Title: Self testing in General Probabilistic Theories

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Collection/Series: Quantum Foundations

Subject: Quantum Foundations

Date: May 26, 2025 - 11:00 AM

URL: https://pirsa.org/25050047

Abstract:

This talk will consist of two parts. In the former I discuss published work [LD, Ligthart, Gross, PRA, 2024], and in the latter some new related results.

Part 1 -- Although there exist theories with "stronger bipartite entanglement" than quantum mechanics (QM), in sense that they have a larger CHSH value than Tsirelson's bound for QM, all such theories known tend to come at a cost, namely, they have strictly weaker bipartite measurements. Thus it has been conjectured that if one looks at scenarios where the correlations depend both on bipartite states and bipartite measurements, e.g. entanglement swapping, such theories cannot beat QM. However, in our recent work [LD, Ligthart, Gross, PRA, 2024], we constructed a General Probabilistic Theory (GPT) -- Oblate Stabilizer Theory (OST) -- that can both achieve a CHSH value of 4 (the mathematical maximum), and maintain this CHSH value after arbitrarily many rounds of entanglement swapping, effectively ruling out this conjecture.

Part 2 -- One particularly non-intuitive feature of OST (for those in the know) is the presence of a "spurious extra dimension" in the local theory: Even though the CHSH violation involves only a two-dimensional section of local state space, we failed to make the entanglement swapping property work without going to three dimensions. In ongoing work, we managed to identify the mechanism behind this phenomenon. To this end, we have introduced a notion of self-testing for GPTs, and, using this we have established a GPT version of the "no-pancake" theorem that says that there is no completely positive map that maps the Bloch sphere to a two-dimensional section. Further, under reasonable assumptions, we have also managed to establish the uniqueness of OST, and provide a prescription for the construction of GPTs capable of stable iterated entanglement swapping.

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Self-testing General Probabilistic Theories

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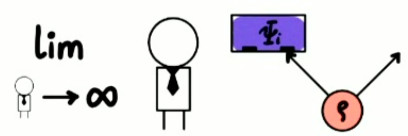


Contents

The Problem

- The CHSH experiment : 

- CHSH value : $\mathbb{E}(A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1) \leq 2 \Leftarrow$ Non-contextual.

A. Aspect, et al.

Measured a CHSH value – $\mathbb{E}(A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1) \approx 2\sqrt{2}$ – in their lab.

Nature is contextual!

The Problem

But, how contextual?

- Classical theories : $\text{CHSH} \leq 2$, Non-contextual.
- Quantum theory : $\text{CHSH} \leq 2\sqrt{2} \leftarrow$ Tsirelson

- Boxworld theory : $\text{CHSH} = 4$, without signalling.

Why $2\sqrt{2}$!?

General Probabilistic Theories

Mathematical formalism that generalizes the operational aspects of QM.

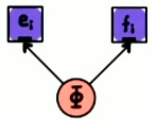
Idea: Model experiments as a two-step process

1. Preparation $\rightarrow$ States (e.g. QM : Density operators)
2. Measurement $\rightarrow$ Effects (e.g. QM : POVM elements)

Generalize by retaining only operational features

- Probabilistic mixtures = **Convexity** $\Rightarrow$ both states and effects live in convex sets.
- The states states must pair positively with effects $\rightarrow$ **Polar duality**.

Tension

Take for example CHSH: 

- Bigger CHSH value $\Rightarrow$ Stronger correlations.
- Stronger correlations $\Rightarrow$ Bigger state set.
- Bigger state set + positivity $\Rightarrow$ weaker measurements by polar duality.

Therefore, stronger CHSH correlations $\Rightarrow$ weaker measurement.

Boxworld : Strongest correlations (PR-boxes), but weakest measurements (separable measurements).

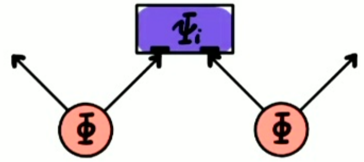
Quantum : Both strong correlations and strong measurements...

Conjecture

Weilenmann and Colbeck PRL + PRA (2020)

If we pick a scenario that probes the strength of both bipartite states and bipartite measurements, then you cannot beat QM.

Which scenario?

- Step 1: Perform *entanglement swapping* – 

Conjecture

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LD's Master Thesis

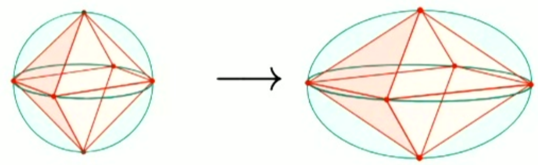
DG: Soo. . . can you prove that?

Counter-example

Oblate Stabilizer Theory

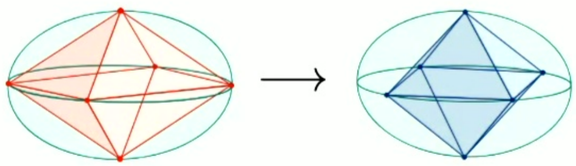
The construction of the local theory

- Step 1: Take qubit stabilizer polytope and *oblatify* its equatorial plane.



Outside Bloch sphere! Negative eigenvalues!
Boring solution: Shrink measurements down. Instead:

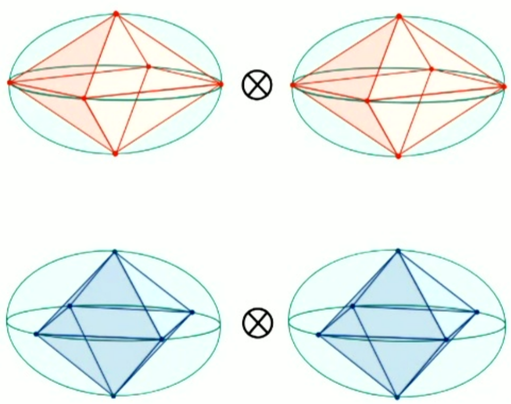
- Rotate the state polytope by $\pi/4$ about z-axis to get the measurement bases



Oblate Stabilizer Theory

The construction of the bipartite theory

Product states + effects



Entangled states + effects

Basic recipe: Bell states

$$\mathcal{R}^{\dagger}_{\pi/4} \otimes \mathbb{1} \; |\Phi^{+}\rangle\langle\Phi^{+}| \; \mathcal{R}_{\pi/4} \otimes \mathbb{1}$$

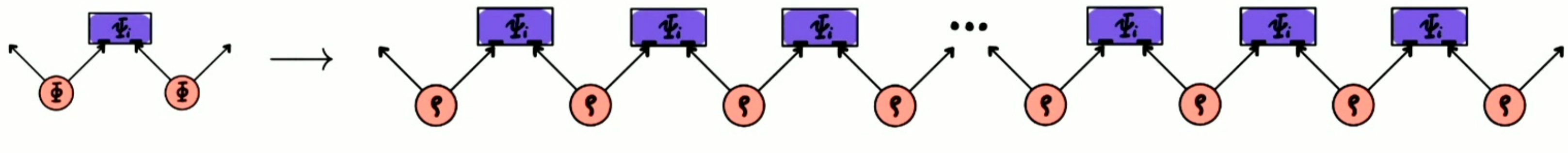
Idea: Gate teleportation.
s.t. $\mathcal{R}_{\pi/4}(\cdot)\mathcal{R}^{\dagger}_{\pi/4} \leftarrow \frac{\pi}{4}$ rot.

Local + Bipartite theory = CHSH 4 + iterated entanglement swapping.

Oblate Stabilizer Theory

In particular: OST beats the Iterated CHSH game with probability 1. What is the Iterated CHSH game? Glad you asked.

Entanglement swapping $\rightarrow$ Ennnnnnnnnnnnnnnnnnnnntanglement swapping



In short:

- Now you have N Bobs who all perform a bipartite measurement.
- Based on the outcome of the Bobs, Alice and Charlie choose which CHSH observable to violate.

Self-testing (Ongoing)

The story of the z-axis

The z-axis of OST is. . . weird.

- The local z-measurements do not participate in the CHSH experiment, and hence seemed to be superfluous.
- Nonetheless, we could not get rid of the z-direction from our theory without introducing negativity in the *bipartite* theory.
- Although OST did the job, at the time (foreshadowing) we did not fully understand its nuances.

GPT Self-testing

Idea: Generalize quantum self-testing.

Quantum self-testing

For some special correlations (e.g. CHSH $= 2\sqrt{2}$, Tilted CHSH, $\ldots$), the quantum model realizing the correlation is essentially unique.

Essentially unique $=$ (Unique model) $\otimes$ (Junk you can ignore), up to isomorphisms.

How to generalize?

- Need only a small set of states and effects to describe scenario
  e.g. CHSH $\rightarrow$

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How to generalize?

- Need only a small set of states and effects to describe scenario
  e.g. CHSH $\rightarrow \rho, \{e_i, \underset{\text{not } e_i}{\underset{\uparrow}{\neg e_i}}\}_{i=0,1}, \{f_i, \neg f_i\}_{i=0,1}.$ $\leftarrow$ "Assignment".
- Ignore everything outside of this assignment $\rightarrow$ Effective GPT.

GPT Self-testing

Result

There exist special correlations such that the *effective GPT* realizing them is unique.

GPTs satisfying –

(1) CHSH is violated.

(2) $\rho(e_i, \underline{1}) = \rho(f_i, \underline{1}) = \frac{1}{2}$.

(3) $\rho(A_0B_0) = \rho(A_0B_1) = \rho(A_1B_0) = \rho(-A_1B_1)$.

can be self-tested.

Lemma

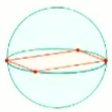
$\mathrm{CHSH} = 4 \Rightarrow$ (1), (2), and (3).

Lemma

QM + $\mathrm{CHSH} = 2\sqrt{2} \Rightarrow$ (1), (2), and (3).

The Unique Model

In these cases, the resulting effective GPT has the following properties:

1. The local states and effects of both Alice and Bob are characterized by a squares. (cf.  XY effects from QM)
2. All these squares are isomorphic with respect to ρ. (cf. Bell state from QM)
3. There is a representation of D_4 on each of these spaces. (more on next slide)
4. The state ρ is unique. (w.r.t the effective GPT)

The D_4 representation

CHSH has built-in symmetries realized by *relabelling* the scenario, i.e.,

1. We can relabel the settings of a measurement machine,
2. Relabel the outcomes w.r.t a given setting.

What Group?
$\mathbb{Z}_2 \wr \mathbb{Z}_2 \cong D_4$

This group is represented on effective GPT

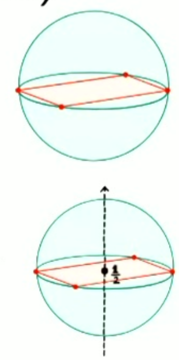
Representation (with character)

$$\chi = \underset{\substack{\uparrow \\ \text{trivial rep}}}{\chi_1} + \underset{\substack{\uparrow \\ \text{2D irrep}}}{\chi_5}$$

(cf. QM)

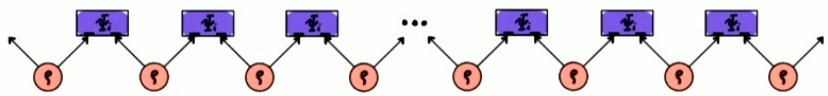
2D irrep →

trivial rep →

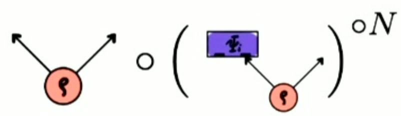


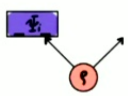
Shift in perspective

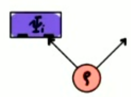
Recall Iterated CHSH

 $\to$ bipartite state

But we can also view this as



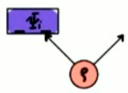
But, what is  ?

 : Effects $\to$ Effects.

This is the dynamics (cf. teleportation) that we get for free with any bipartite GPT!

The cart before the horse

Before I give you the details (the horse), I already give you the final result (the cart). On a high level:

- The maps  themselves are subject to *positivity constraints*.
- Due to D_4 rep, these constraints imply *positive character*.
- There are only *five* representations that satisfy all the constraints.

Through the lens of Dynamics

Let us take another look at the iterated CHSH game with our newfound tools:

We need one more crucial step before the big reveal

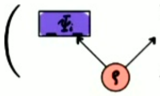
"Alice and Charlie pick CHSH observable based on Bobs' outcome."

$$\equiv$$

They relabel their setup to violate the correct inequality.

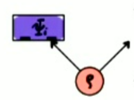
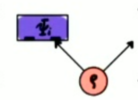
Through the lens of Dynamics

Insight: The iterated CHSH game is a collection of GPT self-testing scenarios,

$\{e_i, \neg e_i\}_{i=0,1},\ \{f_j, \neg f_j\}_{j=0,1},\ \rho \circ \Big($  $\Big)^{\circ N} \circ \kappa,$

one for each outcome combination, for every possible number $N \in \mathbb{N}$ of Bobs.

But, by GPT self-testing, ρ is unique! Therefore, on the effective GPT

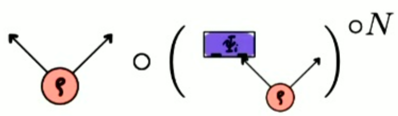
$\rho \circ \Big($  $\Big)^{\circ N} \circ \kappa = \rho \quad \Rightarrow \quad \Big($  $\Big)^{\circ N} \circ \kappa = \mathrm{id}.$

The story of the z-axis (revisited)

Okay... But it is still true that only the "equatorial plane" participates in the CHSH test. So now lets see what happens if we get rid of all "extra dimensions".

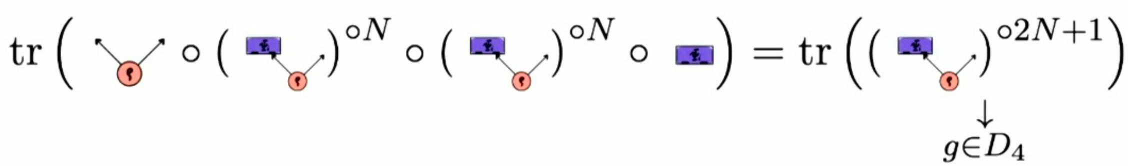
In this case the entire GPT lives on the span of the assignment, and the results of the previous slides hold without any restriction.

Now, since

1.  must be valid states,

The story of the z-axis (revisited)

What is their pairing look like? Up to some technical details. . .



$$= (\chi_1 + \chi_5)(g).$$

And. . .

χ_ϑ	3	1	−1	1	1

$\rightarrow$ It is negative!

No pancakes!

Theorem (No pancakes)*

There are no GPTs with local dimension ≤ 3 that both satisfy the GPT self-testing conditions and also retain their CHSH value in the iterated CHSH game.

* If you demand *local tomography!*

Indeed, Local tomography $\Rightarrow$ No pancakes. But, as we all know

- One man's *Modus Ponens* :

$$\text{Local tomography} \vDash \text{No pancakes.}$$

- Is another man's *Modus Tollens* :

$$\text{Pancakes} \vDash \text{No local tomography!}$$

Theorem (No/All pancakes)

The structure of iterated entanglement swapping forces upon us extra *non-local* dimensions regardless of whether they can be *observed locally*.

All the solutions

Result

The structure induced by teleportation dynamics is strong enough to be able to extend the self-testing formalism to iterated CHSH.

Skipping the technical details,

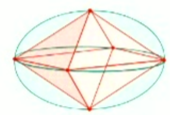
$$\text{Find iterating GPTs} \underset{\substack{\uparrow \\ \text{minimality}}}{\equiv} \text{Find reps of } D_4 \text{ with positive character}^*$$

* Satisfying

- Trivial rep occurs only once (needed to properly normalize measurements).

All the solutions

The solutions are...

- $\dim = 4: \quad \chi_1 + \chi_2 + \chi_5 \xrightarrow{\text{CHSH}=4}$  OST!

  $\chi_1 + \chi_3 + \chi_5$
  $\chi_1 + \chi_4 + \chi_5$
  Q: What is max CHSH if you demand transitivity? A: $2\sqrt{2} \leftarrow$ Master Thesis finally done!

- $\dim = 6: \ \chi_1 + \chi_2 + \chi_3 + \chi_4 + \chi_5$

- $\dim = 8: \ \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5$ [Regular rep]

No other solutions!

$\mathscr{K}_{D_4}$	$\mathscr{K}_{\mathbb{1}}$	$\mathscr{K}_{\xi}$	$\mathscr{K}_{\xi^2}$	$\mathscr{K}_{\eta}$	$\mathscr{K}_{\xi\eta}$	irrep
$\lvert\mathscr{K}_{D_4}\rvert$	1	2	1	2	2	
$\lvert\mathscr{C}_{D_4}\rvert$	8	4	8	4	4	
χ_1	1	1	1	1	1	trivial
χ_2	1	1	1	−1	−1	$D_4/\mathbb{Z}_4$
χ_3	1	−1	1	1	−1	$(-1)^{\pi(g)}$
χ_4	1	−1	1	−1	1	$D_4/\mathbb{Z}_4 \otimes (-1)^{\pi(g)}$
χ_5	2	0	−2	0	0	planar rotations

Outlook

We accomplished what we set out to do:

- Introduced framework for self-testing of GPTs.
- Provided justification for the existence of locally "inert" extra dimensions.
- Showed that Oblate Stabilizer Theory is the minimal solution.

But we have also did more:

- Unconvinced ourselves of local tomography.
- Showed that quantum is indeed optimal $(\dim = 4)$ if we impose transitivity.
- Classified all the iterating, self-testable solutions.

And of course, there are plenty more questions:

- What on earth is the GPT that comes from the group algebra!?
- Is there a transitive solution that can beat quantum?
- Is there a scenario that can directly probe transitivity? ...

Thank you!

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