Title: Flatness and spikes in Ponzano-Regge

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Abstract:

The original spinfoam amplitude, Ponzano-Regge, has two properties in seeming contradiction: (1.) It can be written as an integral of a product of Dirac delta functions imposing that holonomies be exactly flat, and (2.) In its original sum-over-spins form, its leading order large spin asymptotics consist in Regge calculus, modified to include an additional local discrete orientation variable for each tetrahedron, which, when fixed inhomogeneously, leads to critical point equations for the edge lengths which do not necessarily imply flatness, but allow spikes. Of course, this apparent contradiction between flatness and spikes appears only for triangulations with bubbles, for which both of these formulations of the model are divergent and ill-defined anyway, and this may be the resolution of the paradox. However, we explore the possibility of another resolution of this paradox which may also have relevance for the semiclassical regime of 4D spinfoams, in which a similar sum over local orientations appears.

FLATNESS AND SPIKES IN PONZANO-REGGE

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QUANTUM GRAVITY SEMINAR

OUTLINE

The point: To explore a tension to see what can be learned

Ponzano-Regge:

a. In connection formulation: Manifest flatness

b. Large spin asymptotics: Locally oriented Regge!

c. Equations of motion for fixed local orientations: Non-flatness!

Possible resolutions for flatness vs non-flatness?

a. Contradiction seems to arise only when model diverges and so is ill-defined anyway. So, strictly speaking, there is no contradiction. Satisfactory?

b. Perhaps more careful handling of discrete nature of spins would avoid non-flatness for large spins.

c. Is connection possibly sensitive to orientation, so that connection is flat even though geometry is not?

d. Critical point equation from varying local orientations not yet considered. Continuum theory suggests this might imply homogeneity of orientations, imposing flatness also for large spins.

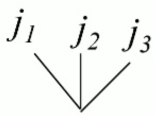
4D Spinfoams: If (d.) is the resolution, might something similar happen in 4D spinfoams? 4D Large spin asymptotics also gives locally oriented Regge, and tetrad gravity EOM are equivalent to GR only with homogeneity of orientation!!

From spin to connection formulation: Manifest flatness

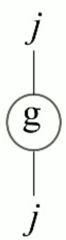
Diagrammatic notation elements:

$\rho_j(g) : V_j \to V_j$ denotes spin j irrep of $SU(2)$. $j \in \mathbb{N}/2$, $g \in SU(2)$.

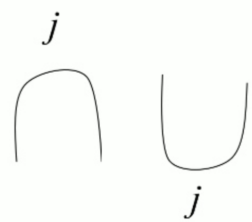
$$\dim\left(V_{j_1} \otimes V_{j_2} \otimes V_{j_3}\right) = \begin{cases} 1 & \text{if } j_1 + j_2 > j_3 \text{ \& cyclic and } j_1 + j_2 + j_3 \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$



denotes specific element of $\mathrm{Inv}\left(V_{j_1} \otimes V_{j_2} \otimes V_{j_3}\right)$ with phase convention chosen



denotes $\rho_j(g) : V_j \mapsto V_j$



bilinear form 'ϵ' on V_j, and its inverse, used to contract, raise and lower indices.

In spinorial realization $V_j = \{\psi^{A_1 \cdots A_{2j}} = \psi^{(A_1 \cdots A_{2j})}\}$, $\epsilon_{(A_1 \cdot A_{2j})(B_1 \cdots B_{2j})} = \epsilon_{A_1(B_1}\epsilon_{|A_1|B_1} \cdots \epsilon_{|A_{2j}|B_{2j})}$

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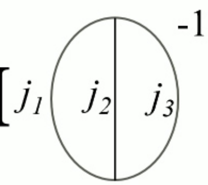
From spin to connection formulation: Manifest flatness

Given a 3D triangulation Δ with edges ℓ, triangles t, and tetrahedra σ,

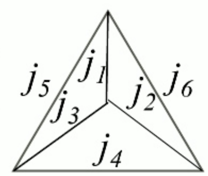
$$W_{PR} = \sum_{\{j_\ell\}}{}' \prod_\ell (-1)^{2j_\ell}(2j_\ell+1) \prod_t (-1)^{j_1+j_2+j_3} \prod_\sigma \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

N.B. $2j_\ell,\ j_1+j_2+j_3 \in \mathbb{N}$, so signs are well-defined!

$$= \sum_{\{j_\ell\}}{}' \prod_\ell (-1)^{2j_\ell}(2j_\ell+1) \prod_t$$



$$\prod_\sigma$$

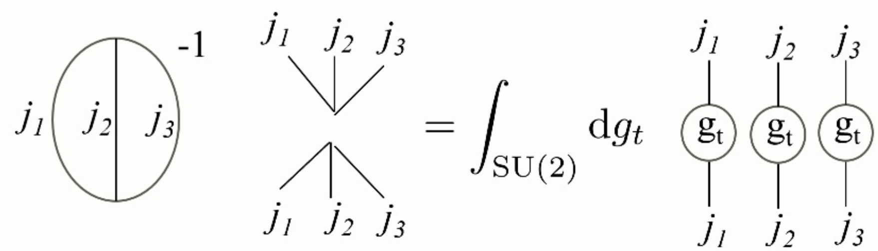


where $\{j_\ell\}' := \{j_\ell\}_{\ell\in \mathrm{int}\Delta} \subset \mathbb{N}/2$.

Assume, for simplicity, no boundary.

Using the following identity once at each triangle

- gives an integral over a $g_t \in$SU(2) at each triangle t, &
- reduces the graph to a product of one loop per edge ℓ, each containing the g_t's around that edge.



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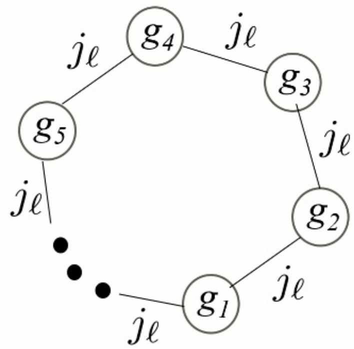
From spin to connection formulation: Manifest flatness

$$W_{PR} = \sum_{\{j_\ell\}} \int \left(\prod_t \mathrm{d}g_t \right) \prod_\ell (2j_\ell + 1)$$



$T_\ell := \{t | \ell \in t\}$

$$= \int \left(\prod_t \mathrm{d}g_t \right) \prod_\ell \sum_j (2j+1) \mathrm{Tr}_j(h_\ell) = \int \left(\prod_t \mathrm{d}g_t \right) \prod_\ell \delta(h_\ell)$$

$$\text{“} = \int \mathcal{D}\omega \delta(F(\omega)) = \int \mathcal{D}\omega \mathcal{D}e \exp\left(i \int e \wedge F(\omega) \right) = \int \mathcal{D}\omega \mathcal{D}e \exp\left(iS[e,\omega] \right) \text{”}$$

also shows first order formalism underlying Ponzano-Regge.

Large spin asymptotics: Locally oriented Regge!

Setting $j_\ell = \lambda j_\ell^o$ ($\in \mathbb{N}/2$) for j_ℓ^o fixed.

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \underset{\lambda\to\infty}{\sim} \frac{1}{\sqrt{3\pi \mathrm{V}}} \cos\left(\sum_{a=1}^{6} j_a \Theta_a + \frac{\pi}{4}\right)$$

[Ponzano and Regge (1968);
Dowdall, Gomes, and Hellmann (2009);
Christodoulou, Långvik, Riello, Röken, and Rovelli (2012)]

where V is the volume of the tetrahedron with edge lengths λj_a and
Θ_a is the *external* dihedral angle at edge a (angle between the normals to the two triangles at a).

$$W_{PR} \sim \sum_{\{j_\ell\}'} \prod_\ell (-1)^{2j_\ell}(2j_\ell+1) \prod_t (-1)^{\sum_{\ell\in t} j_\ell} \prod_\sigma \sum_{\mu_\sigma=\pm 1} \frac{1}{\sqrt{12\pi \mathrm{V}(\sigma)}} \exp i\mu_\sigma \left(\sum_{\ell\in\sigma} j_\ell \Theta_\ell(\sigma) + \frac{\pi}{4}\right)$$

$$= \sum_{\{j_\ell\}'} \sum_{\{\mu_\sigma\}} \prod_\ell (e^{i\pi})^{2j_\ell}(2j_\ell+1) \prod_t (e^{-i\pi})^{\sum_{\ell\in t} j_\ell} \prod_\sigma \frac{1}{\sqrt{12\pi \mathrm{V}(\sigma)}} \exp i\mu_\sigma \left(\sum_{\ell\in\sigma} j_\ell (\pi - \theta_\ell(\sigma)) + \frac{\pi}{4}\right)$$

where $\theta_\ell(\sigma)$ is the *internal* dihedral angle in σ at ℓ (angle inside σ between the planes of the two triangles at ℓ).

(This choice to express the -1's as exponentials is a generalization of that in Chistodoulou et al. and agrees for their triangulation.)

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Large spin asymptotics: Locally oriented Regge!

$$W_{PR} \sim \sum_{\{j_\ell\}'} \sum_{\{\mu_\sigma\}} \left(\prod_\sigma \frac{1}{\sqrt{12\pi \mathrm{V}(\sigma)}} \right) \exp i \left(\sum_\ell j_\ell \left(\left(2 - |T_\ell| + \sum_{\sigma \in \Sigma_\ell} \mu_\sigma \right) \pi - \sum_{\sigma \in \Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) \right) + \frac{\pi}{4} \sum_\sigma \mu_\sigma \right)$$

$$=: \sum_{\{j_\ell\}'} \sum_{\{\mu_\sigma\}} \left(\prod_\sigma \frac{1}{\sqrt{12\pi \mathrm{V}(\sigma)}} \right) \exp i \left(S_{R,\mu} + \frac{\pi}{4} \sum_\sigma \mu_\sigma \right)$$

where T_ℓ and Σ_ℓ respectively denote the set of triangles and tetrahedra containing ℓ, and

$$S_{R,\mu} := \sum_\ell j_\ell \left(\left(2 - |T_\ell| + \sum_{\sigma \in \Sigma_\ell} \mu_\sigma \right) \pi - \sum_{\sigma \in \Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) \right)$$

- Since Ponzano-Regge has an $SU(2)$ connection formulation, it's underlying structure is that of a first order theory with triad e and connection.
- **The sign μ_σ appearing here is the discrete analogue of sgn**$(\det(e))$**.**

Large spin asymptotics: Locally oriented Regge!

Using the fact that, for $\ell \in \mathrm{int}\Delta$, $|T_\ell| = |\Sigma_\ell|$, and, for $\ell \in \partial\Delta$, $|T_\ell| = |\Sigma_\ell| + 1$, for $\mu_\sigma \equiv +1$ this 'locally oriented Regge action' $S_{R,\mu}$ becomes

$$\boxed{S_{R,+1} = \sum_{\ell \in \mathrm{int}\Delta} j_\ell \left(2\pi - \sum_{\sigma \in \Sigma_\ell} \Theta_\ell(\sigma) \right) + \sum_{\ell \in \partial\Delta} j_\ell \left(\pi - \sum_{\sigma \in \Sigma_\ell} \Theta_\ell(\sigma) \right) = S_{\mathrm{Regge}}}$$

Exactly the Regge action, including correct boundary terms, for a general triangulation!

The choice in writing signs as exponentials

In foregoing derivation,

- We made a choice to write $(-1)^{2j_\ell} = (e^{i\pi})^{2j_\ell}$ for each ℓ and $(-1)^{\sum_{\ell \in t} j_\ell} = (e^{-i\pi})^{\sum_{\ell \in t} j_\ell}$ for each t.
- If we had made the reverse choice $(-1)^{2j_\ell} = (e^{-i\pi})^{2j_\ell}$ and $(-1)^{\sum_{\ell \in t} j_\ell} = (e^{i\pi})^{\sum_{\ell \in t} j_\ell}$, then we would be led to an alternative action $\tilde{S}_{R,\mu}$ such that $\tilde{S}_{R,-} = -S_{\text{Regge}}$.
- Note this choice is just a choice of how to write the Ponzano-Regge amplitude. Thus, it cannot affect the asymptotics of Ponzano-Regge. Ponzano-Regge is a well-defined model and so has only one asymptotics!

However,

- we will next consider the critical point equations from varying the j_ℓ's, which makes sense only if we first extend the action to continuous values of the j_ℓ's, beyond half-integers.
- This extension does depend on the choice of how the signs are written as exponentials.
- Hence, the resulting actions and critical point equations will depend on this choice.
- Seems to contradict the fact that Ponzano-Regge, and hence its asymptotics, cannot depend on this choice.
- Nevertheless, following the literature, we assume that the resulting asymptotics tell us something heuristic about Ponzano-Regge.

Equations of motion for fixed local orientations: Non-flatness!

$$S_{R,\mu} = \sum_{\ell} j_\ell \left(\left(2 - |T_\ell| + \sum_{\sigma \in \Sigma_\ell} \mu_\sigma \right) \pi - \sum_{\sigma \in \Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) \right)$$

Critical point equation from varying and internal j_ℓ:
Recalling that Regge showed that the term from variation of the deficit angle vanishes, and using that $|T_\ell| = |\Sigma_\ell|$ for internal ℓ, we have

$$\sum_{\sigma \in \Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) = \left(2 - |T_\ell| + \sum_{\sigma \in \Sigma_\ell} \mu_\sigma \right) \pi \quad \Rightarrow \quad \boxed{\sum_{\sigma \in \Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) = \left(2 + \sum_{\sigma \in \Sigma_\ell} (1 - \mu_\sigma) \right) \pi}$$

giving flatness, $\sum_{\sigma \in \Sigma_\ell} \theta_\ell(\sigma) = 2\pi$, **only** for $\mu \equiv 1$.

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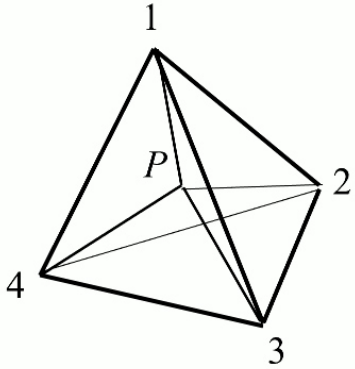
Equations of motion for fixed local orientations: Non-flatness!

Simplest triangulation with spike: 4-1 Pachner move ($^4\tau$ triangulation):

vertices:	4 boundary $a = 1, 2, 3, 4$ 1 internal P
tetrahedra:	4, σ_a, labeled by the vertex a not contained.
edges:	6 boundary ℓ_{ab} 4 internal $\ell_a := \ell_{aP}$
triangles:	4 boundary $t_a \in \sigma_a$ 6 internal $t_{ab} = \sigma_a \cap \sigma_b$



- $|T_{\ell_a}| = |T_{\ell_{ab}}| = 3$
- $\Sigma_{\ell_a} = \{\sigma_b\}_{b \neq a} \quad \Rightarrow \quad |\Sigma_{\ell_a}| = 3$

Critical point equations from varying each internal spin j_ℓ:

$$\sum_{\sigma \in \Sigma_\ell} \mu_\sigma \theta_\ell(\sigma) = \left(2 - |T_\ell| + \sum_{\sigma \in \Sigma_\ell} \mu_\sigma\right)\pi = \left(\sum_{\sigma \in \Sigma_\ell} \mu_\sigma - 1\right)\pi$$

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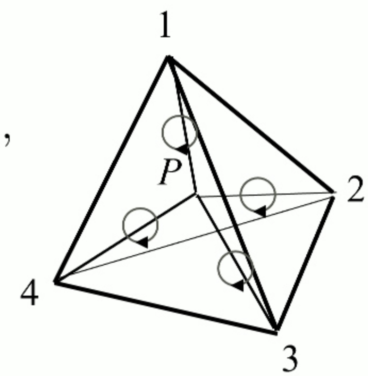
Equations of motion for fixed local orientations: Non-flatness!

For $\mu_1 = \mu_2 = \mu_3 = \mu_4 = +1$:

$$2\pi = \theta_{\ell_1}(\sigma_2) + \theta_{\ell_1}(\sigma_3) + \theta_{\ell_1}(\sigma_4)$$
$$2\pi = \theta_{\ell_2}(\sigma_1) + \theta_{\ell_2}(\sigma_3) + \theta_{\ell_2}(\sigma_4), \text{ etc.}$$

Flatness around all 4 internal ℓ_a, as expected.

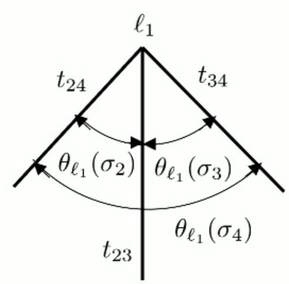


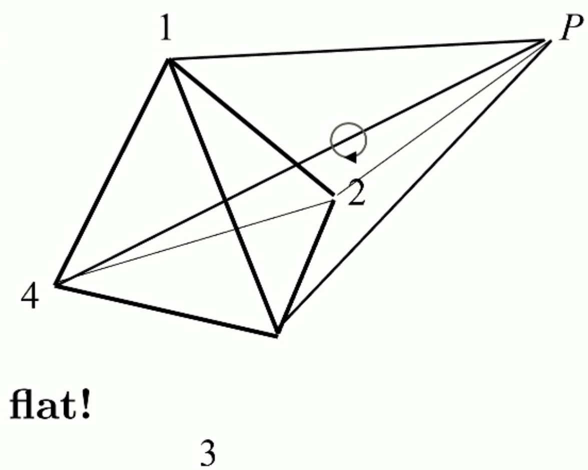
For $\mu_1 = \mu_2 = \mu_3 = +1$, $\mu_4 = -1$:

$$0 = \theta_{\ell_1}(\sigma_2) + \theta_{\ell_1}(\sigma_3) - \theta_{\ell_1}(\sigma_4)$$
$$0 = \theta_{\ell_2}(\sigma_1) + \theta_{\ell_2}(\sigma_3) - \theta_{\ell_2}(\sigma_4)$$
$$0 = \theta_{\ell_3}(\sigma_1) + \theta_{\ell_3}(\sigma_2) - \theta_{\ell_3}(\sigma_4)$$
$$2\pi = \theta_{\ell_4}(\sigma_1) + \theta_{\ell_4}(\sigma_2) - \theta_{\ell_4}(\sigma_3)$$

Flatness around ℓ_4,
but not around ℓ_1, ℓ_2, ℓ_3!
Spike!

E.g., in plane $\perp \ell_1$:



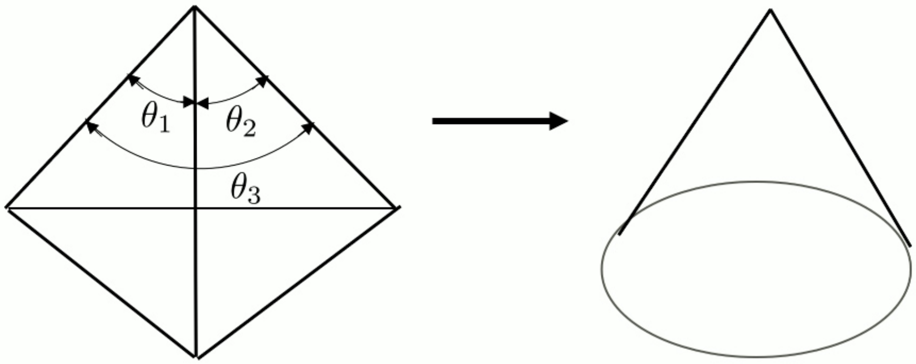


Not flat!

Equations of motion for fixed local orientations: Non-flatness!

2D analogue:

$\theta_1 + \theta_2 = \theta_3$



Conical singularity - not flat!

Key point: If interior dihedral angles around a hinge **don't sum to** 2π,
then the geometry in a neighborhood of the hinge is not embeddable into $\mathbb{R}^n$ and so is **not flat!**

Both exact flatness and arbitrarily curved spikes? Contradiction? Resolution?

1. **Spikes generally correspond to bubbles for which model is ill-defined**

    - **In connection formulation**, Redundant δ's: **Divergence**
    - **In spin formulation**, unbounded sums over internal spins in spikes: **Divergence**

    The reasons for the divergence are opposite in the two formulations: Too much flatness vs. spikes! Strange!

    - However, **because both formulations are ill-defined in this case, there is no strict mathematical contradiction.**
    - Does this satisfy us?

2. **More care about discrete nature of spins**. We already saw one contradiction from treating the spins as continuous — the dependence on exponential expression of sign factors. **Might greater care about discrete nature of spins somehow resolve this flatness vs. spikes tension?**

Both exact flatness and arbitrarily curved spikes? Contradiction? Resolution?

3. **Is the connection at spikes flat, even if geometry is not?**

- Geometry (uniquely determined by the j_ℓ's) is flat at ℓ if and only if $\sum_{\sigma\in\Sigma_\ell} \theta_\ell(\sigma) = 2\pi$.
- But the equation of motion from $S_{R,\mu}$ is $\sum_{\sigma\in\Sigma_\ell} \theta_\ell(\sigma) = \left(2 - \sum_{\sigma\in\Sigma_\ell} (\mu_\sigma - 1)\right)\pi$.
- Could this somehow be the condition for flatness for the **spin-connection** determined by the **triad** e, which knows about orientation?
- **Is the spin-connection even sensitive to the orientation of the triad?** Consider $\tilde{e}^i_a = \mu e^i_a$. Then

$$\omega(\tilde{e})^{ij}_a = 2\tilde{e}^{b[i}\partial_{[a}\tilde{e}^{j]}_{b]} + \tilde{e}_{ak}\tilde{e}^{bi}\tilde{e}^{dj}\partial_{[d}\tilde{e}^k_{b]} = \ldots$$
$$= 2\mu(\partial_b\mu)e^{b[j}e^{i]}_a + \omega(e)^{ij}_a$$

In a coordinate patch in a neighborhood of a sign change, choose coordinates (x, y, z) such that $\mu = \text{sgn}(x)$. Then $\mu\partial_b\mu = 2\text{sgn}(x)\delta(x)\partial_b x = 0$ **if** we regularize $\text{sgn}(x)$ symmetrically. Then $\omega(\mu e) = \omega(e)$, **so it seems ω is not sensitive to μ.**

Both exact flatness and arbitrarily curved spikes? Contradiction? Resolution?

4. **Variation of the local orientation variables?** We have considered critical point equations from variation of the spins. **But we have not considered varying μ_σ.**

   - Is essentially discrete - no continuum approximation possible. Can stationary phase theorem be extended to handle this?
   - **Consideration of continuum triad gravity** suggests the corresponding equation of motion minimizing the action imposes **homogeneity of the μ_σ.**
   - **Would lead to flatness also in the spin formulation**, bringing it in line
     - not only with connection formulation,
     - **but also classical 3D gravity, where the equation of motion is flatness.**

Triad gravity

First order formulation

$$S[e,\omega] := \int e \wedge F(\omega)$$

$$\delta S[e,\omega] = \int (\delta e \wedge F(\omega) + e \wedge d_\omega \delta\omega) = \int (\delta e \wedge F(\omega) + d_\omega e \wedge \delta\omega)$$

$\Rightarrow$ E.O.M.

- $d_\omega e = 0 \quad \Rightarrow \quad \omega = \omega(e)$
- $F(\omega) = 0$ **Flatness**

Second order formulation

$$S[e] := S[e,\omega(e)] = \int e \wedge F(\omega(e)) = \int \mu(x) R[g_{ab}] \sqrt{\det g(x)} d^3x$$

where $\mu(x) := \mathrm{sgn}(\det(e(x)))$ and $g_{ab}(x) := e^i_a(x) e_{bi}(x)$.

- **Varying** $g_{ab}(x)$: E.O.M. says g_{ab} is flat except possibly where $\mu(x)$ changes sign.
- Thus, if $\mu(x)$ is inhomogeneous, S is not necessarily zero on-shell. If it is homogeneous, S is zero on-shell.
  - **Homogeneous $\mu(x)$ minimizes the action.**
  - Also **recovers consistency with first order formulation**, yielding **flatness everywhere.**

Variation of μ and Homogeneity of orientations?

- $\mu(x)$ is discrete, so stationary phase theorem doesn't apply to its variation.
- **Can we nevertheless somehow conclude that minimization of the action by homogeneous $\mu(x)$ implies inhomogeneous $\mu(x)$ are suppressed in the second order path integral?**
- It seems it must be so, in order to have consistency with first order path integral.

Homogeneous $\mu(x)$ is also necessary to obtain geometric flatness, which is the Einstein equation for 3D.

Relevance for 4D spinfoams

- In 4D, asymptotics of spin-foams also yields locally oriented Regge calculus.
- There, too, to obtain correct Einstein equations for the geometry, the orientation must be homogeneous.
- If we can make precise an argument that inhomogeneous orientations are suppressed in Ponzano-Regge, maybe we could make a similar arguent in 4D?

THANK YOU

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