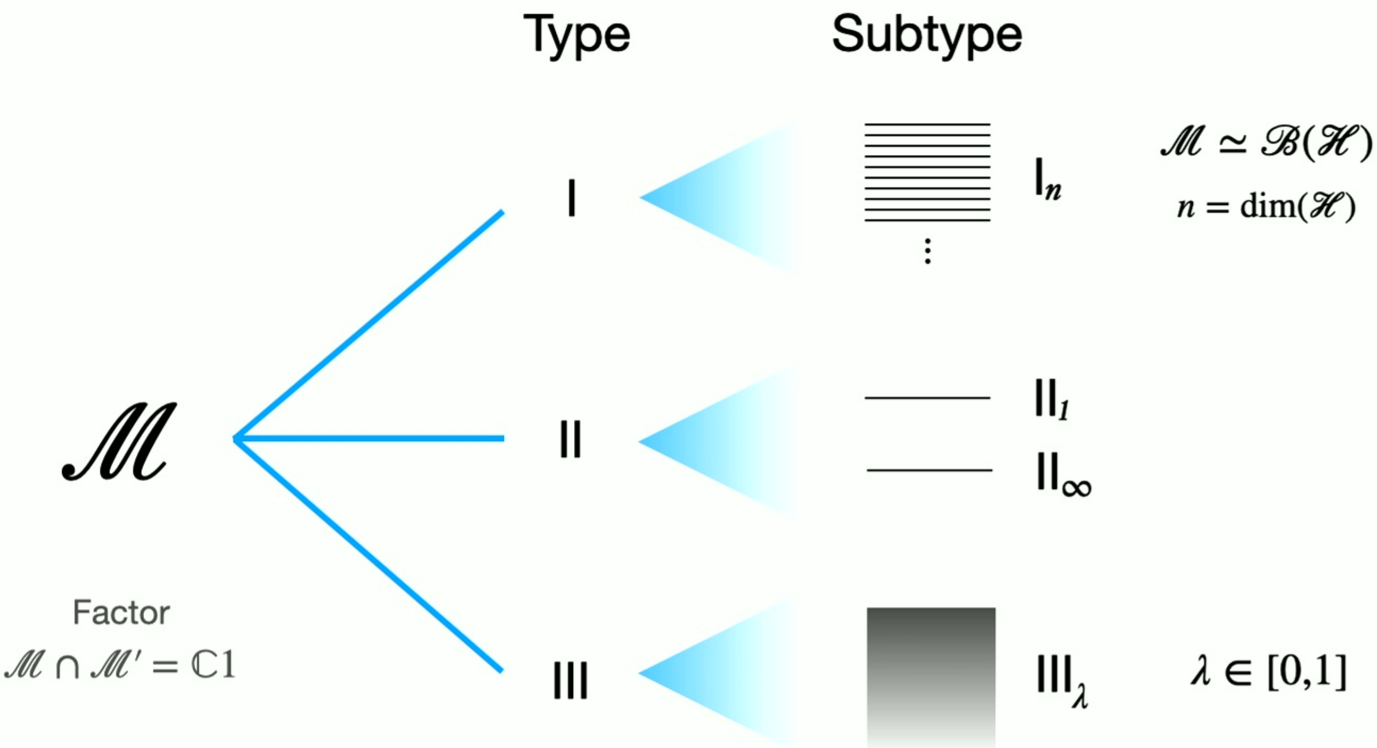
Types of factors



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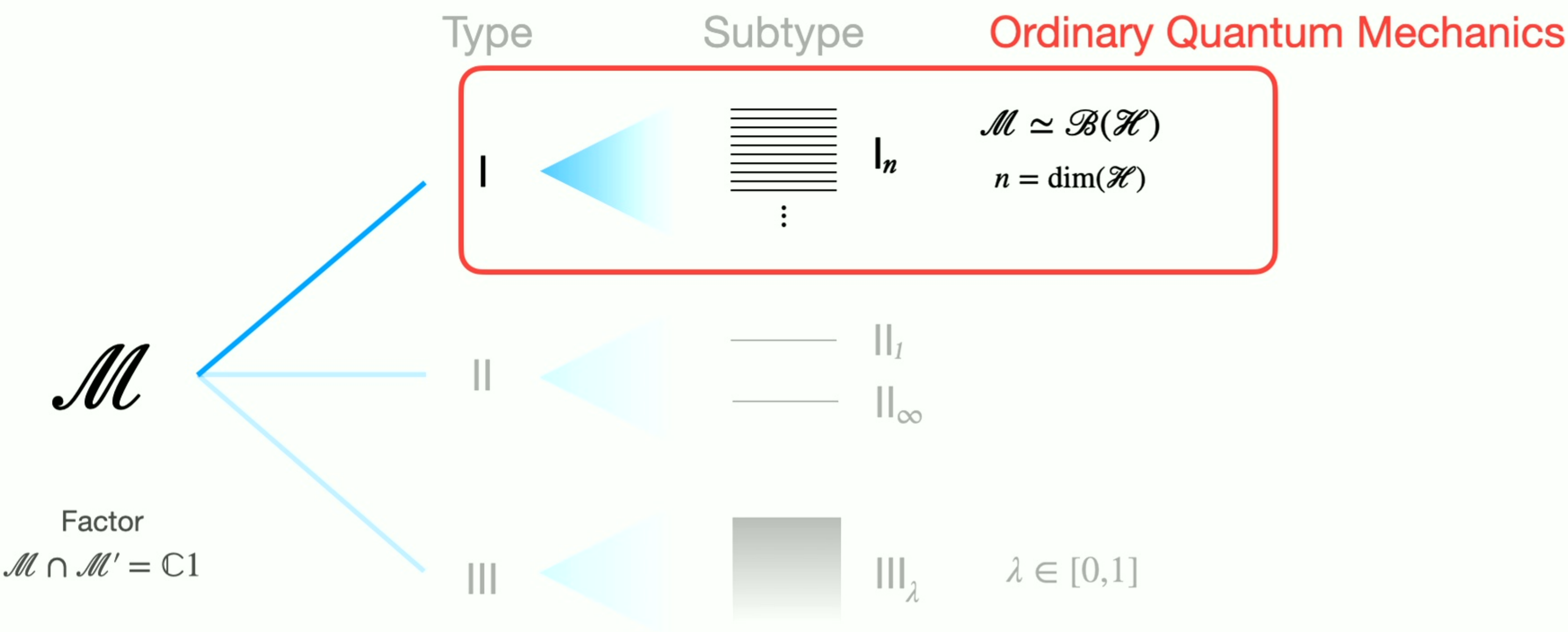

Main Theorem: The type classification is in 1-to-1 correspondence with operational properties in entanglement theory.

	Defining property	Subtypes characterised by
Type I	finite 1-shot entanglement	Schmidt rank
Type II	LOCC not trivial	maximally entangled states
Type III	LOCC trivial	embezzlement of entanglement

L. v. L., A. Stottmeister, R. F. Werner, H. Wilming, “Embezzlement of Entanglement, Quantum Fields, and the Classification of von Neumann Algebras,” arXiv.2401.07299.
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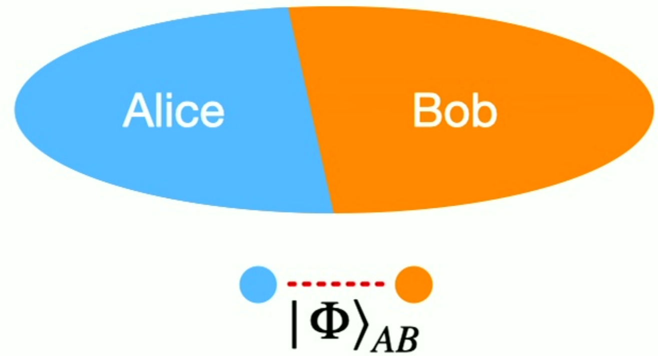
One-shot entanglement



Full system is in a pure state $|\Psi\rangle \in \mathscr{H}$

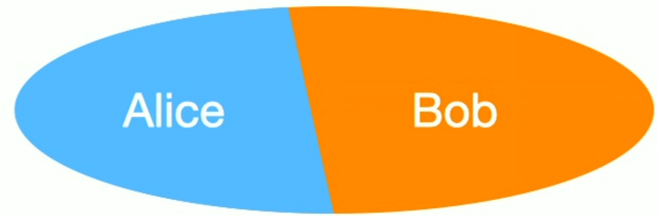
A bipartite state $|\Phi\rangle_{AB}$ can be distilled from (in one shot) if

$$|\Psi\rangle \otimes |0\rangle_A |0\rangle_B \longrightarrow \boxed{\textbf{LOCC}} \longrightarrow |\Psi'\rangle \otimes |\Phi\rangle_{AB}$$

for some $|\Psi'\rangle \in \mathscr{H}$.

$|\Psi\rangle$ has **infinite 1-shot entanglement** if every finite-dim $|\Phi\rangle_{AB}$ can be distilled.

One-shot entanglement



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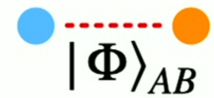

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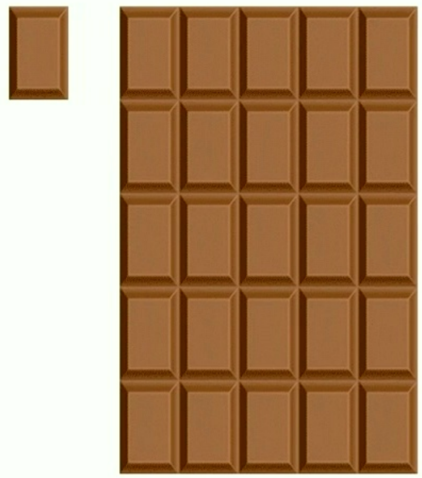
Embezzlement

Noun. /ɪmˈbez.əl.mənt/

The crime of secretly taking money that is in your care or that belongs to an organization or business you work for.

Cambridge Dictionary

Embezzlement of chocolate?



user585825, How many whole pieces can be taken out in this way? (Infinite chocolate bar problem), URL (version: 2018-08-20): https://math.stackexchange.com/q/2889188

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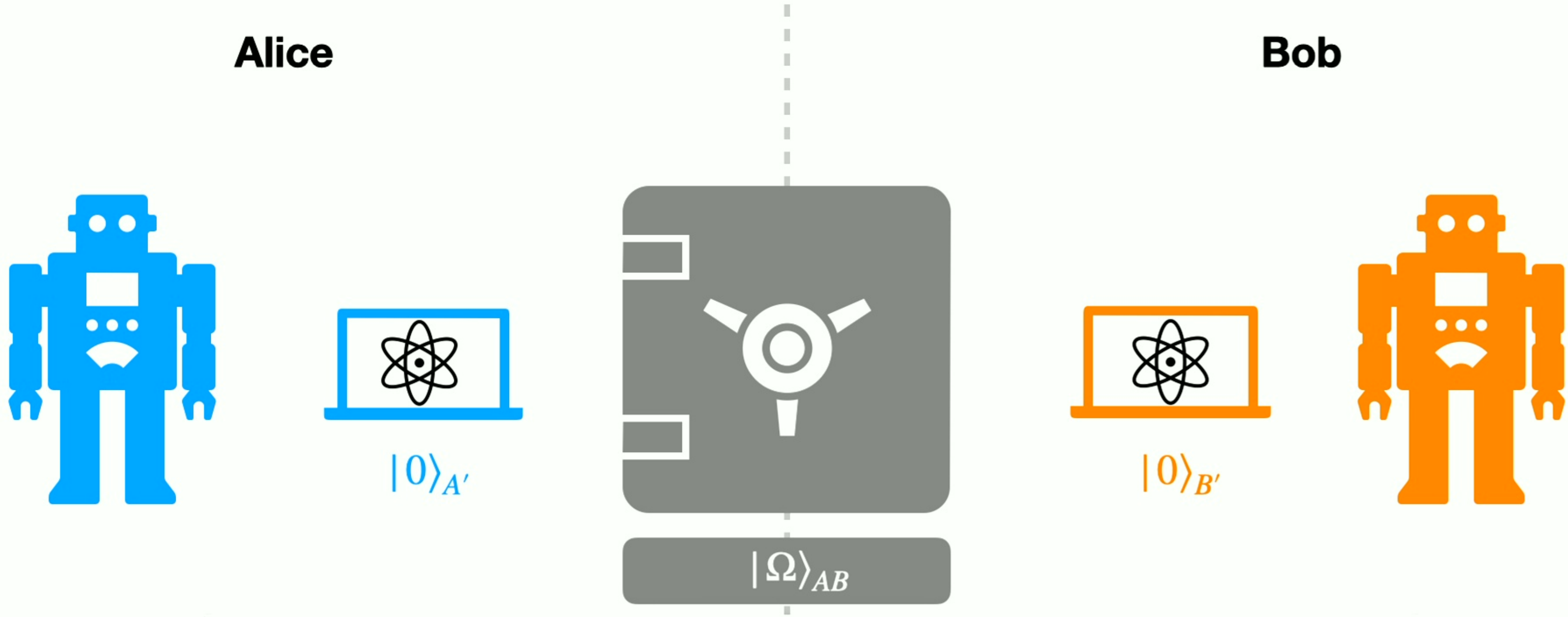
Embezzlement of entanglement
Noun. /ɪmˈbez.əl.mənt əv ɪnˈtæŋ.gəl.mənt/

The crime of secretly taking entanglement that is in your care or that belongs to an organization or business you work for.

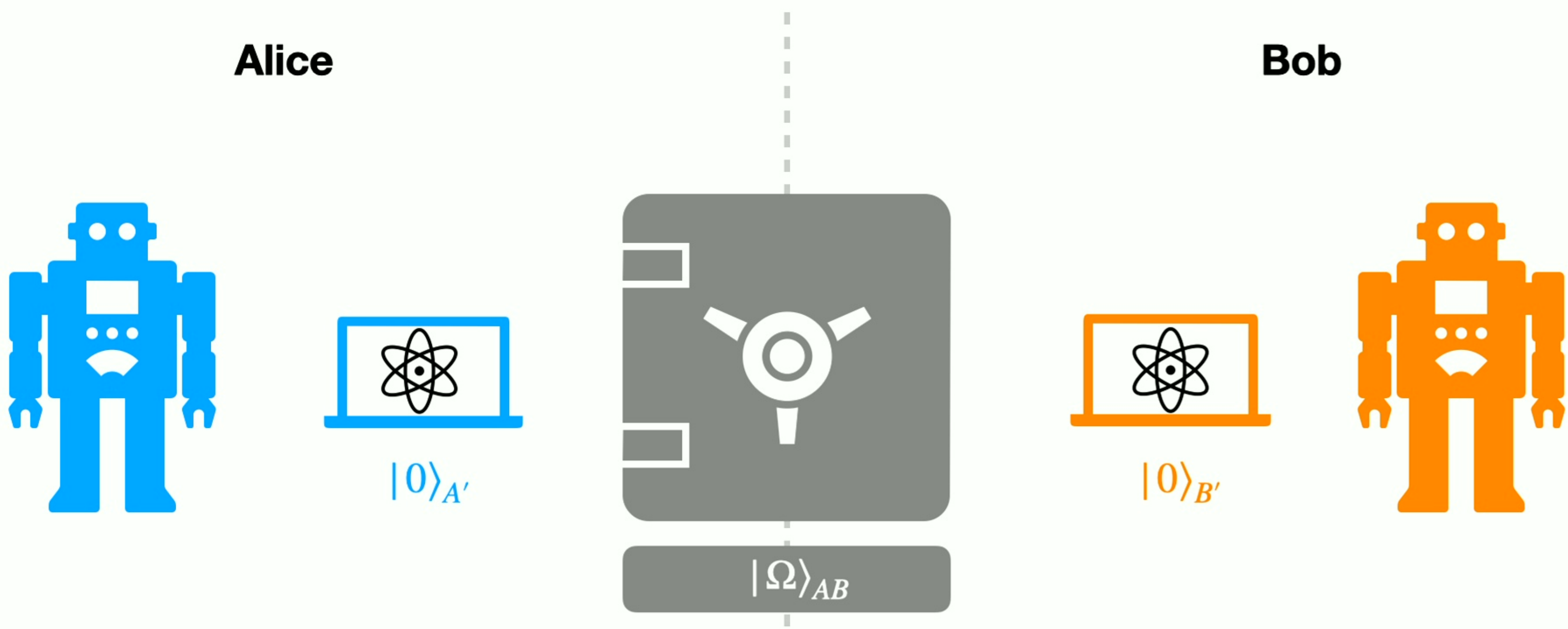
Cambridge Dictionary (next edition)

W. van Dam and P. Hayden, "Universal Entanglement Transformations without Communication", *Physical Review A* 67, no. 6 (2003).

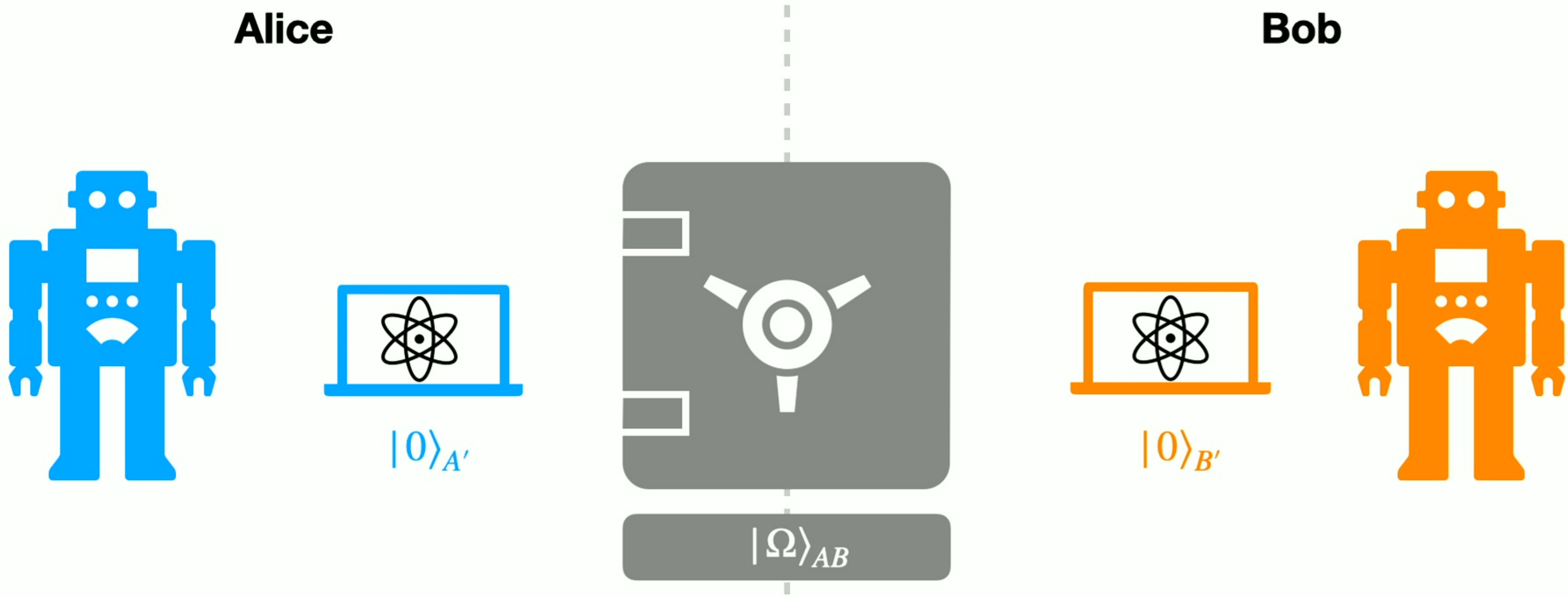


The crime of secretly taking entanglement [...]

To be sure that no record of the crime exists, should be done without classical communication!



Ideally: $U_{AA'}U_{BB'}(|\Omega\rangle_{AB} \otimes |0\rangle_{A'}|0\rangle_{B'}) = |\Omega\rangle_{AB} \otimes |\Psi\rangle_{A'B'}$



Ideally: $U_{AA'}U_{BB'}(|\Omega\rangle_{AB} \otimes |0\rangle_{A'}|0\rangle_{B'}) = |\Omega\rangle_{AB} \otimes |\Psi\rangle_{A'B'}$ **Impossible!**

Richard Cleve, Li Liu, and Vern I. Paulsen, "Perfect Embezzlement of Entanglement", *Journal of Mathematical Physics* 58, no. 1 (2017)

L. v. L., A. Stottmeister, R. F. Werner, H. Wilming., "Pure state entanglement in von Neumann algebras", arXiv.2401.07299

Embezzling states

For a bipartite system $(\mathscr{M}_A, \mathscr{M}_B)$ on $\mathscr{H}$:

Embezzling state: $|\Omega\rangle_{AB} \in \mathscr{H}$ such that <u>for all</u> $|\Psi\rangle_{A'B'} \in \mathbb{C}^n \otimes \mathbb{C}^n$ and all $\varepsilon > 0$

$$|\Omega\rangle_{AB} \otimes |0\rangle_A |0\rangle_B \longrightarrow \boxed{\textbf{Local unitaries}} \longrightarrow \approx_\varepsilon |\Omega\rangle_{AB} \otimes |\Psi\rangle_{A'B'}$$

L.v.L., A. Stottmeister, R.F. Werner, H. Wilming. "Embezzlement of Entanglement, Quantum Fields, and the Classification of von Neumann Algebras," arXiv.2401.07299.

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W. van Dam and P. Hayden, “Universal Entanglement Transformations without Communication”, *Physical Review A* 67, no. 6 (2003): 060302

Quantifying Embezzlement

worst-case error:

$\kappa(\Omega) = \inf \ \epsilon > 0$ s.t. every $|\Psi\rangle$ can be embezzled from $|\Omega\rangle$ up to error $< \epsilon$ *

$\longrightarrow \quad 0 \leq \kappa(\Omega) \leq 2 \qquad \kappa(\Omega) = 0 \iff |\Omega\rangle$ is embezzling

Best worst-case error:		Worst worst-case error:
$\kappa_{\min} = \inf_{|\Omega\rangle} \kappa(\Omega)$	$\leq \quad \kappa(\Omega) \quad \leq$	$\kappa_{\max} = \sup_{|\Omega\rangle} \kappa(\Omega)$

$\kappa_{\min}$ and $\kappa_{\max}$ depend only on $(\mathscr{M}_A, \mathscr{M}_B, \mathscr{H})$!

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* with respect to the norm distance of the reduced states

Embezzlement and the classification

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
$\kappa_{\min}$	2	2	0 or 2	**0**	**0**
$\kappa_{\max}$	2	2	2	$2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$	**0**

L. v. L., A. Stottmeister, R.F. Werner, H.Wilming, "Embezzlement of Entanglement, Quantum Fields, and the Classification of von Neumann Algebras," arXiv:2401.07299.

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$\kappa_{\min}$	2	2	0 or 2	**0**	**0**
$\kappa_{\max}$	2	2	2	$2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$	**0**

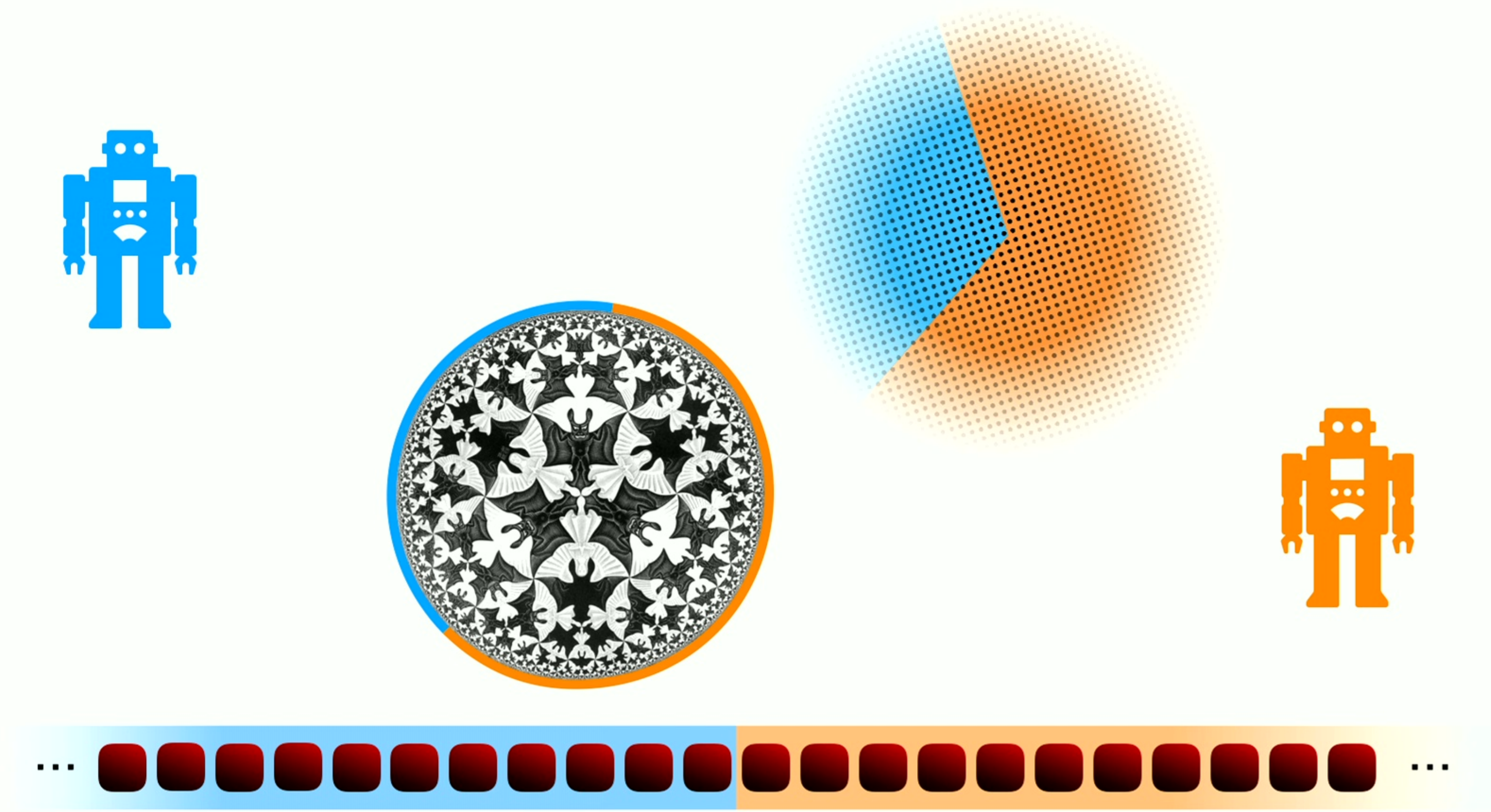
Embezzlement reveals Connes' classification of type III factors: $\lambda = \left(\frac{2-\kappa_{\max}}{2+\kappa_{\max}}\right)^2$

L. v. L., A. Stottmeister, R.F. Werner, H.Wilming, "Embezzlement of Entanglement, Quantum Fields, and the Classification of von Neumann Algebras," arXiv:2401.07299.

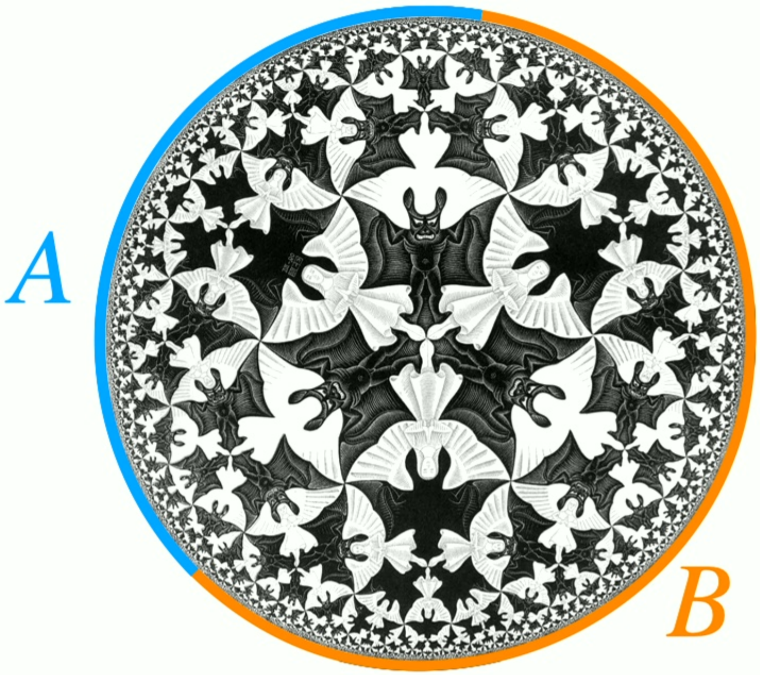
Main Theorem: The type classification is in 1-to-1 correspondence with operational properties in entanglement theory.

The classification of von Neumann algebras is a classification of infinite quantum systems in terms of their entanglement properties.

Algebraic properties in concrete models



Vacuum sector of a CFT on the circle



$A \mapsto \mathscr{M}(A)$ vN algebras on the vacuum sector $\mathscr{H}$ covariant under the improper Moebius group.

- each $\mathscr{M}(A)$ is a type III_1 factor
- Haag duality holds: $\mathscr{M}(A) = \mathscr{M}(B)'$

F. Gabbiani, J. Fröhlich. “Operator algebras and conformal field theory”, Comm. in Math. Phys. 155.3 (1993)

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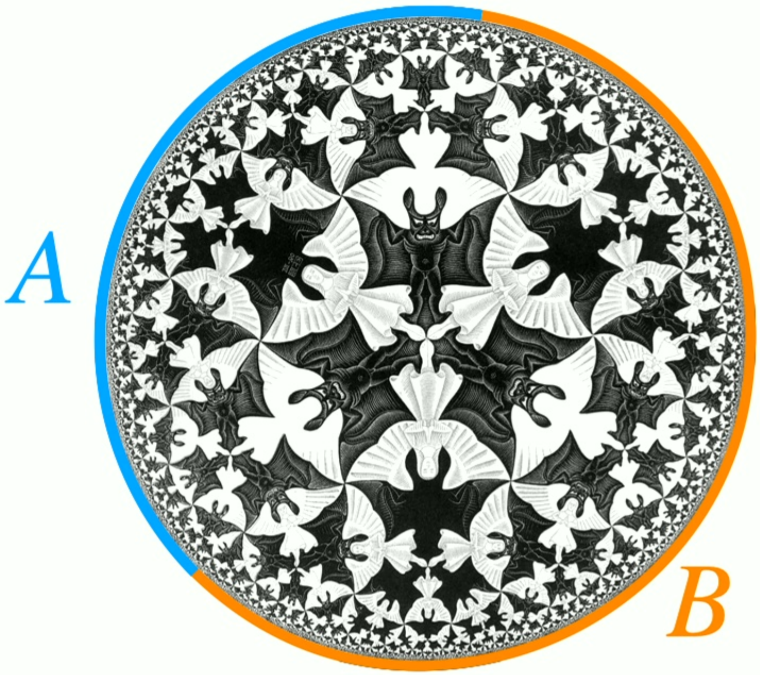
Vacuum sector of a CFT on the circle



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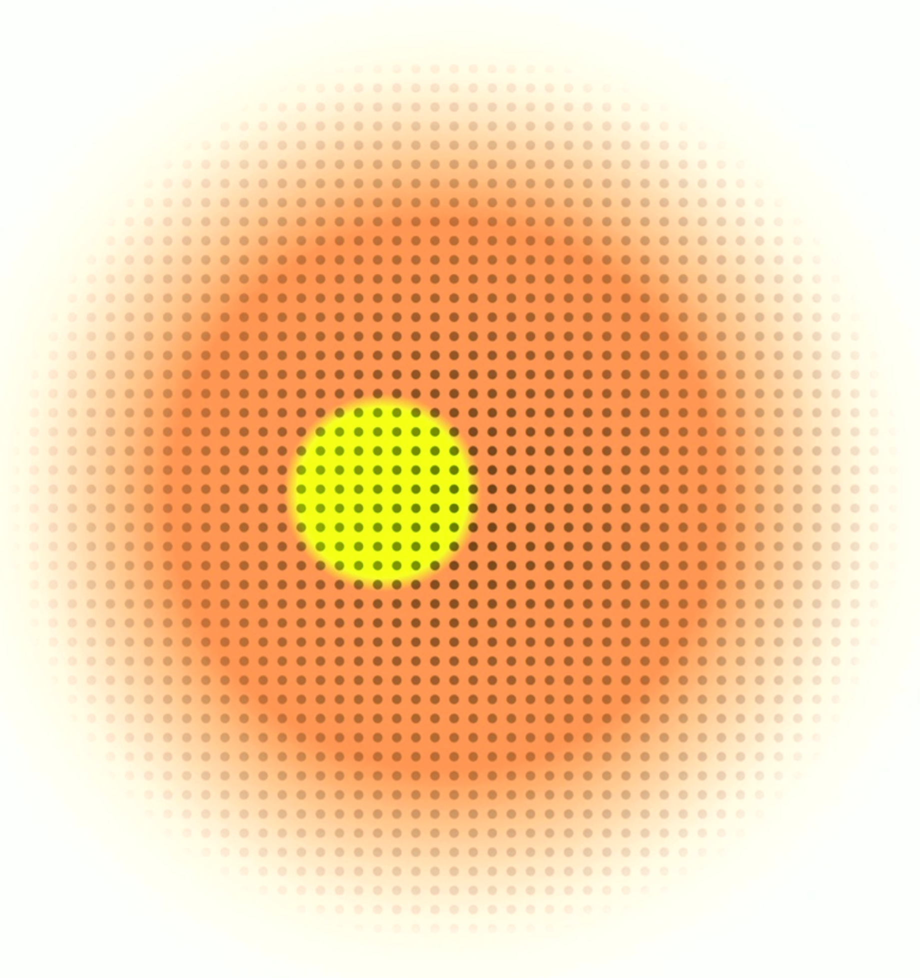
covariant under the improper Moebius group.

- each $\mathscr{M}(A)$ is a type III_1 factor

- Haag duality holds: $\mathscr{M}(A) = \mathscr{M}(B)'$

$\longrightarrow$ vacuum state (and every other pure state in the vacuum sector) is embezzling.

F. Gabbiani, J. Fröhlich. "Operator algebras and conformal field theory", Comm. in Math. Phys. 155.3 (1993)



Ground state sector of a spin system

Given a ground state $|\Omega\rangle$ the sector Hilbert space $\mathscr{H}$, consists of those pure states that can be approximated arbitrary well by acting on $|\Omega\rangle$ with operators of **finite** support.

The sector is the GNS rep. of the pure state ω on the C*-algebra $\mathscr{A}_\Gamma = \bigotimes_{x\in\Gamma} M_2(\mathbb{C})$.

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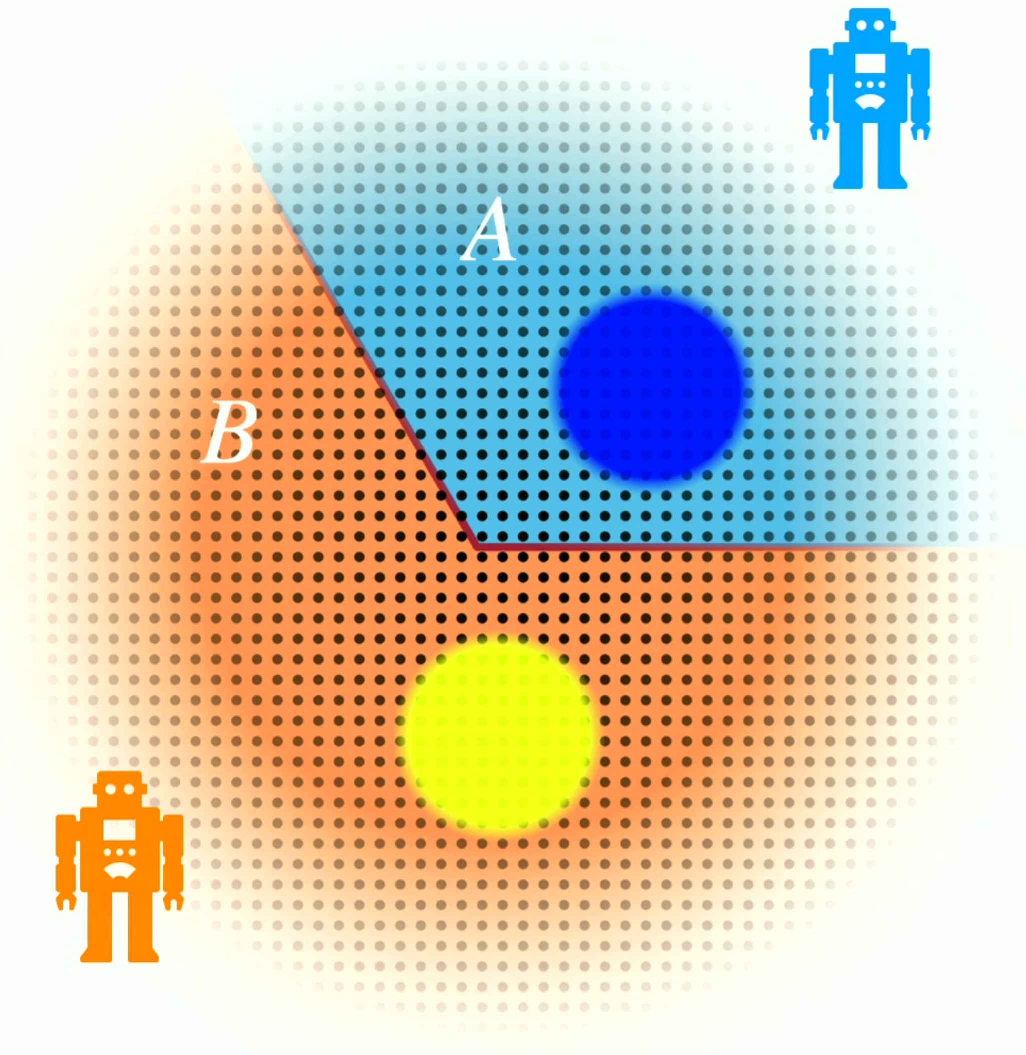

local von Neumann algebras

Operators localized in A or B generate **commuting** von Neumann algebras $\mathscr{M}_A$ and $\mathscr{M}_B$ acting on $\mathscr{H}$.

↓

describe perations with approximately finite support

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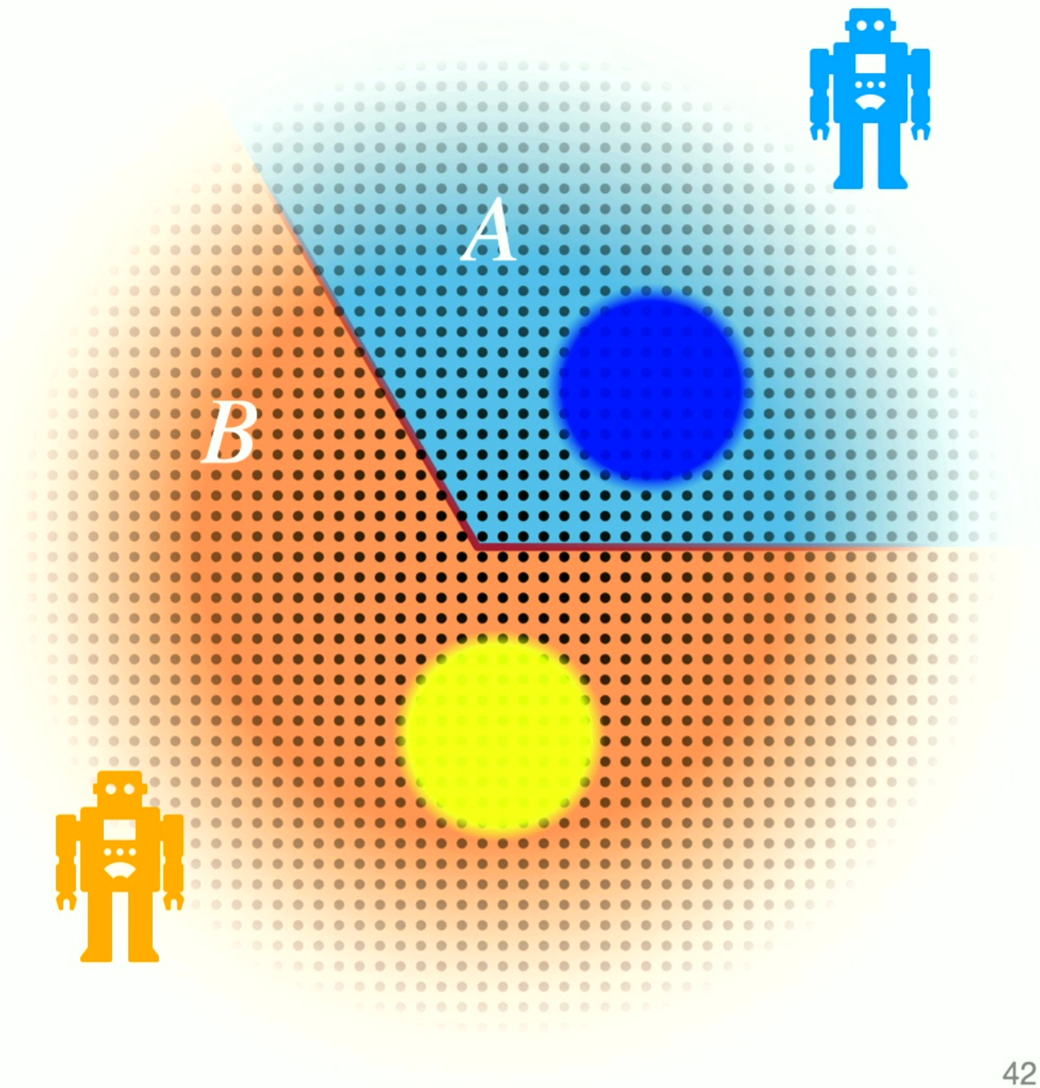

$|\Psi\rangle_{AB}$ → **LOCC protocol** → $|\Phi\rangle_{AB}$

possible to arbitrary precision acting on **finite regions** only.

⇕

$|\Psi\rangle_{AB}$ → **LOCC protocol** → $|\Phi\rangle_{AB}$

possible to arbitrary precision acting with operators from $\mathscr{M}_A$ and $\mathscr{M}_B$.

J. Crann et al., "State Convertibility in the von Neumann Algebra Framework," Communications in Mathematical Physics 378, no. 2 (2020): 1123–56

L. v. L., A. Stottmeister, R.F. Werner, H. Wilming, "Pure state entanglement and von Neumann algebras", arXiv:2409.17739

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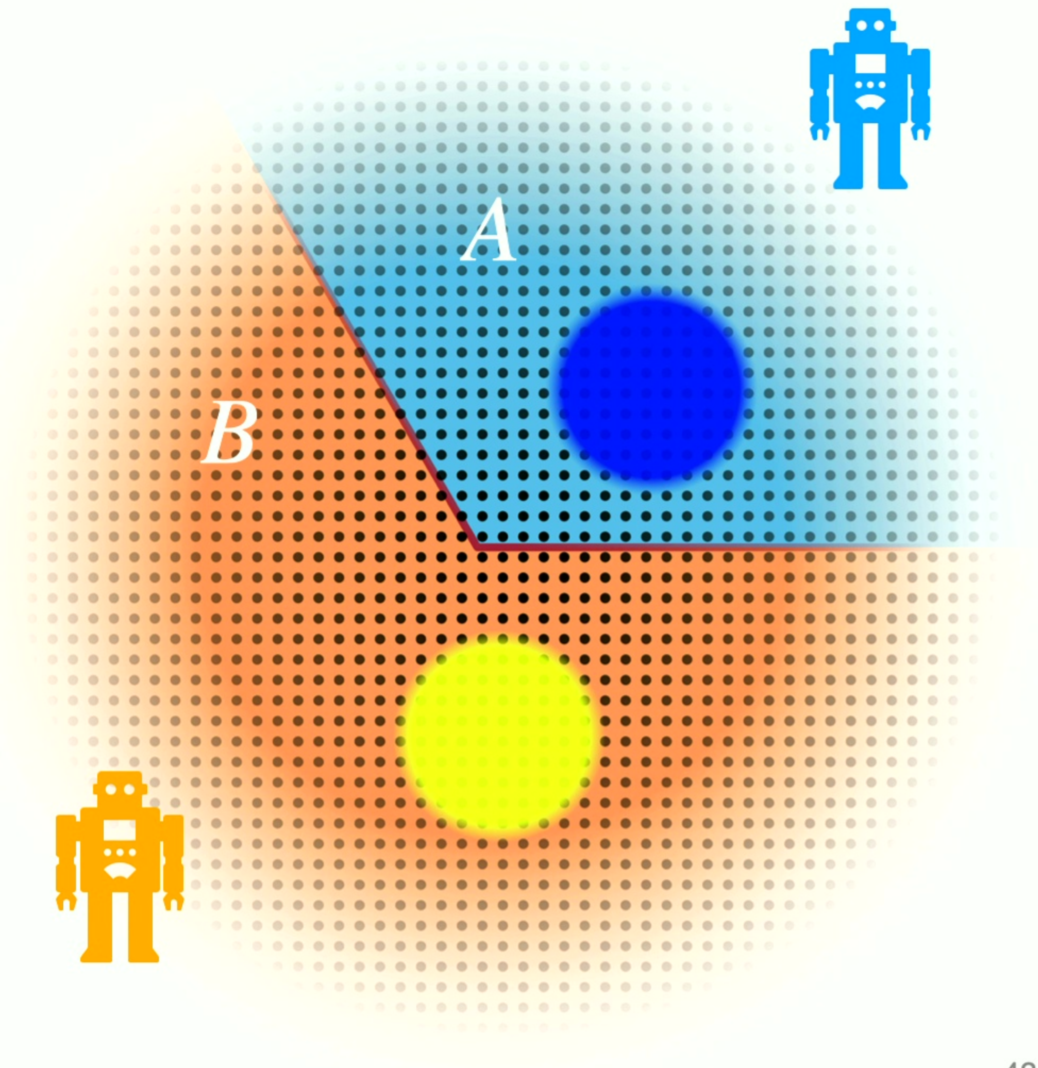
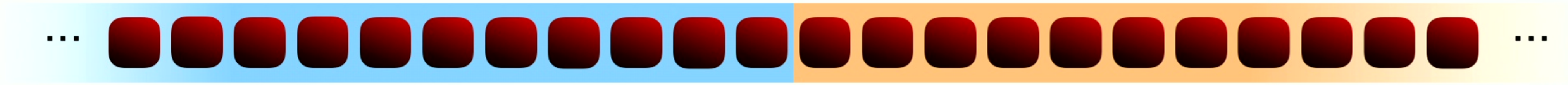


- Purity of ω implies that $\mathscr{M}_{A/B}$ are **factors** with

$$\mathscr{M}_A \vee \mathscr{M}_B = B(\mathscr{H})$$

E.g. $H = -\frac{1}{2}\sum (X_n X_{n+1} + Y_n Y_{n+1})$

Example: **Spin chains**



Theorem (Matsui). For a gapped spin chain, the ground state sector yields **type I** factors (this implies Haag duality).

T. Matsui, "Boundedness of Entanglement Entropy and Split Property of Quantum Spin Chains," *Reviews in Mathematical Physics* 25, no. 09 (2013): 1350017

Theorem. For a **critical** translation-invariant free fermion chains, the ground state sector has **type** $\mathbf{III}_1$ factors in Haag duality.

L. v. L., A. Stottmeister, Henrik Wilming, "Critical fermions are universal embezzlers". *Nature Physics* (2025).

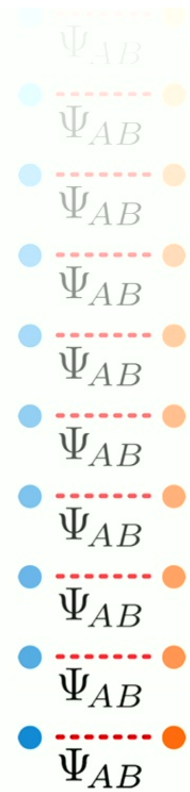
→ Jordan-Wigner

type III_1 + Haag duality holds for critical XY chain & transverse-field Ising model

Infinitely many copies of $|\Psi_{AB}\rangle$

ground state sector of $H = 1 - \sum P_n$, P_n = projection onto n-th $|\Psi_{AB}\rangle$

$$\mathscr{H} = \bigotimes_{n=1}^{\infty} \left(\mathbb{C}^d \otimes \mathbb{C}^d; |\Psi_{AB}\rangle\right), \quad \mathscr{M}_A = \bigotimes_{n=1}^{\infty} \left(M_d(\mathbb{C}) \otimes \mathbf{1}; |\Psi_{AB}\rangle\right), \quad \mathscr{M}_B = \ldots$$



> Haag duality $\mathscr{M}_A = \mathscr{M}_B'$ holds!
>
> Type depends on the Schmidt spectrum of $|\Psi_{AB}\rangle$

Qubit case:

$$|\Psi_{AB}\rangle \propto |1\rangle_A |1\rangle_B + \sqrt{\lambda}\,|2\rangle_A |2\rangle_B \implies$$

λ	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
Type of $\mathscr{M}_{A/B}$	I_∞	III_λ	II_1

Option A

$$\Psi_{AB} \propto |1\rangle_A|1\rangle_B + |2\rangle_A|2\rangle_B + |3\rangle_A|3\rangle_B$$

Type II_1

Option B

$$\Psi_{AB} \propto |1\rangle_A|1\rangle_B + 2|2\rangle_A|2\rangle_B + 3|3\rangle_A|3\rangle_B$$

Type III_1

Is there an operational difference?

Both have infinite 1-shot entanglement,
but only B can embezzle.

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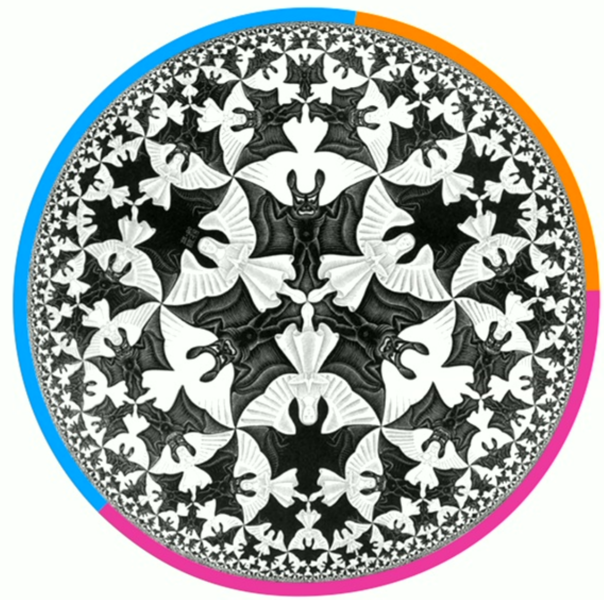
Open problem in multi-partite entanglement theory

For all $n \in \mathbb{N}$, there are vN algebraic N-partite systems where all pure state are embezzling!

D. Leung, B. Toner, J. Watrous. "Coherent state exchange in multi-prover quantum interactive proof systems". Chicago Journal of Theoretical Computer Science 19.1

L. v. L., A. Stottmeister, H. Wilming. *Multipartite Embezzlement of Entanglement* (2024).



Does multipartite mbz appear in physics?

Are CFTs on the circle tripartite mbz?