Title: Edge-colored graphs and exponential integrals

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Abstract:

We show that specific exponential integrals serve as generating functions of labeled edge-colored graphs. Based on this, we derive asymptotics for the number of edge-colored graphs with arbitrary weights assigned to different vertex structures. The asymptotic behavior is governed by the critical points of a polynomial. As an application, we discuss the Ising model on a random graph and show how its phase transitions arise from our formula.

Edge-Colored Graphs & Exponential Integrals

joint work w/ M. Borinsky & C. Meroni

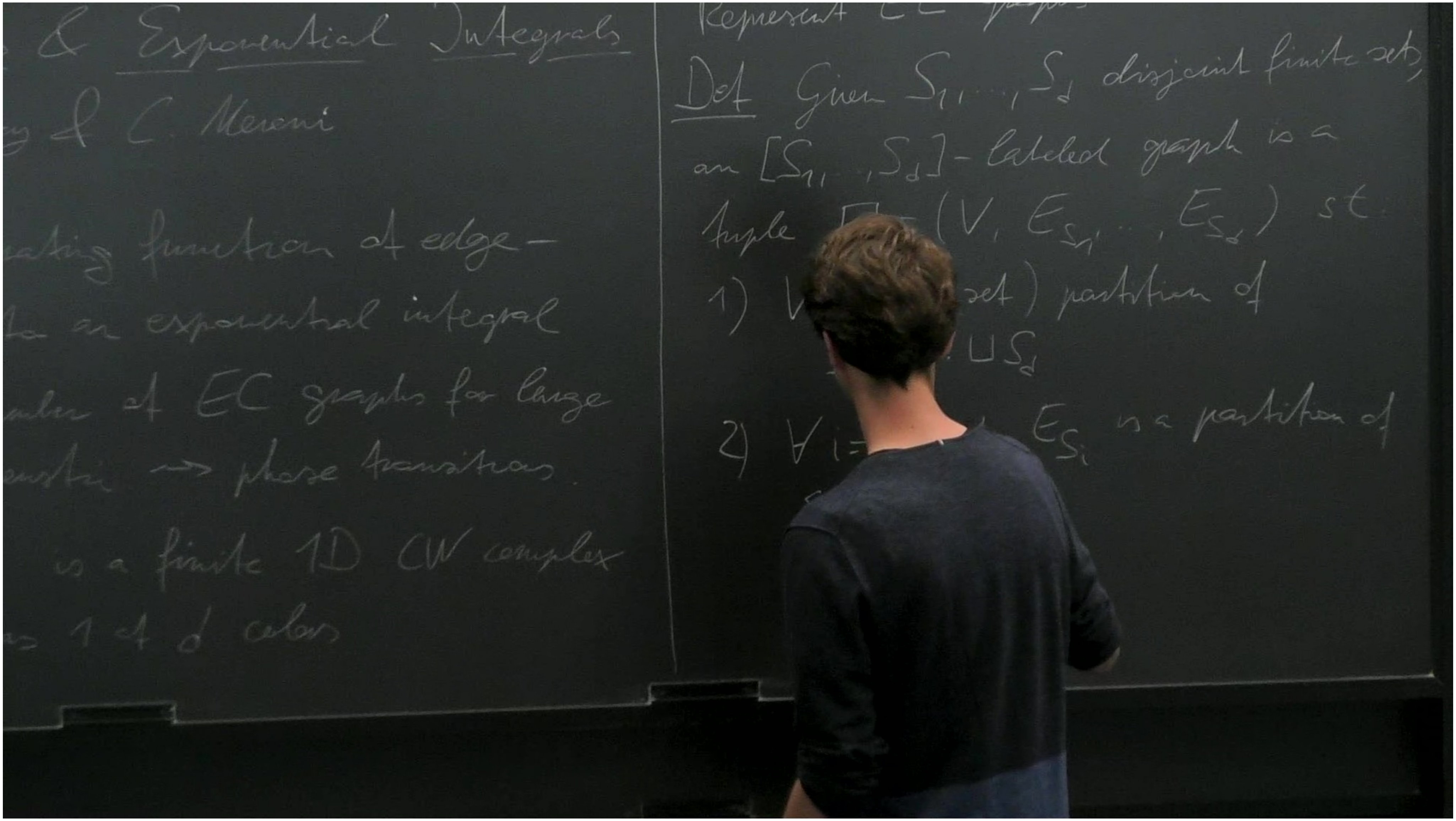
Goals 1) relate the generating function of edge-colored (EC) graphs to an exponential integral

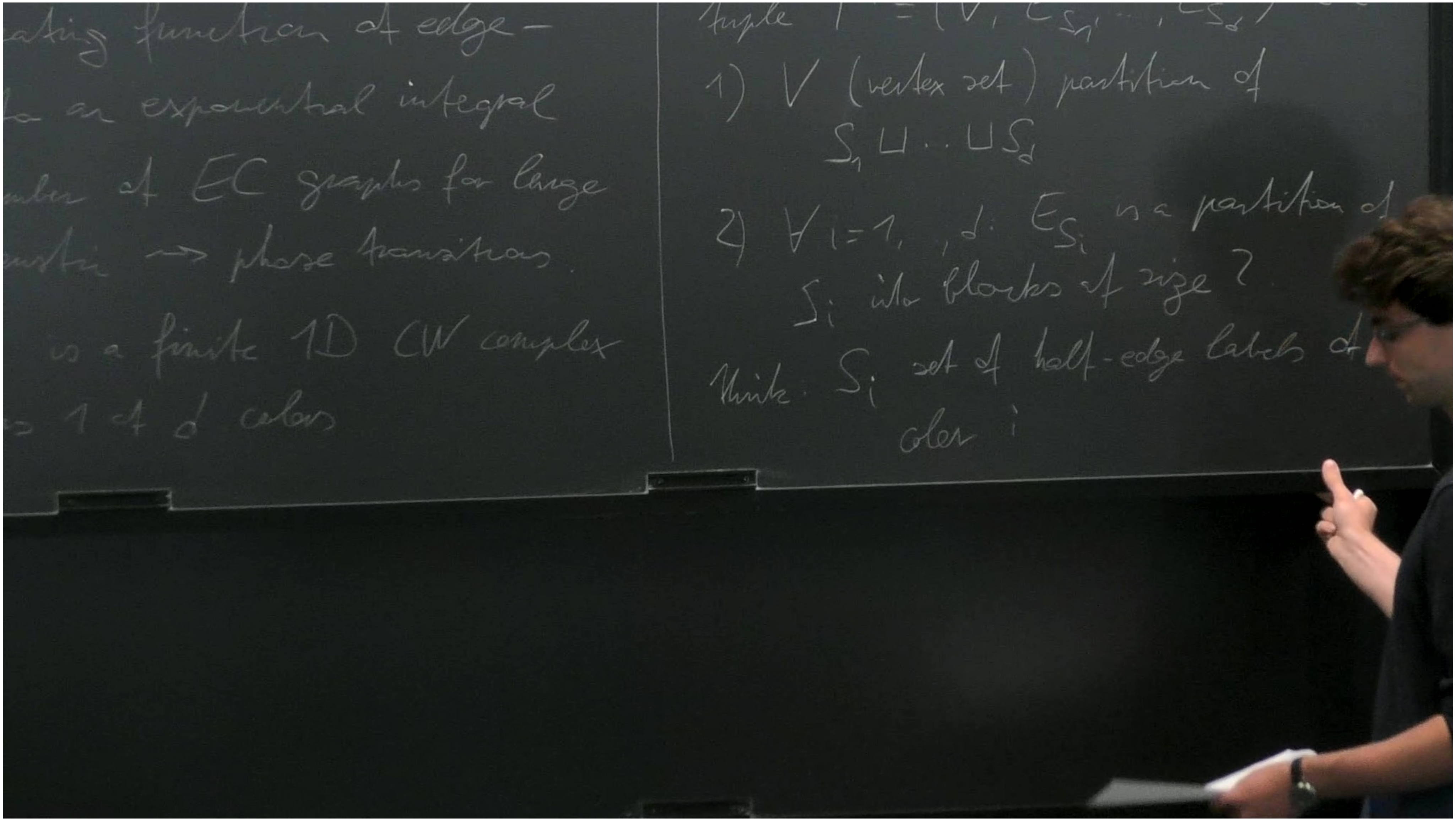
2) find the asymptotic number of EC graphs for large negative Euler characteristic $\rightsquigarrow$ phase transitions

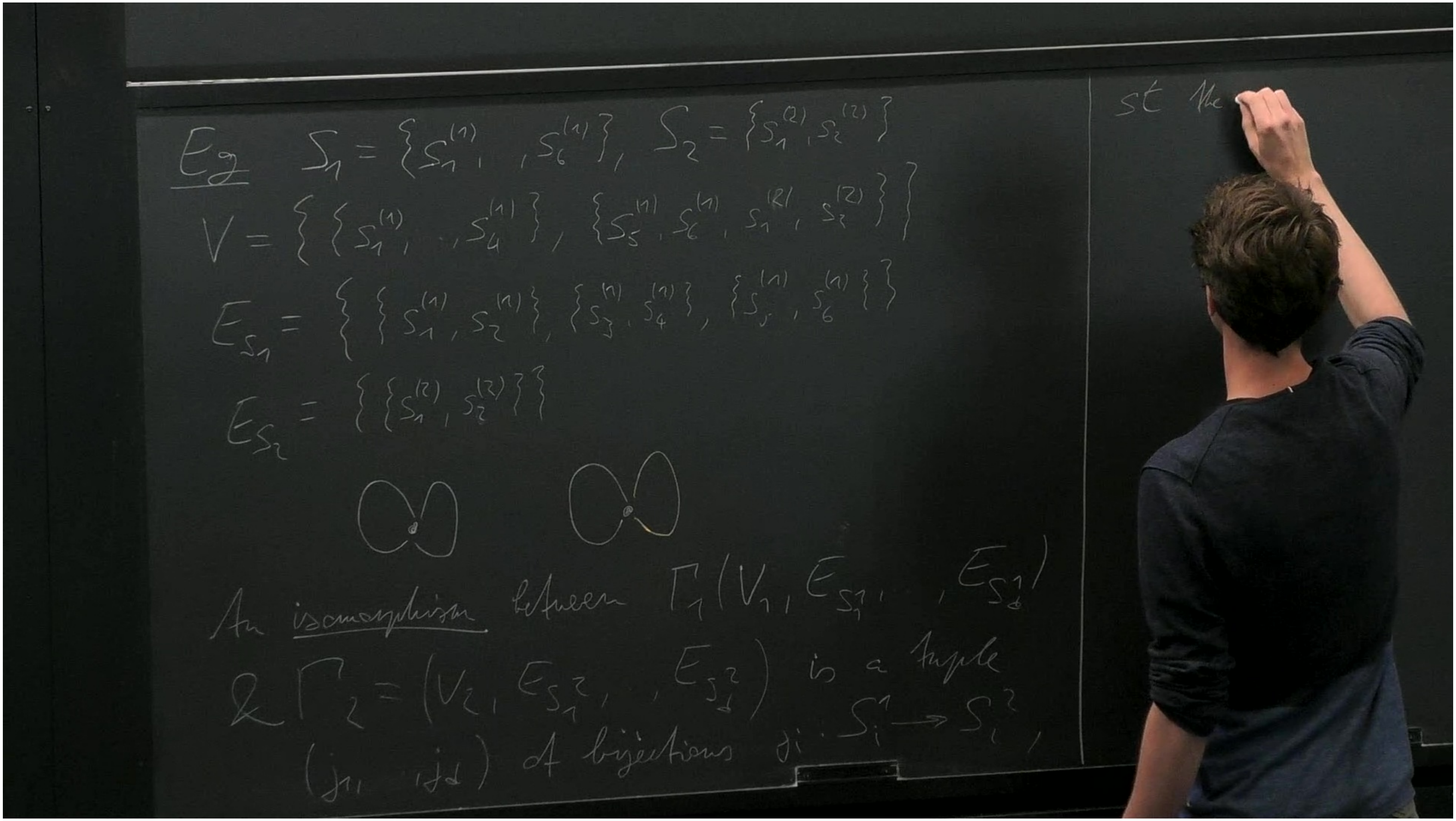
§ EC Graphs A graph G is a finite 1D CW complex

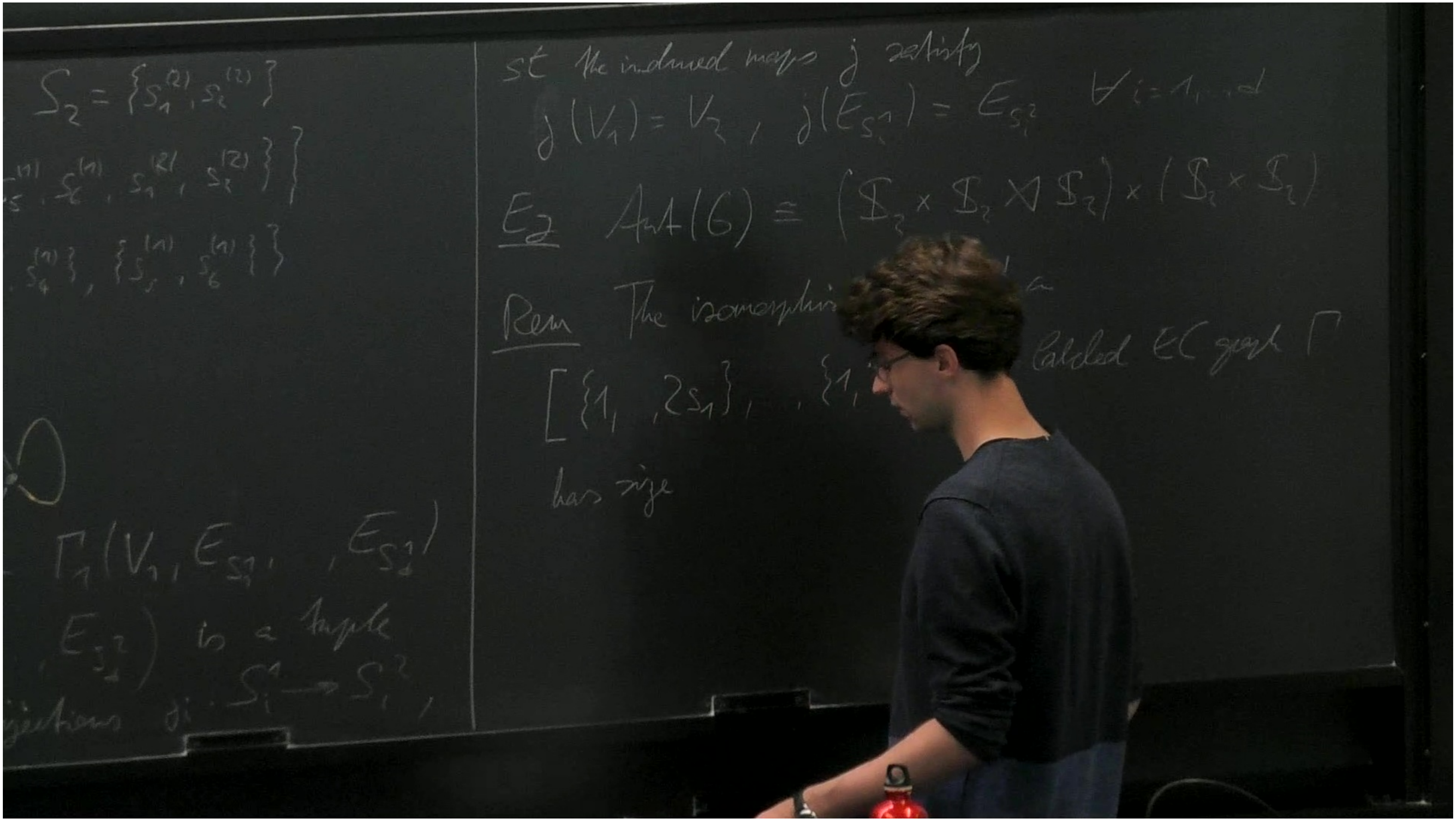
it's EC if each edge has 1 of d colors

Represent EC


& Exponential Integrals
& C. Meroni
function of edge-
an exponential integral
of EC graphs for large
→ phase transitions.
is a finite 1D CW complex
1 of d colors
Represent
Def Given S_1, ..., S_d disjoint finite sets,
an [S_1, ..., S_d]-labeled graph is a
tuple (V, E_S1, ..., E_Sd) s.t.
1) V set) partition of
⊔ S_d
2) ∀ i = E_Si is a partition of


ating function of edge-
to an exponential integral
mber of EC graphs for large
metric → phase transitions
is a finite 1D CW complex
1 of d colors
tuple Γ = (V, E_{S_1}, …, E_{S_d})
1) V (vertex set) partition of
S_1 ⊔ .. ⊔ S_d
2) ∀ i = 1, .., d: E_{S_i} is a partition of
S_i into blocks of size 2.
think: S_i set of half-edge labels of
color i


Eg $S_1 = \{s_1^{(1)}, \ldots, s_6^{(1)}\}$, $S_2 = \{s_1^{(2)}, s_2^{(2)}\}$
$V = \{\{s_1^{(1)}, \ldots, s_4^{(1)}\}, \{s_5^{(1)}, s_6^{(1)}, s_1^{(2)}, s_2^{(2)}\}\}$
$E_{S_1} = \{\{s_1^{(1)}, s_2^{(1)}\}, \{s_3^{(1)}, s_4^{(1)}\}, \{s_5^{(1)}, s_6^{(1)}\}\}$
$E_{S_2} = \{\{s_1^{(2)}, s_2^{(2)}\}\}$
An isomorphism between $\Gamma_1(V_1, E_{S_1^1}, \ldots, E_{S_d^1})$
& $\Gamma_2 = (V_2, E_{S_1^2}, \ldots, E_{S_d^2})$ is a tuple
$(j_1, \ldots, j_d)$ of bijections $j_i : S_i^1 \to S_i^2$,
st the


$S_2 = \{s_1^{(2)}, s_2^{(2)}\}$
st the induced maps j satisfy
$j(V_1) = V_2$, $j(E_{S_i^1}) = E_{S_i^2}$ $\forall i = 1, \dots, d$
Eg $Aut(G) \cong (S_3 \times S_2 \rtimes S_2) \times (S_2 \times S_2)$
Rem The isomorphism
labelled EC graph Γ
$[\{1, \dots, 2s_1\}, \dots, \{1,$
has size
$\Gamma_1(V_1, E_{S_1^1}, \dots, E_{S_d^1}$
$E_{S_d^2})$ is a tuple
$j_i : S_i^1 \to S_i^2$,

$S_4^{(1)}\}, \{S_j^{(1)}, S_6^{(1)}\}\}$

$(V_1, E_{S_1^1}, \ldots, E_{S_d^1})$

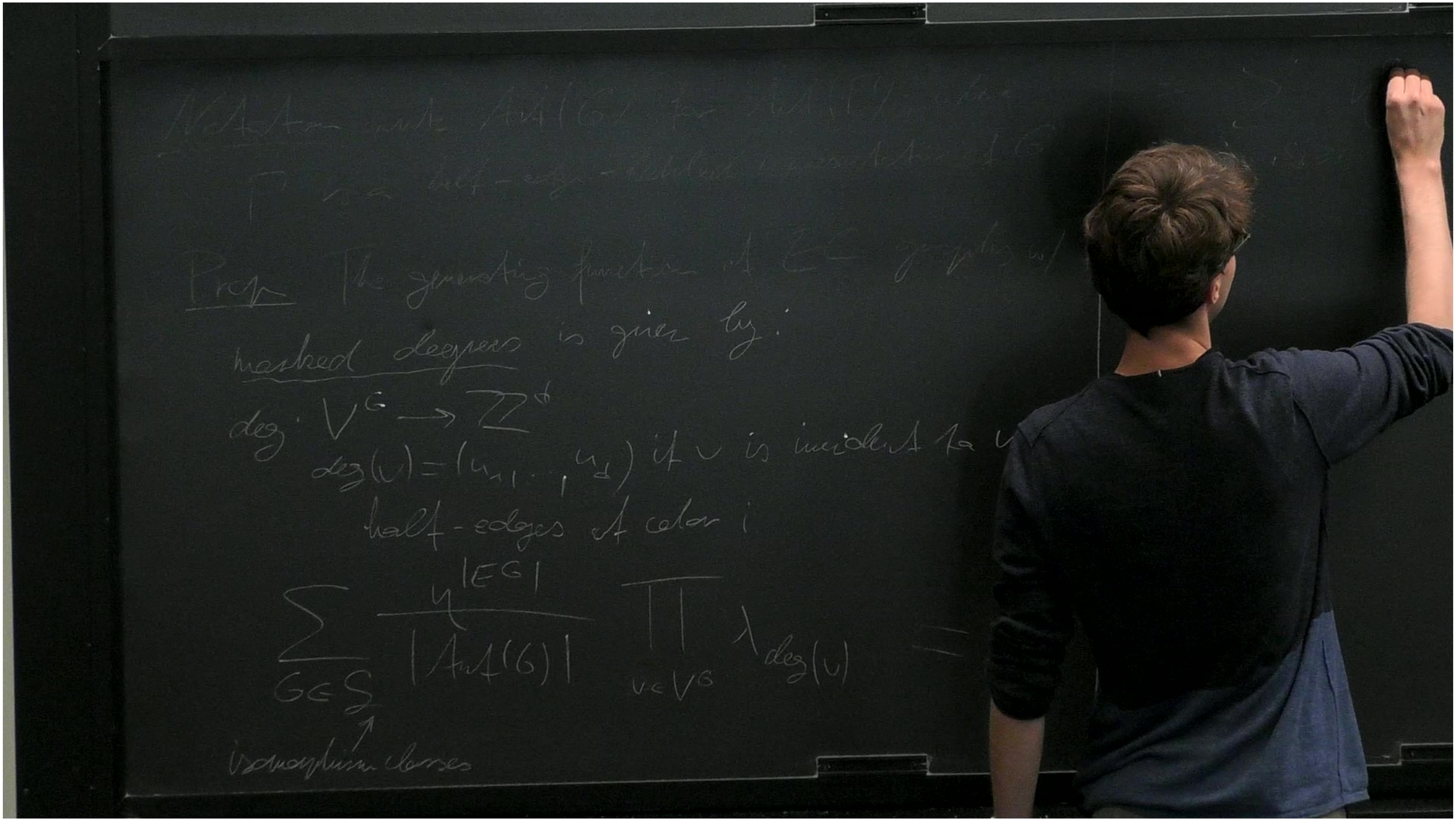
is a tuple

$S_i^1 \to S_i^2$,

<u>Rem</u> The isomorphism class of a $[\{1,\ldots,2s_1\},\ldots,\{1,\ldots,2s_d\}]$-labelled EC graph Γ has size

$$\frac{(2s_1)!\cdots(2s_d)!}{|\mathrm{Aut}(\Gamma)|}$$

group action by $\mathfrak{S}_{2s_1}\times\cdots\times\mathfrak{S}_{2s_d} \rightsquigarrow$ orbit-stabilizer Thm



Prop The generating function of graphs w/
marked degrees is given by:
$\deg: V^G \to \mathbb{Z}^d$
$\deg(v) = (n_1, \dots, n_d)$ if v is incident to n_i
half-edges of color i
$\sum_{G \in S} \frac{u^{|E_G|}}{|\mathrm{Aut}(G)|} \prod_{v \in V^G} \lambda_{\deg(v)} =$
isomorphism classes

color

$$[x_1^{\alpha_1} \cdots x_d^{\alpha_d}] \exp\left(\sum_{\substack{n_i \geq 0 \\ n_1 + \dots + n_d \geq 1}} \lambda_{(n_1, \dots, n_d)} \frac{x_1^{n_1} \cdots x_d^{n_d}}{n_1! \cdots n_d!} \right)$$

coeff. extraction

incident to v

Eg $d = 2$, $\quad [\hbar^1] \displaystyle\sum_{G \in \mathcal{G}} \frac{\hbar^{|E^G|}}{|\mathrm{Aut}(G)|} \prod_{v \in V^G} \lambda_{\deg(v)}$

$=$ 

$$+ \frac{1}{2} \lambda_{(2,0)} + \frac{1}{2} \lambda_{(1,0)}^2 + \frac{1}{2} \lambda_{(0,2)} + \frac{1}{2} \lambda_{(0,1)}^2$$

$\deg(v)$

$(j_1, \dots, j_d)$ of bijections $j_i : S_i \to S_i$,

Proof of Prop: $(s_1)! \dots (s_d)! \, [x_1^{s_1} \dots x_d^{s_d}] \exp\left(\sum_{\substack{u_i \geq 0 \\ u_1 + \dots + u_d \geq 1}} \lambda_{(u_1, \dots, u_d)} \frac{x_1^{u_1} \dots x_d^{u_d}}{u_1! \dots u_d!} \right)$

$$= \sum_{\substack{n_{(u_1, \dots, u_d)} \in \mathbb{Z}_{\geq 0} \\ \sum_{u_i} u_i n_{(u_1, \dots, u_d)} = s_i}} \frac{s_1! \dots s_d!}{\prod_{(u_1, \dots, u_d)} n_{(u_1, \dots, u_d)}! \, (u_1! \dots u_d!)^{n_{(u_1, \dots, u_d)}}} \cdot \prod_{u_i} \lambda_{(u_1, \dots, u_d)}^{n_{(u_1, \dots, u_d)}}$$

$|S_i| = s_i$

$\bigcirc$ = # partitions of $S_1 \sqcup \dots \sqcup S_d$ with $n_{(u_1, \dots, u_d)}$ many blocks containing u_i elements from S_i

[grp action $\mathfrak{S}_{s_1} \times \dots \times \mathfrak{S}_{s_d}$

$(2s_i - 1)!! = \#$ matchings in $\{1, \ldots, 2s_i\}$

$\frac{(2s_1)! \ldots (2s_d)!}{|Aut(G)|} = \#$ representatives Γ of G $\square$

§ Integrals Take $D \subseteq \mathbb{R}^d$ a n'hood of 0,

$g : D \to \mathbb{R}$ st :

1) g is analytic near 0 w/ converging power series exp

$$g(x_1, \ldots, x_d) = -\frac{1}{2}\sum x_i^2 + \sum_{\substack{n_1, \ldots, n_d \geq 0 \\ n_1 + \ldots + n_d \geq 3}} \Lambda_{(n_1, \ldots, n_d)} \frac{x_1^{n_1} \ldots x_d^{n_d}}{n_1! \ldots n_d!}$$

$[x_1^{s_1} \ldots x_d^{s_d}] \exp\left(\sum_{\substack{n_i \geq 0 \\ n_1 + \ldots + n_d \geq 1}} \lambda_{(n_i)} \frac{x_1^{n_1} \ldots x_d^{n_d}}{n_1! \ldots n_d!}\right)$

$\ldots s_d!$

$\ldots_{n_1 \ldots n_d)}! (n_1! \ldots n_d!)^{n_{(n_1, \ldots, n_d)}}$

$\lambda_{(n_1, \ldots, n_d)}^{n_{(n_1, \ldots, n_d)}}$ $|S_i| = s_i$

of $S_1 \sqcup \ldots \sqcup S_d$ with

blocks containing n_i elements

group action $\mathfrak{S}_{s_1} \times \ldots \times \mathfrak{S}_{s_d}$

2) $\sup_{x \in D} g(x) = g(0)$ is unique

3) $I(z) := \left(\frac{z}{2\pi}\right)^{d/2} \int_D \exp(z g(x))\,dx$ exists for $z > 0$

Thm For large z, $I(z)$ has an asymptotic expansion

$$I(z) \sim \sum_{n \geq 0} A_n z^{-n}, \quad \text{where } A_n = \sum_{G \in \mathcal{G}^*_{-n}} \frac{1}{|\mathrm{Aut}(G)|} \prod_{v \in V_G} \Lambda_{\deg(v)}$$

admissible graphs

$(|\deg(v)| \geq 3)$,

$\chi(G) = -n$

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admissible graphs $(|\deg(v)| \geq 3)$, $\chi(G) = -n$

<u>Proof idea</u> Laplace expansion of $I(z)$.

Prop. $\sqrt{\frac{z}{2\pi}} \int_{\mathbb{R}} e^{-zx^2/2} x^{2s}\, dx = (2s-1)!!$

$(zg(x))dx$ exists for $z>0$

an asymptotic expansion

$A_n = \sum_{G \in \mathcal{G}^*_{-n}} \frac{1}{|\mathrm{Aut}(G)|} \prod_{v \in V_G} \Lambda_{\deg(v)}$

admissible graphs

$I(z)$, $(|\deg(v)| \geq 3)$,

$\chi(G) = -n$

$= (2s-1)!!$

!!

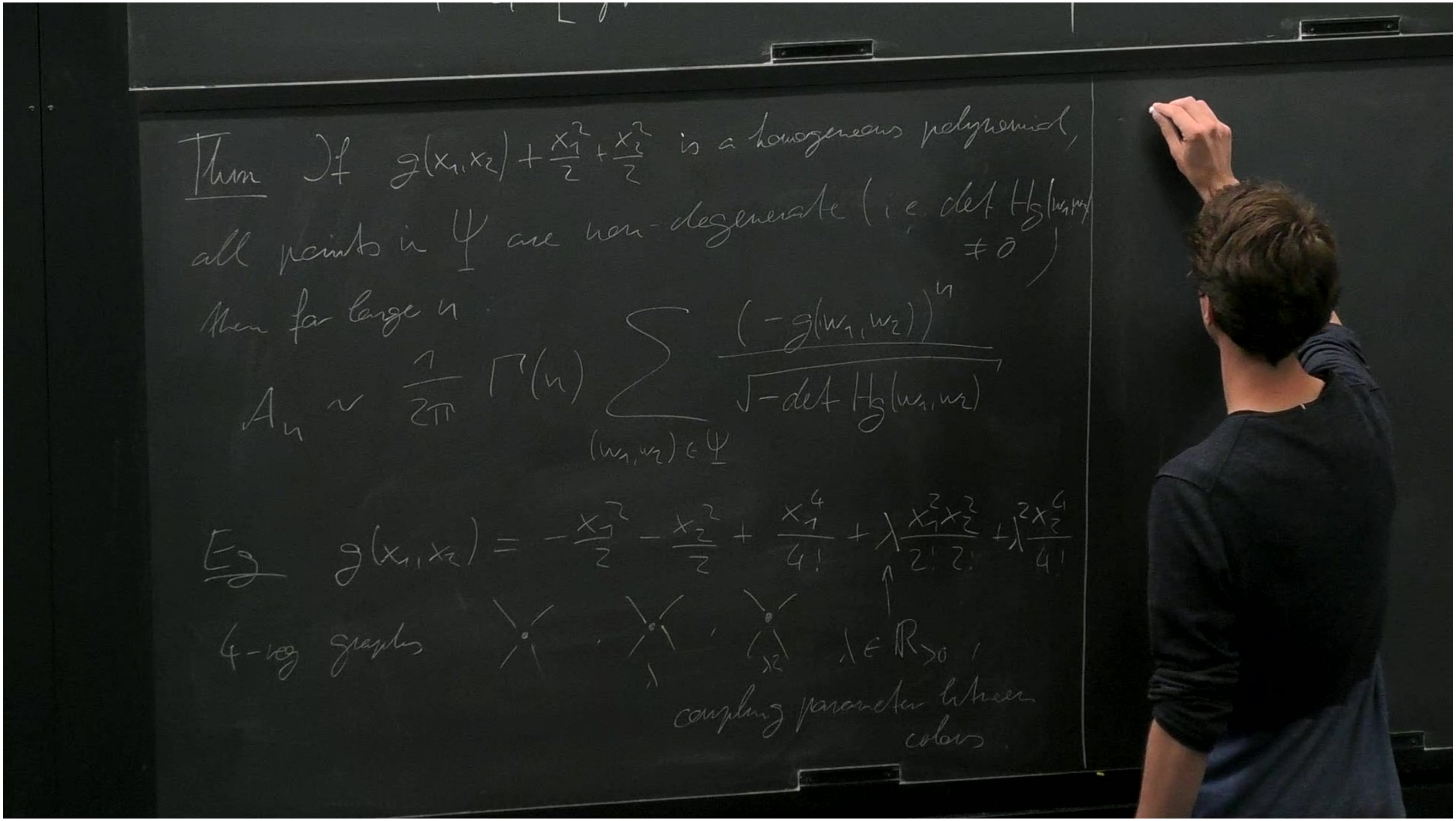
§ Asymptotics Restrict to $\underline{d=2}$

($d>2$ ongoing work) $g \in \mathbb{R}[x_1, x_2]$

$\Psi = \{ (w_1, w_2) \in (\mathrm{cint}_{\mathbb{C}\cdot\mathbb{R}^2}\, g) \setminus \{0\}$:

$(w_1, w_2) = c \cdot (\tilde{w}_1, \tilde{w}_2)$, $c \in \mathbb{C}$, $(\tilde{w}_1, \tilde{w}_2) \in \mathbb{R}^2$

$\|(w_1, w_2)\| \leq \|(w_1', w_2')\| \quad \forall (w_1', w_2') \in (\mathrm{cint}_{\mathbb{C}\cdot\mathbb{R}^2}\, g) \setminus \{0\} \}$


Thm If $g(x_1,x_2)+\frac{x_1^2}{2}+\frac{x_2^2}{2}$ is a homogeneous polynomial,
all points in Ψ are non-degenerate (i.e. $\det H_g(w_1,w_2) \neq 0$)
then for large n
$A_n \sim \frac{1}{2\pi}\Gamma(n)\sum_{(w_1,w_2)\in\Psi}\frac{(-g(w_1,w_2))^n}{\sqrt{-\det H_g(w_1,w_2)}}$
Eg $g(x_1,x_2) = -\frac{x_1^2}{2}-\frac{x_2^2}{2}+\frac{x_1^4}{4!}+\lambda\frac{x_1^2x_2^2}{2!\,2!}+\lambda^2\frac{x_2^4}{4!}$
4-reg graphs
λ
λ^2
$\lambda \in \mathbb{R}_{>0}$,
coupling parameter between colors

... action $S_{s_1} \times \dots \times S_{s_d}$

$\frac{x_2^2}{2}$ is a homogeneous polynomial,

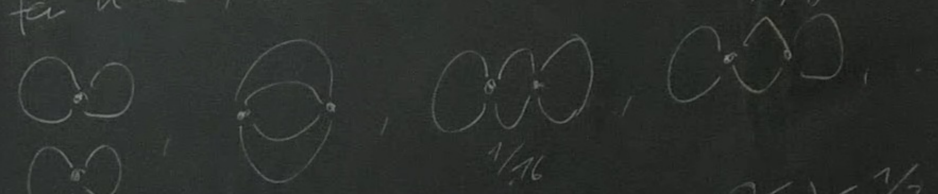
non-degenerate (i.e. $\det H_g(w_1,w_2) \neq 0$)

$$\sum_{(w_1,w_2)\in\Psi} \frac{(-g(w_1,w_2))^n}{\sqrt{-\det H_g(w_1,w_2)}}$$

$$- \frac{x_2^2}{2} + \frac{x_1^4}{4!} + \lambda \frac{x_1^2 x_2^2}{2!\,2!} + \lambda^2 \frac{x_2^4}{4!}$$

$\lambda \in \mathbb{R}_{>0}$, coupling parameter between colors

for $n=2$, $\exists$ 15 such graphs:



$$\Psi = \begin{cases} (\pm\sqrt{6}, 0) \\ \left(\pm\sqrt{(5-3\lambda)/4\lambda}, \dots\right) \\ (0, \pm\sqrt{6}/\lambda) \end{cases}$$

$0 < \lambda < 1/3$

$1/3 < \lambda < 3$

$\lambda > 3$

$\rightsquigarrow$ phase transitions at $\lambda = 1/3$, $\lambda = 3$

critical Ising model of random 4-reg graph

for nonorientable graph G, define partition

funct $Z(G,\lambda) = \sum \lambda^{|E^\gamma|}$

γ Eulerian subgraphs of G

$\deg(v) \equiv 0 \ (2) \ \forall v \in \gamma$

order edges of G according to whether they belong to γ

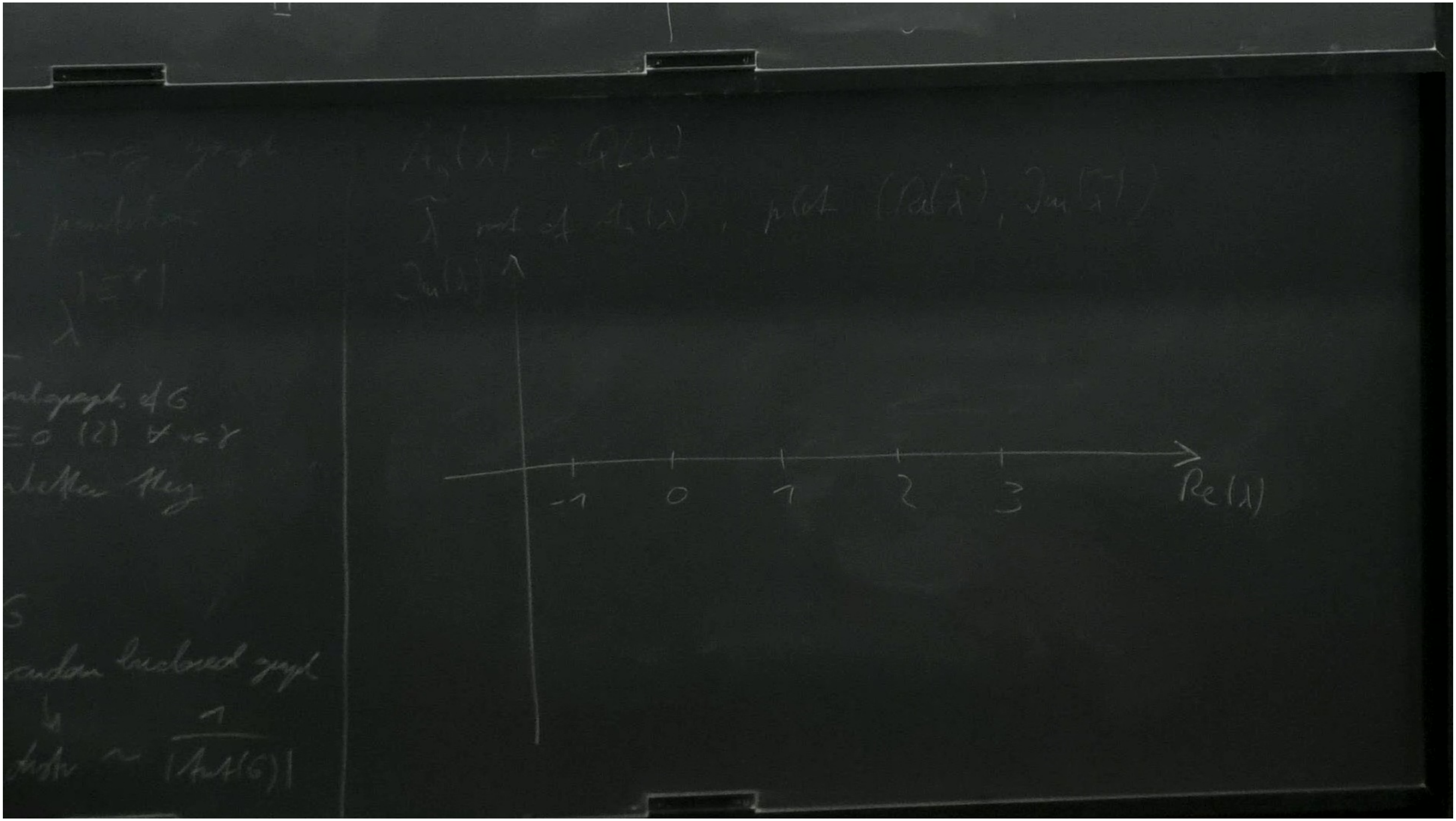
$(G,\gamma) \longleftrightarrow$ bicoloring of G

$$A_4 = \sum_{\substack{G \text{ 4-reg} \\ \chi(G) = -4 \\ \text{vertex weakness by } \partial}} \frac{Z(G,\lambda)}{|\mathrm{Aut}(G)|} \rightsquigarrow \text{random bicolored graph}$$

distr $\sim \frac{1}{|\mathrm{Aut}(G)|}$

$A_4(\lambda) \in \mathbb{Q}[\lambda]$

$\tilde{\lambda}$ root of A_4


-1
0
1
2
3
Re(λ)

... graph
... generating

$|E^c|$

λ

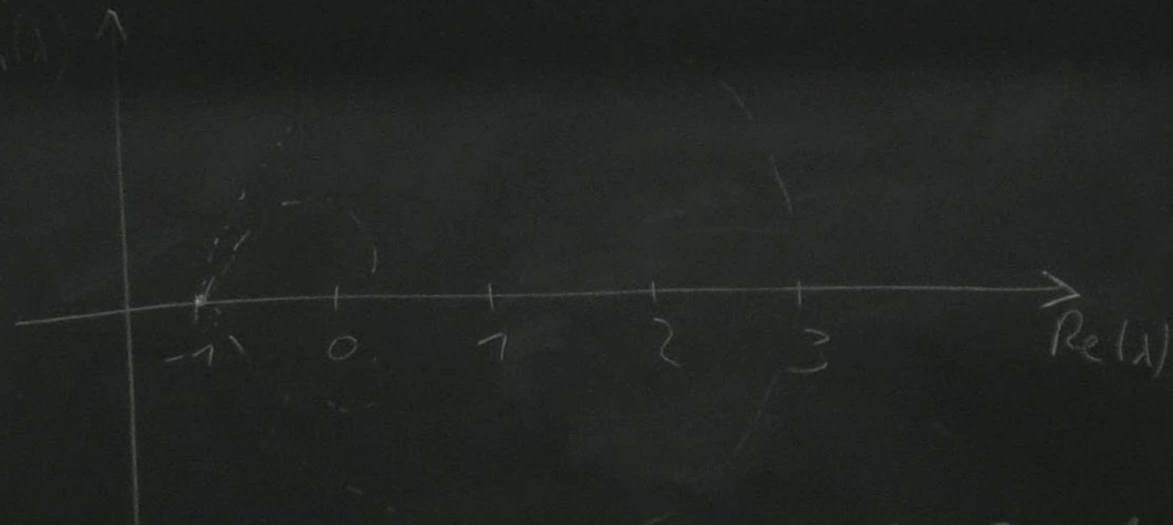
...ulograph of G
$=0$ (2) $\forall$ ...

random bicubed graph

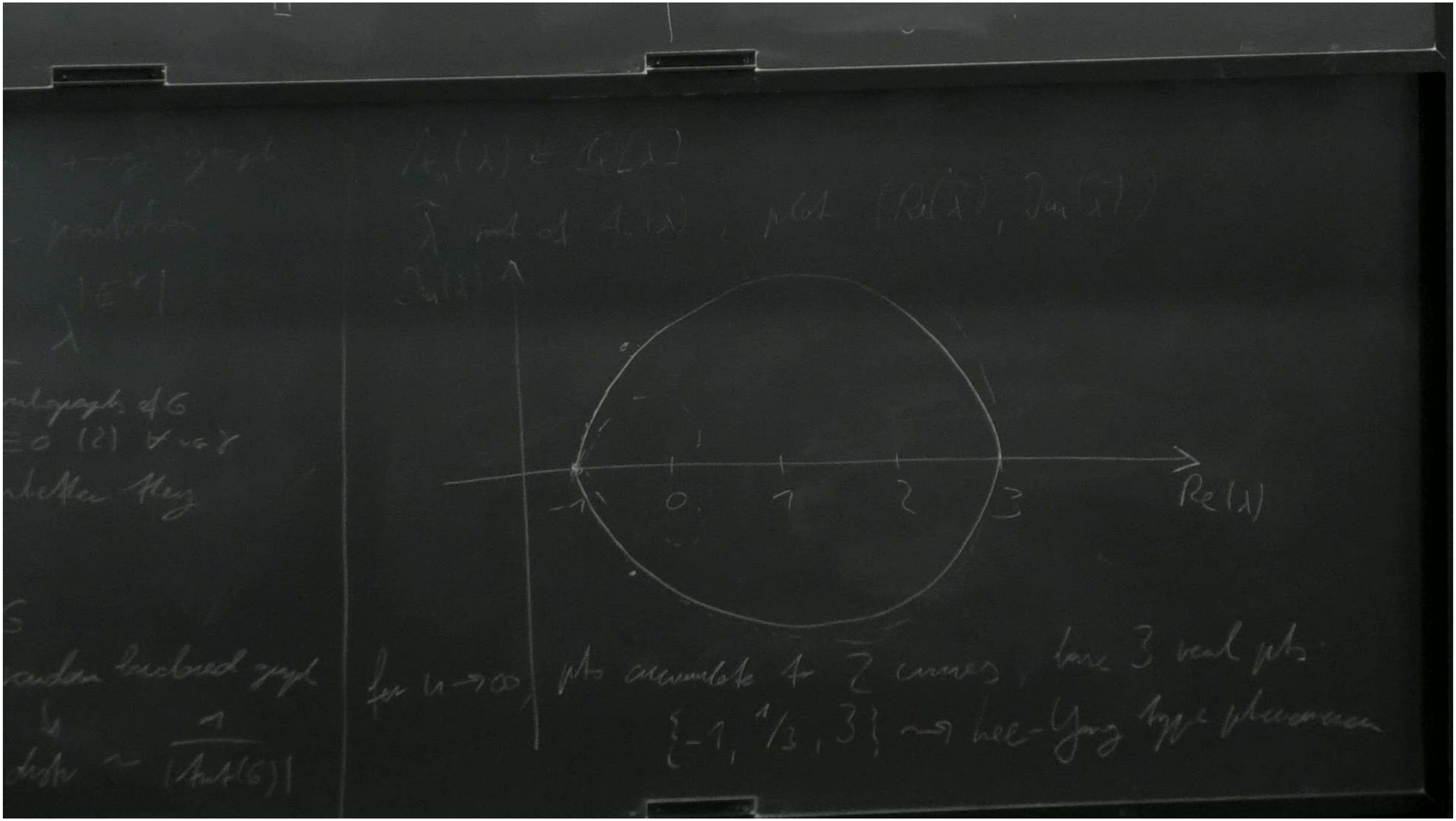
$\sim \frac{1}{|\mathrm{Aut}(G)|}$

$A_n(\lambda) \in \mathbb{Q}[\lambda]$

$\tilde{\lambda}$ root of $A_n(\lambda)$, plot $(\mathrm{Re}(\tilde{\lambda}), \mathrm{Im}(\tilde{\lambda}))$



for $n \to \infty$, pts accumulate to 2 curves, have 3 real pts
$\{-1, 1/3, 3\} \rightsquigarrow$ Lee-Yang type phenomenon



Re(λ)
-1
0
1
2
3
for n→∞, pts accumulate to 2 curves, have 3 real pts
{-1, 1/3, 3} ⇝ Lee-Yang type phenomena