Title: 2-dimensional topological field theories via the genus filtration

Speakers: Jan Steinebrunner

Collection/Series: Mathematical Physics

Subject: Mathematical physics

Date: May 22, 2025 - 11:00 AM

URL: https://pirsa.org/25050034

Abstract:

By a folk theorem (non-extended) 2-dimensional TFTs valued in the category of vector spaces are equivalent to commutative Frobenius algebras. Upgrading the bordism category to an (infinity, 1)-category whose 2-morphism are diffeomorphisms, one can study 2D TFTs valued in higher categories, leading for example to (derived) modular functors and cohomological field theories.

I will explain how to describe such more general (non-extended) 2D TFTs as algebras over the modular infinity-operad of surfaces. In genus 0 this yields an E_2^{SO}-Frobenius algebra and I will outline an obstruction theory for inductively extending such algebras to higher genus. Specialising to invertible TFTs, this amounts to a genus filtration of the classifying space of the bordism category and hence the Madsen--Tillmann spectrum MTSO_2. The aforementioned obstruction theory identifies the associated graded in terms of curve complexes and thereby yields a spectral sequence starting with the unstable and converging to the stable cohomology of mapping class groups.

2-dimensional TFTs via the genus filtration

Jan Steinebrunner

parts are joint work with Shaul Barkan

① Functorial topological field theories

Study non-extended 2D TFTs

$$F : (\mathrm{Bord}_2, \sqcup) \longrightarrow (\mathrm{Vect}_{\mathbb{C}}, \otimes)$$

$$M \xrightarrow{W} N \quad \longmapsto \quad F(M) \xrightarrow{F(W)} F(N)$$

Classical theorem (Abrams)

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2, \mathrm{Vect}_{\mathbb{C}}) \simeq \left\{ \begin{array}{l} \text{commutative Frobenius algebras} \\ (A, \tau) \quad \tau : A \to \mathbb{C} \text{ non-deg. trace} \end{array} \right\}$$

① Functorial topological field theories

Study non-extended 2D TFTs

surface category

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$$\longmapsto \quad F(M) \xrightarrow{F(W)} F(N)$$

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$M \xrightarrow{W} N \quad \longmapsto \quad F(M) \xrightarrow{F(W)} F(N)$

time evolution operators

state space

$F(M \sqcup N) \cong F(M) \otimes F(N)$

Classical theorem (Abrams)

$A = F(S^1)$

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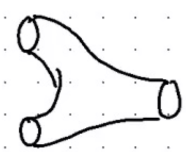
μ $\quad A \otimes A \longrightarrow A$

$A \longrightarrow \mathbb{C}$

Categorify: $\mathrm{Vect}_{\mathbb{C}} \rightsquigarrow \mathrm{LinCat}_{\mathbb{C}}$ = 2-category of linear categories



$\rightsquigarrow \quad \underset{\substack{\parallel \\ F(S^1)}}{A} \boxtimes A \longrightarrow A$ braided monoidal structure

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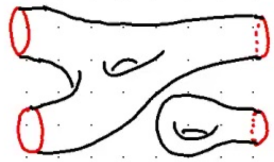

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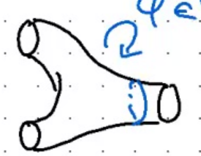

$\varphi \in \mathrm{Diff}(\Sigma)$ $\rightsquigarrow$

$$\underset{\overset{\|}{F(S^1)}}{A} \boxtimes A \xrightarrow{F(\varphi)} A$$

ribbon braided monoidal structure

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② Higher/derived targets

Suppose $F: \mathrm{Bord}_2 \longrightarrow \mathrm{LinCat}_{\mathbb{C}}$ is a modular functor
where $A = F(S^1)$ is not semi-simple

} $\exists$ Examples from VOA

$\hookrightarrow$ can form interesting derived category $\mathcal{D}(A) = \mathrm{Ch}(A)[\simeq_{q.i}^{-1}]$

? $\rightsquigarrow$ "derived modular functor"

$$\mathrm{Bord}_2 \longrightarrow \mathrm{dgCat} \text{ or stable } \infty\text{-cat}/_{\mathbb{C}}$$

Here Bord_2 is an $(\infty,1)$-category:

objects: closed 1-mfds

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objects: closed 1-mfds

morphisms: 2-mfds

invertible
2-morphisms: diffeomorphisms

3-morphisms: isotopies

$\vdots$

Witten, Kontsevich-Manin, ...

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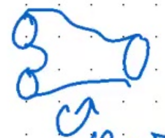
⋮



morphism

$\varphi \in \mathrm{Diff}(\Sigma)$ 2-morphism

Knows about moduli spaces

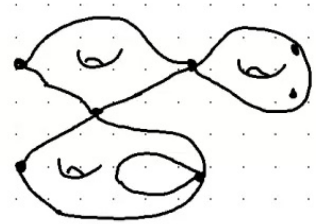
$$\mathcal{M}^{fr}_{g,n} \simeq B\mathrm{Diff}_\partial(\Sigma_{g,n})$$

Witten, Kontsevich-Manin, ...

Cohomological field theories fit into this framework

$$\mathrm{Bord}_2 \rightsquigarrow \overline{\mathrm{Bord}_2}$$

$$\mathcal{M}^{fr}_{g,k} \rightsquigarrow \overline{\mathcal{M}}_{g,k}$$



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isms: isotopies

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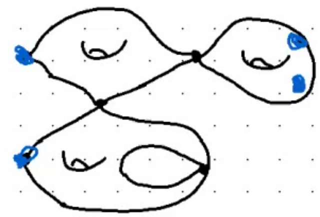

$$\mathrm{CohFT} \simeq \mathrm{Fun}^{\otimes}(\overline{\mathrm{Bord}}_2, \mathrm{Ch}_{\mathbb{Q}})$$

Thm (Deshmukh '22) " $\overline{\mathrm{Bord}}_2 \sim \mathrm{Bord}_2 \cup \{\ \}$ "
$\{\ \}$

c) Invertible TFTs (later)

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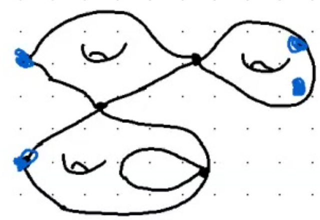

$$\mathrm{CohFT} \simeq \mathrm{Fun}^{\otimes}(\overline{\mathrm{Bord}_2}, \mathrm{Ch}_{\mathbb{Q}})$$ Kontsevich '03

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c) Invertible TFTs (later)

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③ Modular ∞-operads   j/w Shaul Barkan, generalizing Getzler-Kapranov, Costello, Hackney-Robertson-Yau

Sketch defn A modular ∞-operad $\mathcal{O}$ consists of

- a space of colours $\mathrm{col}(\mathcal{O}) \circlearrowleft$ + involution
- for $k \geq 0$ a space of k-ary operations $\mathcal{O}(k) \circlearrowleft \Sigma_k$ with a map $\mathcal{O}(k) \longrightarrow \mathrm{col}(\mathcal{O}) \times \dots \times \mathrm{col}(\mathcal{O})$
- gluing maps $\mathcal{O}(k) \underset{\mathrm{col}\,\mathcal{O}}{\times} \mathcal{O}(\ell) \longrightarrow \mathcal{O}(k+\ell-2)$
- self-gluing maps $\mathcal{O}(k) \underset{\mathrm{col}\,\mathcal{O} \times \mathrm{col}\,\mathcal{O}}{\times} \mathrm{col}\,\mathcal{O} \longrightarrow \mathcal{O}(k-2)$

+ units & higher coherence assuring associativity.

↳ allow evaluation of every suitably labeled graph.

Formal definition

Example (geometric) Mfd_d manifold modular operad

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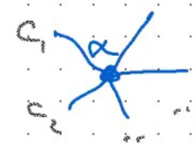

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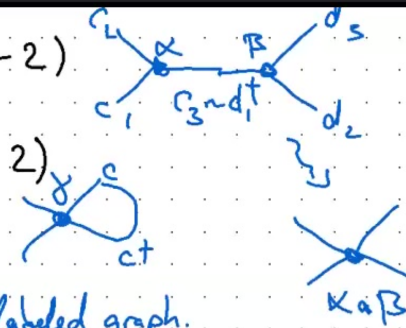
Formal definition

Example (geometric) Mfd_d manifold modular operad gluing maps $\mathcal{O}(k) \times_{\mathrm{col}\,\mathcal{O}} \mathcal{O}(\ell) \longrightarrow \mathcal{O}(k+\ell-2)$



- self-gluing maps $\mathcal{O}(k) \times_{\mathrm{col}\,\mathcal{O} \times \mathrm{col}\,\mathcal{O}} \mathrm{col}\,\mathcal{O} \longrightarrow \mathcal{O}(k-2)$

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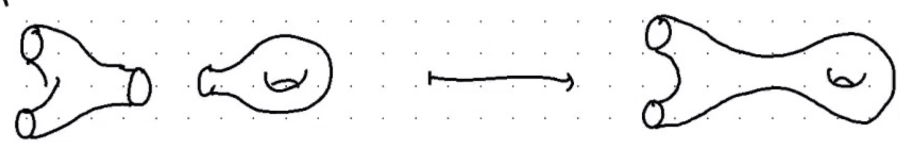
Formal definition $\mathrm{ModOp} \subset \mathrm{Fun}(\mathrm{Gr}, \mathcal{S})$

Example (geometric) Mfd_d manifold modular operad

colours: connected closed $(d-1)$-mfds

k-ary operations: connected compact d-mfds with k ordered boundary components



$$\mathrm{Mfd}_2(3) \times_{\mathrm{col}\,\mathrm{Mfd}_2} \mathrm{Mfd}_2(1) \longrightarrow \mathrm{Mfd}_2(2)$$

Note: Mfd_2 has a connected space of colours

$\mathrm{col}\,\mathrm{Mfd}_2 =$

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$\kappa \circ \beta$

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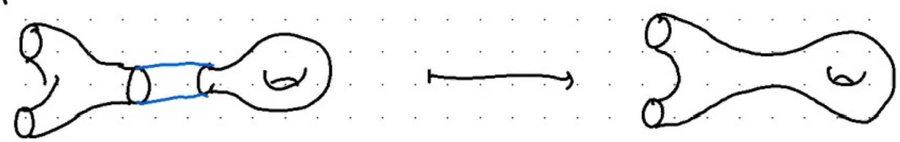

$\mathrm{Mfd}_2(3) \underset{\mathrm{col}\,\mathrm{Mfd}_2}{\times} \mathrm{Mfd}_2(1) \longrightarrow \mathrm{Mfd}_2(2)$

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$\mathrm{col}\,\mathrm{Mfd}_2$ = moduli of circles $\simeq BSO(2$

Can treat it as 1-coloured modular opera

Example (algebraic):

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Mfd_2 has a connected space of colours

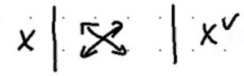
col Mfd_2 = moduli of circles = $BSO(2)^{\circlearrowleft} +$

Can treat it as 1-coloured modular opera

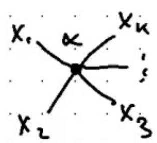
<u>Example</u> (algebraic):

$\mathcal{C}$ sym. monoidal ∞-cat $\rightsquigarrow$ $\mathcal{U}(\mathcal{C})$ modular operad

colours: dualizable objects of $\mathcal{C}$



k-ary operations:



$\rightsquigarrow$ $\alpha : x_1 \otimes \dots \otimes x_k \longrightarrow \mathbb{1}$

contract operations

$\hookrightarrow$

$\hookrightarrow$

Can express TFT in terms of modular ∞-operads

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Mfd_2 has a connected space of colours

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Example (algebraic):

$\mathcal{C}$ sym. monoidal ∞-cat $\rightsquigarrow$ $\mathcal{U}(\mathcal{C})$ modular operad

colours: dualizable objects of $\mathcal{C}$ $\quad x \mid \;\; \mid x^\vee$

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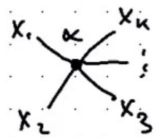

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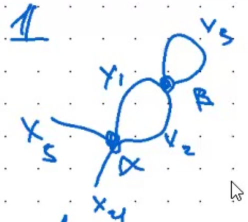

contract operations using duality pairings

$\hookrightarrow$ string calculus for dualizable obj. in s.m. ∞-cat.

Can express TFT in terms of modular ∞-operads

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String calculus for dualizable obj. in s.m. ∞-cat.

Can express TFTs in terms of modular ∞-operads

$$\{F : \mathrm{Bord}_d \longrightarrow \mathcal{C} \text{ sym. mon}\} \overset{\sim}{\longleftrightarrow} \{F' : \mathrm{Mfd}_d \longrightarrow \mathcal{U}(\mathcal{C}) \text{ map of mod. } \infty\text{-operads}\}$$

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_d, \mathcal{C}) \qquad\qquad \mathrm{Alg}^{\mathrm{mod}}_{\mathrm{Mfd}_d}(\mathcal{C})$$

Comes from adjunction
$$\begin{tikzcd} \mathrm{ModOp} \arrow[r, bend left, "\mathrm{Env}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_\infty \arrow[l, bend left, "\mathcal{U}"] \end{tikzcd}$$

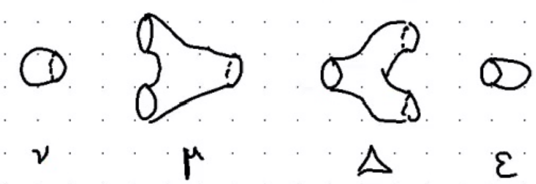
and $\mathrm{Env}(\mathrm{Mfd}_d) = \mathrm{Bord}_d$

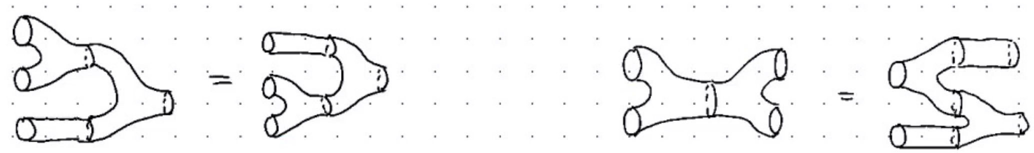
What does this buy us?

- Can focus on connected manifolds
- Duality is built-in
- More algebraic structure on ModOp

(4) Genus 0 and E_2^{SO}-Frobenius algebras (still j/w Barkan)

Observation: All the generators and relations in the definition of a commutative Frobenius algebra have genus 0.





$\rightsquigarrow$ We can try to approximate TFTs by what they do in genus 0

Write $\mathcal{M} := \mathrm{Mfd}_2$ for the surface modular operad

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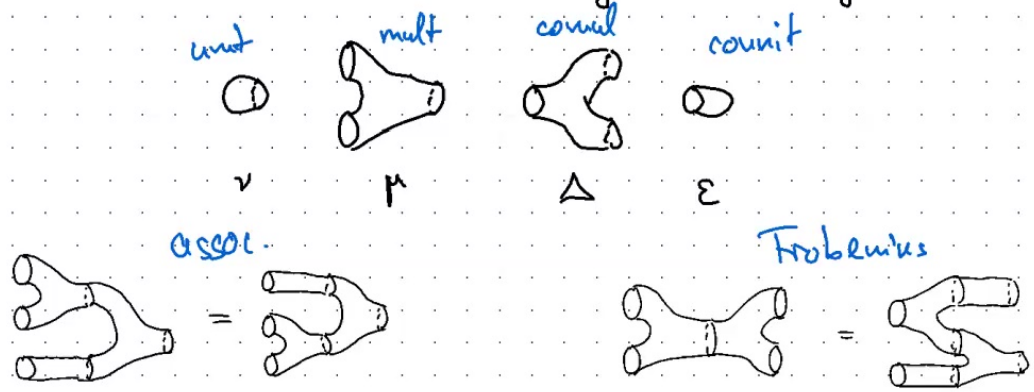

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Write $\mathcal{M} := \mathcal{M}fd_2$ for the surface modular operad

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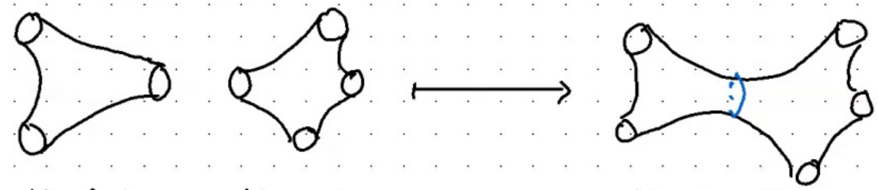
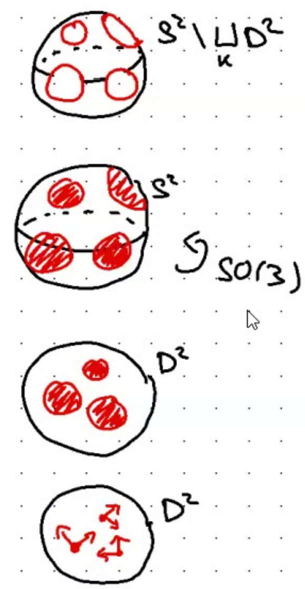
... is ... part $\mathcal{M}_{\leq 0} \subset \mathcal{M}$ is not a modular operad, but still a cyclic operad.

(Don't have self-gluing, have gluing.)



$$\mathcal{M}_{\leq 0}(k) \times \mathcal{M}_{\leq 0}(\ell) \longrightarrow \mathcal{M}_{\leq 0}(k+\ell)$$

Can express this in more elementary terms:



$$\mathcal{M}_{\leq 0}(k) = B\mathrm{Diff}_\partial(S^2 \setminus \sqcup_k D^2)$$

$$\simeq \mathrm{Emb}(\sqcup_k D^2, S^2) /\!/ \mathrm{Diff}(S^2) \simeq SO(3)$$

$$\simeq$$

$$\simeq$$

This is the framed little disk operad

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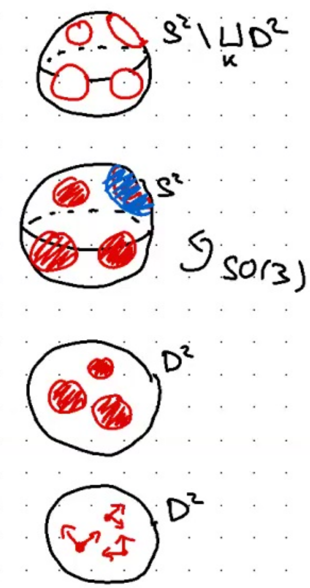

$$\mathcal{M}_{\le 0}(k) = BDiff_\partial(S^2 \setminus \sqcup_k D^2)$$

$$\simeq Emb(\sqcup_k D^2, S^2)//_{Diff(S^2) \simeq SO(3)}$$

$$\simeq Emb(\sqcup_{k-1} D^2, D^2)$$

$$\simeq Conf^{fr}_{k-1}(D^2)$$

This is the framed little disk operad

$$\mathcal{M}_{\le 0} \simeq E_2^{SO} = E_2 //_{SO(2)}$$

Algebras over E_2^{SO} in LinCat are

More generally $\quad Alg_{E_2^{SO}}(\mathcal{C}) \simeq Alg_{E_2}(\mathcal{C})^{SO(2)} \simeq Alg_{E_1}(Alg_{E_1}(\mathcal{C}))^{SO(2)}$

↳ The part of the TFT is usually quite manageable

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$\mathcal{M}_{\leq 0} \simeq E_2^{SO} = E_2 /\!/ SO(2)$

Algebras over E_2^{SO} in LinCat_k are ribbon braided mon. cat.

More generally $\mathrm{Alg}_{E_2^{SO}}(\mathcal{C}) \simeq \mathrm{Alg}_{E_2}(\mathcal{C})^{SO(2)} \simeq \mathrm{Alg}_{E_1}(\mathrm{Alg}_{E_1}(\mathcal{C}))^{SO(2)}$

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However, this only describes the algebra structure over $\mathcal{M}_{\leq 0}$ as an ordinary operad. Cyclic algebras come with an additional trace:

$$\mathrm{Alg}^{cyc}_{\mathcal{M}_{\leq 0}}(\mathcal{C}) = \left\{ (A, \tau) \;\middle|\; A \in \mathrm{Alg}_{E_2}(\mathcal{C}),\ \tau : \int_{S^2} A \xrightarrow{\text{non-degenerate } SO(3)\text{-invariant}} \mathbb{1} \right\}$$

$\left(A \otimes A \longrightarrow \int_{S^2} A \longrightarrow \mathbb{1}\right.$ is a non-degenerate duality pairing. $\left. \rightsquigarrow A^\vee \simeq A \right)$

Here $\int_{S^2} A$ is the factorization homology of A over S^2.
(Assumes that $\mathcal{C}$ has sufficient colimits.)

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non-degenerate
SO(3)-invariant

$$\mathrm{Alg}^{cyc}_{\mathcal{M}_{\leq 0}}(\mathcal{C}) = \left\{ (A,\tau) \;\middle|\; A \in \mathrm{Alg}_{E_2^{SO}}(\mathcal{C}),\quad \tau : \boxed{\int_{S^2} A} \longrightarrow \mathbb{1} \right\}$$

"E_2^{SO}-Frobenius alg."

$\Big(A \otimes A \longrightarrow \int_{S^2} A \xrightarrow{\tau} \mathbb{1}$ is a non-degenerate duality pairing. $\rightsquigarrow A^\vee \simeq A \Big)$

Here $\int_{S^2} A$ is the factorization homology of A over S^2.
(Assumes that $\mathcal{C}$ has sufficient colimits.)

<u>Ex</u> $\mathcal{C} = \mathrm{Vect}_{\mathbb{C}}$ $\quad \int_{S^2} A = A \xrightarrow{\tau} \mathbb{C}$ $\rightsquigarrow$ get commutative Frobenius algebras

$\mathcal{C} = \mathrm{LinCat}_{\mathbb{C}}$ $\xrightarrow{\text{Müller-Woike}}$ get ribbon-braided Grothendieck-Verdier duality

⑤ The genus filtration & obstruction theory

$\mathcal{M}^{(g)}$ = universal approximation of $\mathcal{M}$ from genus $\leq g$

agrees with $\mathcal{M}$ in genus $\leq g$

$= \mathrm{Ind}_g^{\infty}(\mathcal{M}_{|\leq g})$

$$\mathrm{grModOp}^{\leq g} \underset{(-)_{\leq g}}{\overset{\mathrm{Ind}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{grModOp}$$

Ex

$\mathcal{M}^{(g-1)}(\text{arity } 0, \text{ genus } g) \;=\; C(\Sigma_g)/\!/\mathrm{Diff}(\Sigma_g) \;\simeq\; \partial \overline{\mathcal{M}}_g^{BS}$

$C(\Sigma_g)$ = curve complex = $\{$  $\}$

systems

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$\mathcal{M}^{(g)}$ = universal approximation of $\mathcal{M}$ from genus $\leq g$

agrees with $\mathcal{M}$ in genus $\leq g$

$$= \mathrm{Ind}_g^\infty(\mathcal{M}_{|\leq g}) \longrightarrow \mathcal{M}$$

$$\begin{tikzcd} \mathrm{grModOp}^{\leq g} \arrow[r, shift left, hook, "\mathrm{Ind}"] \arrow[r, phantom, "\perp"] & \mathrm{grModOp} \arrow[l, shift left, "(-)_{\leq g}"] \end{tikzcd}$$

Ex

$\mathcal{M}^{(g-1)}$(arity 0, genus $g-1$) $= \mathcal{M}_{g-1}$

$\mathcal{M}^{(g-1)}$(arity 0, genus g) $= C(\Sigma_g) /\!\!/ \mathrm{Diff}(\Sigma_g) \simeq \partial \overline{\mathcal{M}}_g^{BS}$

$C(\Sigma_g)$ = curve complex = $\{$  $\}$

systems of disjoint simple closed curve.

Ex $\mathcal{M}^{(0)} = \mathrm{Ind}_0^\infty(\mathcal{M}_{|\leq 0}) \simeq$ (Giansiracusa)

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Translation to modular operads:

$$\mathrm{Mod}_! : \mathrm{CycOp} \underset{\perp}{\rightleftarrows} \mathrm{ModOp} : \mathrm{Mod}^*$$

$\mathrm{Mod}_! \mathcal{M}_{\leq 0}$ free modular operad on genus 0 part

Giansiracusa: $\mathrm{Mod}_! \mathcal{M}_{\leq 0} \simeq \mathcal{H}bdy \subset \mathrm{Mfd}_3^\partial$

"handlebody modular operad"

$(S^1 \times D^2) \natural (S^1 \times D^2)$

This only sees part of the mapping class group.

↳ get

Note: $\mathcal{H}bdy \longrightarrow \mathcal{M}$ is a bijection on π_0

So if $\mathcal{C}$ is a 1-category, then $\mathrm{Alg}^{\mathrm{mod}}_{\mathcal{H}}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}}(\mathcal{C})$

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$= \mathrm{Ind}_\partial(\mathcal{M}_{1,0}) \simeq \mathcal{H}\mathrm{bdy}$ (Giansiracusa)

↪ "annular functor"

Tower of approximations

$$\mathcal{H}\mathrm{bdy} = \mathcal{M}^{(0)} \longrightarrow \mathcal{M}^{(1)} \longrightarrow \mathcal{M}^{(2)} \longrightarrow \cdots \longrightarrow \mathcal{M} = \mathcal{M}\mathrm{fd}_2$$

Take

$$\mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}}(\mathcal{C}) \longrightarrow \cdots \longrightarrow \mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}^{(2)}}(\mathcal{C}) \longrightarrow \mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}^{(1)}}(\mathcal{C}) \longrightarrow \mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}^{(0)}}(\mathcal{C})$$

$$\mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}}(\mathcal{C}) \cong \mathrm{Fun}^{\otimes}(\mathrm{Bord}_2, \mathcal{C}) \qquad \mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}^{(0)}}(\mathcal{C}) \cong \mathrm{Alg}^{\mathrm{cyc}}_{E_2^{SO}}(\mathcal{C})$$

2D TFTs $\longrightarrow \cdots \quad \cdots \longrightarrow \; ? \longrightarrow$

Q: How much more structure is needed to promote an E_2^{SO}-Frob. alg to a 2D TFT?

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Obstruction theory

$$\mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}^{(g)}}(\mathcal{C}) \quad ?$$
$$\downarrow \qquad\qquad \downarrow$$

Thm (S.) Given $\mathcal{F} \in \mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}^{(g-1)}}(\mathcal{C})$ there is an equivalence

$$\left\{ \begin{matrix} \text{Lifts of } \mathcal{F} \text{ to} \\ \mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}^{(g)}}(\mathcal{C}) \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{tikzcd} \operatorname{colim}_{C \subset \Sigma_g} \int_{\Sigma_g \setminus C} A \arrow[r, "\mathcal{F}"] \arrow[d] & 1 \\ \int_{\Sigma_g} A \arrow[ur, dashed, "Z^{(g)}"'] \end{tikzcd} \quad \begin{matrix} \mathrm{Diff}(\Sigma_g) \\ \text{-equiv.} \\ \text{extension} \end{matrix} \right\}$$

Informally:

The datum of a 2D TFT in $\mathcal{C}$ is equivalent to a family of map

$$Z^{(g)} : \int_{\Sigma_g} A \longrightarrow 1 \qquad \mathrm{Diff}(\Sigma_g)\text{-equivariant}$$

each subject to constraints from genus $\leq g-1$.

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each subject to constraints from genus $\leq g-1$.

Q: What are the $Z^{(g)}$? " universal genus g partition function. "

Suppose we have $F: \mathrm{Bord}_2 \longrightarrow \mathcal{C}$, fix Σ_g

$A = F(S^1)$ algebra of point operators

$\bigsqcup_k D^2 \hookrightarrow \Sigma_g$

$(\Sigma_g \setminus \bigsqcup_k D^2) : \bigsqcup_k S^1 \longrightarrow \emptyset$

$F($

$A^{\otimes} \cong F(\bigsqcup_k S^1) \longrightarrow \mathbb{1}$

As we vary the disks, this assembles into

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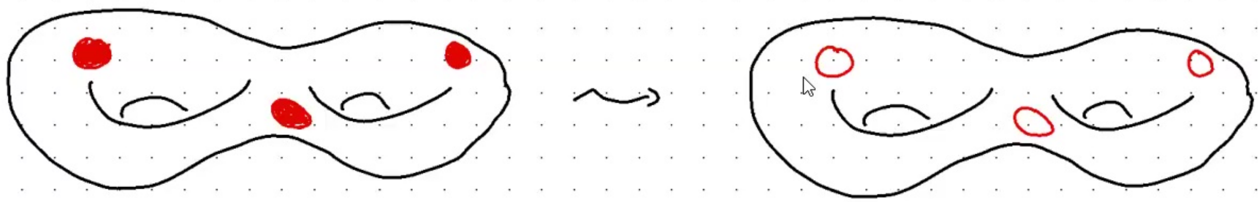
have $F: \mathrm{Bord}_2 \longrightarrow \mathcal{C}$, fix Σ_g

$A = F(S^1)$ algebra of point operators



$\coprod_k D^2 \hookrightarrow \Sigma_g$ $\qquad$ $(\Sigma_g \setminus \coprod_k D^2) : \coprod_k S^1 \longrightarrow \emptyset$

$F($

$A^{\otimes k} \cong F(\coprod_k S^1) \longrightarrow \mathbb{1}$

As we vary the disks, this assembles into

$$Z^{(g)} : \operatorname*{colim}_{\coprod_k D^2 \subset \Sigma_g} A^{\otimes k} = \longrightarrow \mathbb{1}$$ ($\mathrm{Diff}(\Sigma_g)$-equivariant)

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Q: How much more structure is needed to promote an E_2^{SO}-Frob. alg to a 2D TFT?

Convergence

Thm (Harer '85) $C(\Sigma_g) \simeq \bigvee_{\infty} S^{2g-2}$ is $(2g-3)$-connected

$\hookrightarrow$ Get that $M^{(g)} \longrightarrow M^{(g+1)}$ is $(2g-1)$-connected.

Cor If $\mathcal{C}$ is an $(n,1)$-category and $n \leq 2g+1$

then $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_{2,1}, \mathcal{C}) = \mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}^{(g)}}(\mathcal{C})$

Obstruction theory ? $\mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}^{(g)}}(\mathcal{C}) \downarrow$

Thm (S.) Given $F \in \mathrm{Alg}^{\mathrm{mod}}_{\mathcal{M}^{(g-1)}}(\mathcal{C})$ there is an equivalence

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6 Invertible TFTs and a spectral sequence

Recall A TFT $F : \mathrm{Bord}_2 \longrightarrow \mathcal{C}$ is invertible, if there is another $G : \mathrm{Bord}_2 \longrightarrow \mathcal{C}$ s.t. $F \otimes G \cong \mathbb{1}$ (trivial)

$$\Updownarrow$$

$$\mathrm{Bord}_2 \longrightarrow \mathrm{Pic}(\mathcal{C}) = \begin{cases} B\mathbb{C}^\times \\ B^2\mathbb{C}^\times \\ \Omega^{\infty-k} I\mathbb{Z} \end{cases}$$

$$\Updownarrow$$

$$|\mathrm{Bord}_2| \longrightarrow \mathrm{Pic}(\mathcal{C})$$

Used by Freed-Hopkins '16 to compute the group of symmetry protected topological phases.

computed by Galatius-Madsen-Tillmann-Weiss '06

$MTSO(2) = \mathbb{CP}^\infty_{-1}$ well-understood Thom spectrum.

Genus filtration gives filtration of $|\mathrm{Bord}_2|$ by infinite loop spaces.

$$|\mathrm{Env}(\mathcal{M}^{(0)})| \longrightarrow |\mathrm{Env}(\mathcal{M}^{(1)})| \longrightarrow \cdots \longrightarrow |\mathrm{Env}(\mathcal{M})|.$$

Main theorem yields description of associated graded as $\Sigma^{\infty+1}$

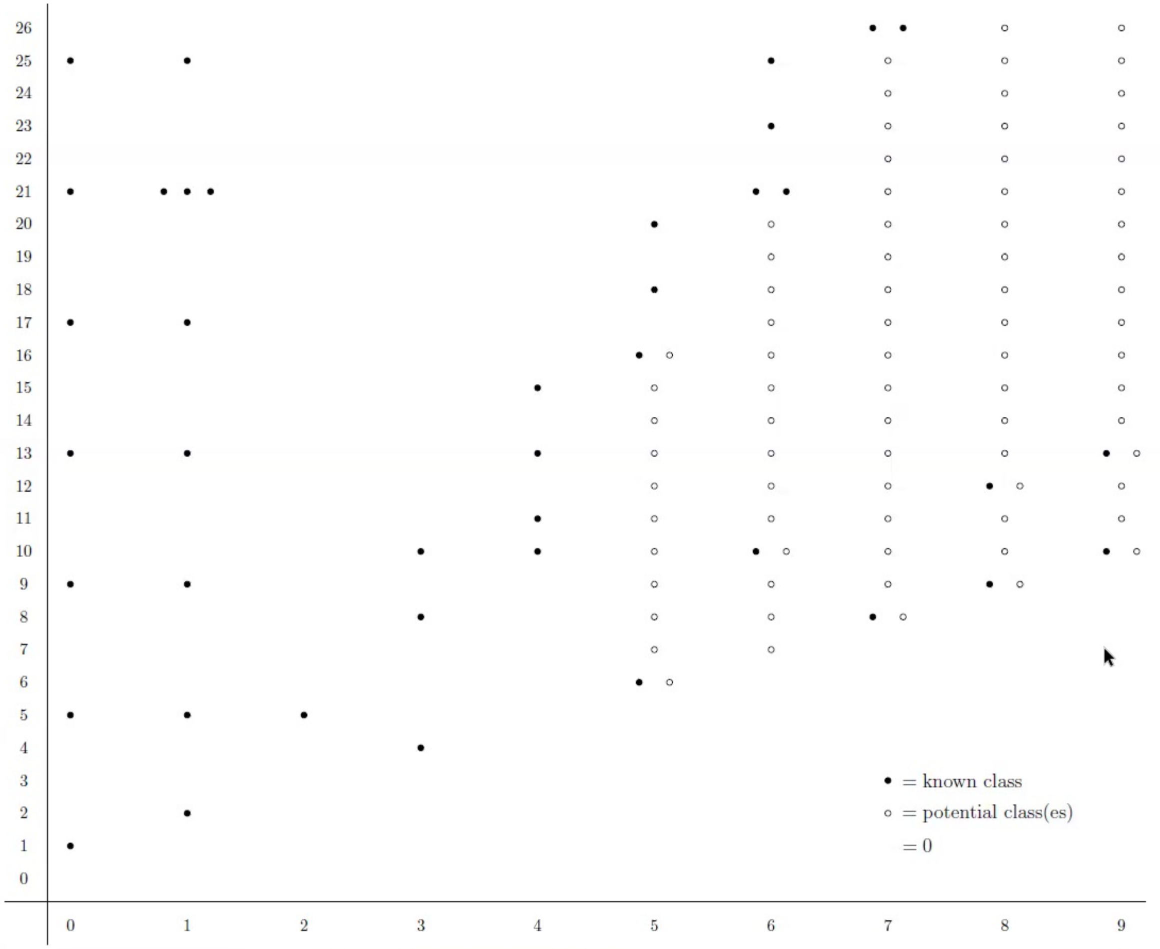
↪ get spectral sequence

$$E^1_{g,n} = \begin{cases} H^n(BSO(3);\mathbb{Q}) & g=0 \\ H^{n+1}(BDiff(S^1\times S^1)/BDiff(S^1\times D^2);\mathbb{Q}) & g=1 \Rightarrow H^{n+g}(\tau_{\geq 0} MTSO(2);\mathbb{Q}) \\ H_c^{n+g}(\mathcal{M}_g;\mathbb{Q}) & g\geq 2 \end{cases}$$

$$H_c^{n+g}(\mathcal{M}_g;\mathbb{Q}) \cong H^{5g-n-5}(\mathcal{M}_g;\mathbb{Q})$$

$$H^{n+g}(\tau_{\geq 0} MTSO(2);\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & n+g \text{ even} \\ 0 & \text{odd.} \end{cases}$$

2. The spectral sequence $E^1_{g,k} = H^{5g-5-k}(B\Gamma_g) \Rightarrow H^{\mathrm{Sp}}_{g+k}(\tau_{\geq 0}\Sigma MTSO_2)$



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$$\mathcal{C} = \mathrm{Ch}_{\mathbb{Q}} \to D(\mathbb{Q}) \circlearrowleft \mathrm{Pic}(\mathbb{Q})$$

$$\wr$$

$$\mathrm{Bord}_2 \rightleftarrows D(\mathbb{Q}) /\!/ \mathrm{Pic}(\mathbb{Q})$$

$$X \longrightarrow Y$$

$$\updownarrow$$

$$L \in \mathrm{Pic}$$

$$X \longrightarrow Y \otimes L$$

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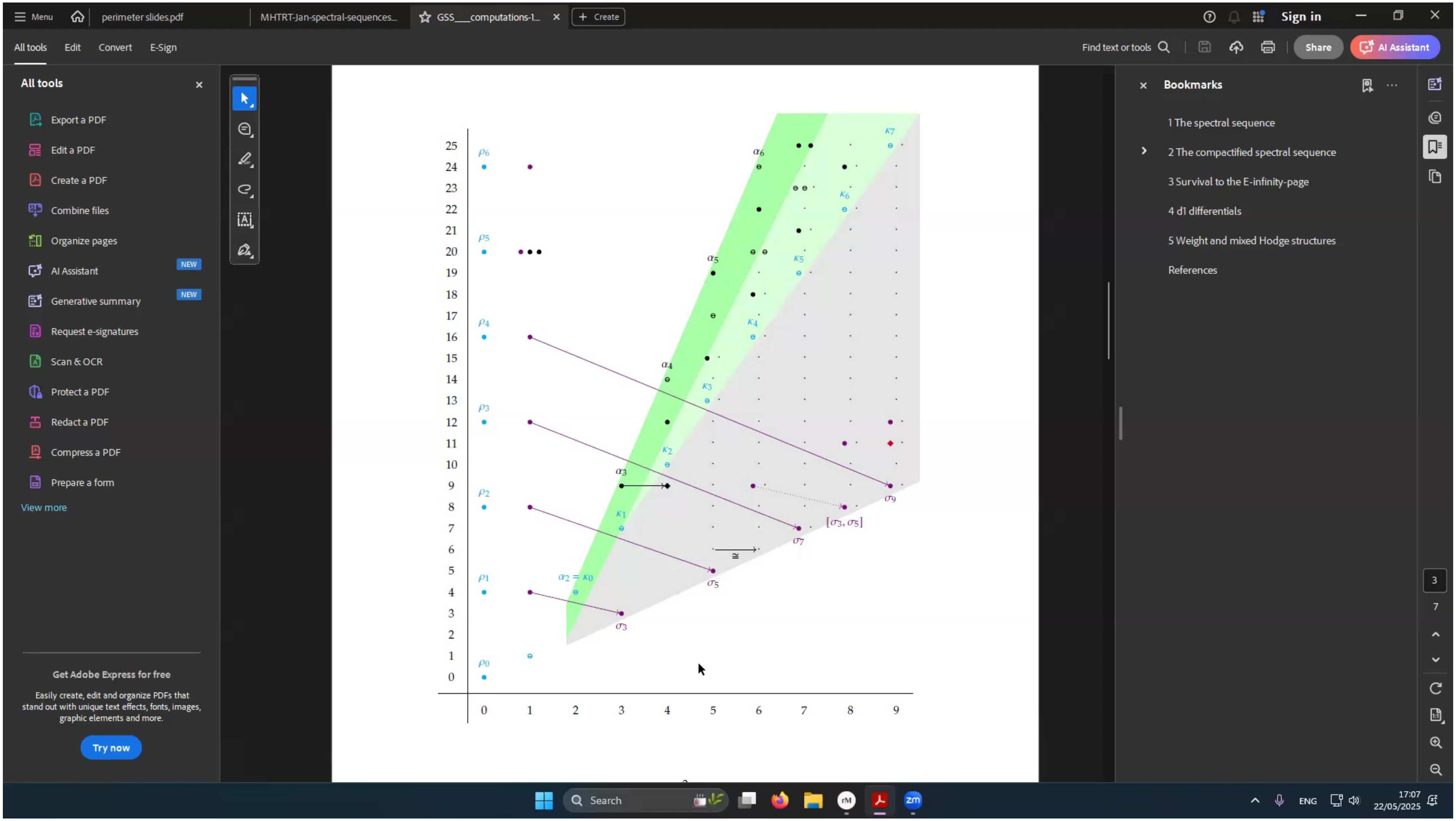
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3 Survival to the E-infinity-page
4 d1 differentials
5 Weight and mixed Hodge structures
References
ρ_0 ρ_1 ρ_2 ρ_3 ρ_4 ρ_5 ρ_6
$\alpha_2 = \kappa_0$ α_3 α_4 α_5 α_6
κ_1 κ_2 κ_3 κ_4 κ_5 κ_6 κ_7
σ_3 σ_5 σ_7 σ_9 $[\sigma_3, \sigma_5]$
$\cong$

$$\mathcal{C} = \mathrm{Ch}_{\mathbb{Q}} \to \mathcal{D}(\mathbb{Q}) \circlearrowleft \mathrm{Pic}(\mathbb{Q})$$

$$\wr$$

$$\mathrm{Bord}_2 \longrightarrow \mathcal{D}(\mathbb{Q}) /\!/ \mathrm{Pic}(\mathbb{Q})$$

$$X \longrightarrow Y$$

$$\updownarrow$$

$$L \in \mathrm{Pic}$$

$$X \longrightarrow Y \otimes L$$

$$H_{14}(M_5) \cong \mathbb{Q}$$