Title: Universal Microscopic Descriptions for Anomalies and Long-Range Entanglement

Speakers: Ryohei Kobayashi

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Abstract:

I will present a unified framework for understanding the statistics and anomalies of excitations—ranging from particles to higher-dimensional objects—in quantum lattice systems. We introduce a general method to compute the quantized statistics of Abelian excitations in arbitrary dimensions via Berry phases of locality-preserving symmetry operations, uncovering novel statistics for membrane excitations. These statistics correspond to quantum anomalies of generalized global symmetries and imply obstructions to gauging, enforcing long-range entanglement. In particular, we show that anomalous higher-form symmetries enforce intrinsic long-range entanglement, meaning that fidelity with any SRE states must exhibit exponential decay, unlike ordinary (0-form) symmetry anomalies. As an application, we identify a new example of (3+1)D mixed-state topological order with fermionic loop excitations, characterized by a breakdown of remote detectability linked to higher-form symmetry anomalies.

Universal Microscopic Descriptions for Anomalies and Long-Range Entanglement

Ryohei Kobayashi (IAS)

w/ Yu-An Chen (PKU), Po-Shen Hsin (KCL), Hanyu Xue (PKU), Yuyang Li (PKU) arXiv: 2412.01886

w/ Po-Shen Hsin (KCL), Abhinav Prem (IAS) arXiv: 2504.10569

Perimeter Institute, Quantum Information Seminar

Statistics of excitations, and Anomalies

Statistics of quasiparticles (anyons): topological order, spin liquids [Wen, Wang=Senthil,…]

Nontrivial statistics often implies nontrivial low-energy spectrum, as only bosons can condense.

Associated with dynamic consequence of 't Hooft anomalies of higher-form symmetries; forbids confined phases

[Gaiotto=Kapustin=Seiberg=Willett,…]

Anomaly and anyon statistics constrain entanglement structure of many-body systems; enforces Long-range entanglement

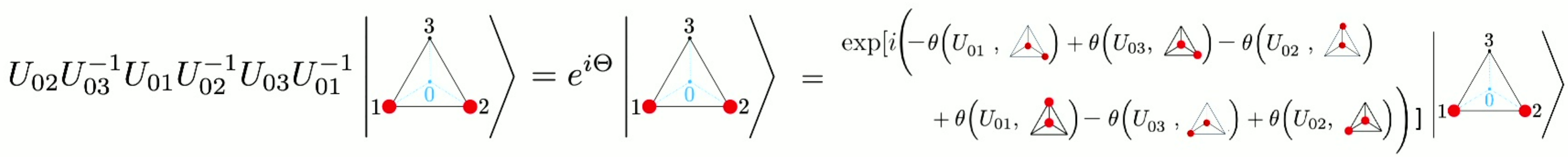
[Bravyi=Hastings=Verstraete, Aharanov=Touati, Li=Lee=Yoshida,…]

Anyons can be non-invertible, but in this talk we are mostly interested in invertible excitations (symmetries).

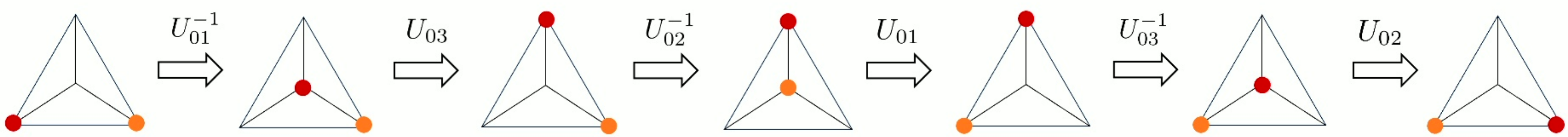
Microscopic definition of statistics

Gapped local Hamiltonian system in (2+1)D: How to define statistics of quasiparticles in microscopic lattice models?

T-junction: [Levin=Wen]



This process indeed does half-braiding of two identical particles:

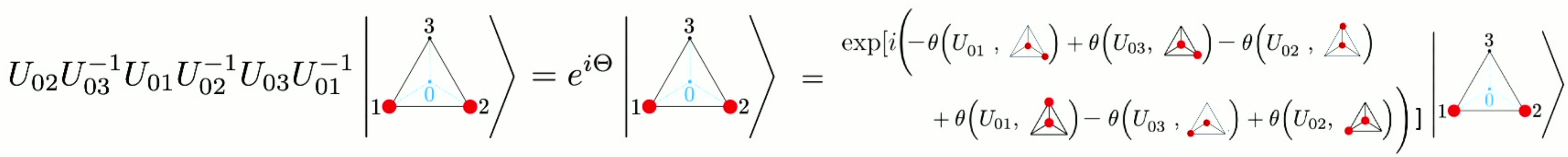


To say it's an invariant, we further need to check stability against perturbations.

Microscopic definition of statistics

Gapped local Hamiltonian system in (2+1)D: How to define statistics of quasiparticles in microscopic lattice models?

T-junction: [Levin=Wen]



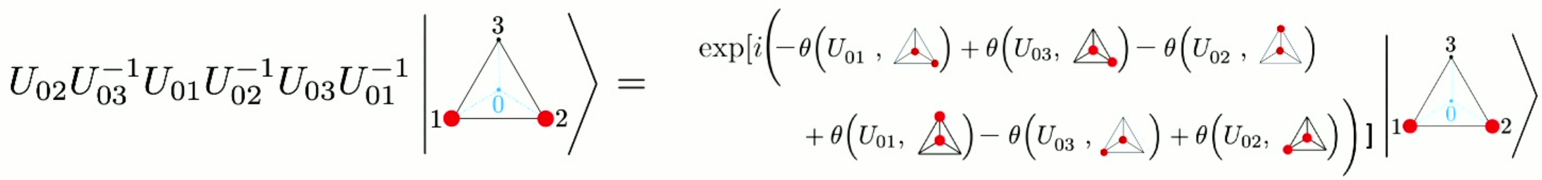
✓ Invariant under choices of unitary by phases, initial excitation configurations

✓ Invariant under perturbations nearby the ends of unitaries

Question: Spins of Abelian anyons should be quantized. Is this T junction a quantized invariant? (cf. Vafa's theorem)

Quantization of T-junction

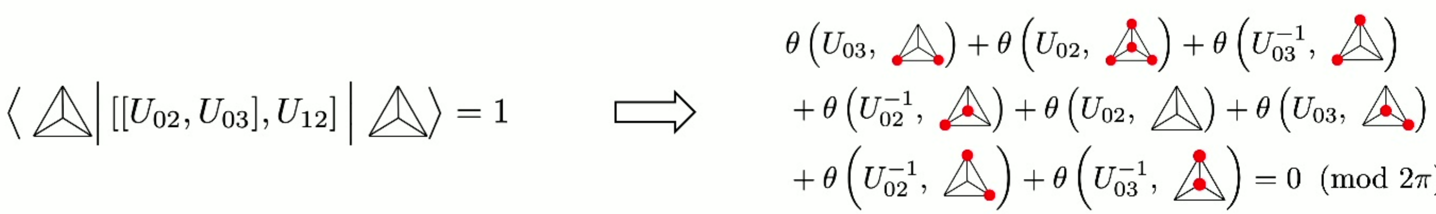
T junction is a quantized invariant. Let's see this explicitly for Abelian anyons with Z2 fusion rule. [RK=Li=Xue=Hsin=Chen]

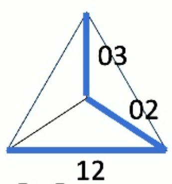


Let's say each unitary is finite depth local circuit.

Key observation is that the triple commutator of operators with no common overlap must vanish:

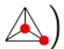
For instance,

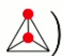
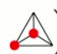




Quantization of T-junction

(4 x T junction) for Z2 Abelian anyons is the combination of triple commutators:

$\exp[4i \Big(\theta\big(U_{01}^{-1},$  $\big) + \theta\big(U_{03},$  $\big) + \theta\big(U_{02}^{-1},$  $\big)$

$+ \theta\big(U_{01},$  $\big) + \theta\big(U_{03}^{-1},$  $\big) + \theta\big(U_{02},$  $\big)\Big)]$

$$= \langle [[U_{02}, U_{03}], U_{12}] \rangle \times \langle [[U_{01}, U_{02}], U_{13}] \rangle \times \langle [[U_{03}, U_{01}], U_{23}] \rangle$$

$$\times \langle [[U_{02}^{-1}, U_{03}^{-1}], U_{12}] \rangle \times \langle [[U_{01}^{-1}, U_{02}^{-1}], U_{13}] \rangle \times \langle [[U_{03}^{-1}, U_{01}^{-1}], U_{23}] \rangle$$

$$\times \langle [[U_{03}, U_{02}], U_{23}] \rangle^2 \times \langle [[U_{02}, U_{01}], U_{12}] \rangle^2 \times \langle [[U_{01}, U_{03}], U_{13}] \rangle^2$$

$$= 1$$

This shows that the spin of Z2 Abelian anyons through T-junction must be quantized as 0, 1/4, 1/2, 3/4.

We will see that such mechanism for quantization is observed in a very general setup.

Quantization of T-junction

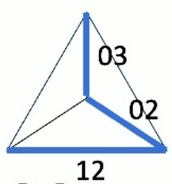
T junction is a quantized invariant. Let's see this explicitly for Abelian anyons with Z2 fusion rule. [RK=Li=Xue=Hsin=Chen]

$$U_{02}U_{03}^{-1}U_{01}U_{02}^{-1}U_{03}U_{01}^{-1}\left|\triangle\right\rangle = \exp[i\Big(-\theta\big(U_{01},\triangle\big)+\theta\big(U_{03},\triangle\big)-\theta\big(U_{02},\triangle\big)+\theta\big(U_{01},\triangle\big)-\theta\big(U_{03},\triangle\big)+\theta\big(U_{02},\triangle\big)\Big)]\left|\triangle\right\rangle$$

Let's say each unitary is finite depth local circuit.

Key observation is that the triple commutator of operators with no common overlap must vanish:

For instance,



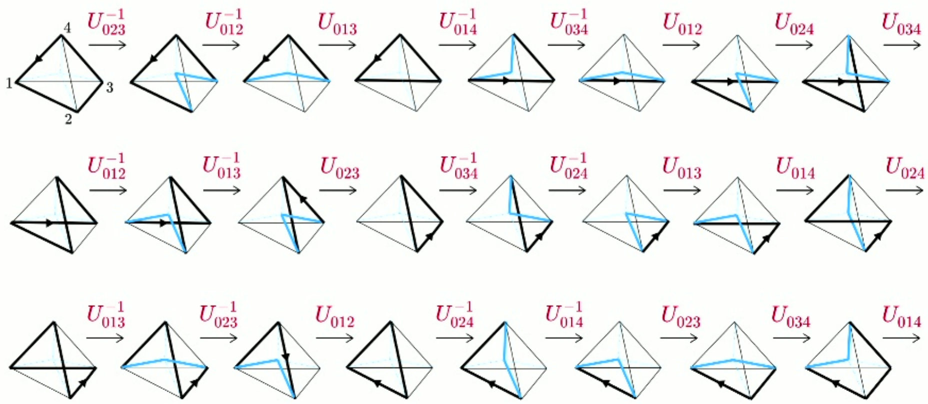
$$\left\langle\triangle\right|[[U_{02},U_{03}],U_{12}]\left|\triangle\right\rangle = 1 \quad\Longrightarrow\quad \begin{aligned}&\theta\left(U_{03},\triangle\right)+\theta\left(U_{02},\triangle\right)+\theta\left(U_{03}^{-1},\triangle\right)\\&+\theta\left(U_{02}^{-1},\triangle\right)+\theta\left(U_{02},\triangle\right)+\theta\left(U_{03},\triangle\right)\\&+\theta\left(U_{02}^{-1},\triangle\right)+\theta\left(U_{03}^{-1},\triangle\right)=0 \pmod{2\pi}\end{aligned}$$

Generalized statistics

Such invariants can be defined in generic space dimensions, with generic invertible extended excitations.

Example: Z2 1-form symmetry in (3+1)D. 24 step unitaries:

$$\begin{aligned}\mu_{24} := \; & U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1} \\ & \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1} \\ & \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1}\end{aligned}$$



"Fermionic loops"

[Thorngren, Chen=Hsin, Fidkowski=Haah=Hastings, RK=Li=Xue=Hsin=Chen]

We will give the general framework for such invariants, and discuss physical consequences.

Framework for Generalized statistics

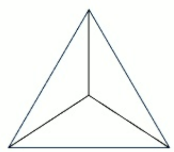
Setup:
- Gapped local lattice system, with tensor product Hilbert space
- Finite invertible p-form symmetry with fusion group G, generated by a finite depth unitary circuit
  (G can be non-abelian w/ p = 0)

End of symmetry operators correspond to the extended excitations.

Input:
- Possible configurations of excitations $\mathcal{A}$ (on a simplicial complex embedded in space): finite group
- Set of symmetry operators $\mathcal{S}$: symmetry generators creating excitation configurations

Example... T junction

- $\mathcal{A}$: G (=ZN) anyon configurations on



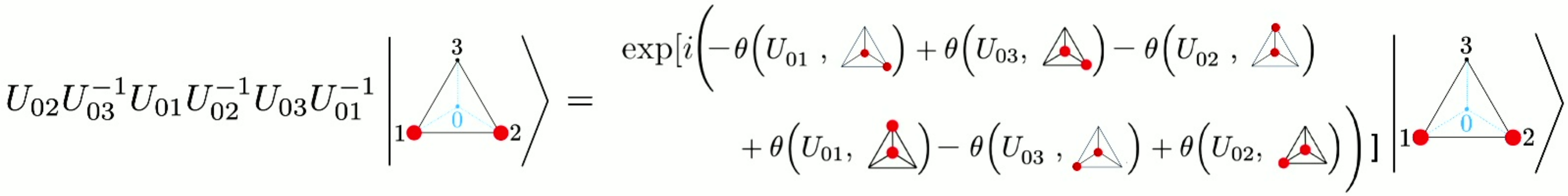
$\mathcal{A} = G^3$

(anyons on four vertices fuse to vacuum)

- $\mathcal{S}$: set of anyon string operators on edges. Six generators of G^6 (# of edges) $\partial : \mathcal{S} \to \mathcal{A}$

Framework for Generalized statistics

Invariant is a sequence of unitaries acting on a state, getting back to the original one



In general, it is sum of the phases $\theta(s,a)$ $\quad s \in \mathcal{S}, a \in \mathcal{A}$

$$U(s)\,|a\rangle = \exp(i\theta(s,a))\,|a+\partial s\rangle$$

It is convenient to introduce a formal sum of the objects $E = \bigoplus_{s\in\mathcal{S},a\in\mathcal{A}} \mathbb{Z}\theta(s,a)$

The invariant is formulated as a specific subgroup $E_{\mathrm{inv}} \subset E$

(Let us restrict ourselves to the Abelian fusion group G in this talk. Can be safely generalized to non-Abelian groups.)

Group of invariants: $E_{\text{inv}} \subset E$

The condition for being an invariant: Linear constraints on integer coefficients $\epsilon(s,a)$ of $E = \bigoplus_{s\in\mathcal{S}, a\in\mathcal{A}} \mathbb{Z}\theta(s,a)$

1. The invariant corresponds to sequence of unitaries, with same initial and final state (Berry phase).

$$\sum_{s\in\mathcal{S}} \epsilon(s,a) - \sum_{s\in\mathcal{S}} \epsilon(s, a-\partial s) = 0, \quad \text{for any } a \in \mathcal{A} \ .$$

2. The invariant has to be stable against phase redefinitions of the unitary operators.

$$\sum_{a\in\mathcal{A}} \epsilon(s,a) = 0, \quad \text{for any } s \in \mathcal{S} \ .$$

3. The invariant has to be stable against perturbations nearby the boundaries of unitary operators.

$$\sum_{\substack{a\in\mathcal{A} \\ a|_{\sigma_j} = a_*^{(j)}}} \epsilon(s,a) = 0 \ , \qquad \sigma_j \in \text{supp}(s)$$

(Stability against perturbations within a j-simplex σ_j)

(uses exponentially decaying correlation length = gapped)

The three types of linear constraints together define $E_{\text{inv}} \subset E$

Trivial invariants from locality: $E_{\mathrm{id}} \subset E_{\mathrm{inv}}$

Some invariants $e \in E_{\mathrm{inv}}$ correspond to the trivial invariants (identity).

Trivial invariants originate from higher commutator:

$$\langle a| \, [[[U(s_1), U(s_2)], \cdots], U(s_n)] \, |a\rangle = 1 \qquad\qquad \mathrm{supp}(s_1) \cap \cdots \cap \mathrm{supp}(s_n) = \varnothing$$

Let $E_{\mathrm{id}} \subset E_{\mathrm{inv}}$ be the group of higher commutators. Then define generalized statistics as

$$T = E_{\mathrm{inv}} / E_{\mathrm{id}}$$

Though E_{inv} is an infinite group (direct sum of integers), the genuine invariant T is a finite Abelian group.

Invariants are torsions, and quantized.

Quantization of Generalized statistics

Let's explicitly show that the invariant $T = E_{\text{inv}}/E_{\text{id}}$ is a finite group (torsion).

First, one can show that the equivalence class $[e] \in E_{\text{inv}}/E_{\text{id}}$ doesn't depend on initial state, i.e., the ratio

$$\frac{\langle a_0| \prod U(s_j)^{\pm} |a_0\rangle}{\langle a_0'| \prod U(s_j)^{\pm} |a_0'\rangle} \in E_{\text{id}}$$ for any pair of initial states.

In other words, it is equal to product of higher commutators, and actually $\frac{\langle a_0| \prod U(s_j)^{\pm} |a_0\rangle}{\langle a_0'| \prod U(s_j)^{\pm} |a_0'\rangle}$ = 1

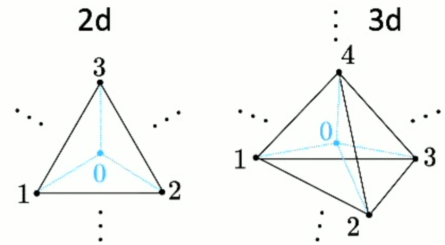
Then, sum up the phase over all choices of initial states:

$$\begin{aligned} |\mathcal{A}|[e] &= \sum_{a_0\in\mathcal{A}} \sum_{(s,a)} \epsilon(s,a)\theta(s,a+a_0) \\ &= \sum_{a_0\in\mathcal{A}} \sum_{(s,a)} \epsilon(s,a-a_0)\theta(s,a) \quad = \sum_{(s,a)} \left(\sum_{a_0\in\mathcal{A}} \epsilon(s,a_0) \right) \theta(s,a) = 0 \end{aligned}$$

$\Rightarrow$ [e] has finite order,

Showing T is a finite group

Conjecture: Generalized Statistics = Group Cohomology



Take a triangulation on a sphere embedded in d dimensional space.

p-dimensional excitation ((d-p-1)-form symmetry) with fusion group G.

The invariants can be systematically evaluated on computer using Smith normal form.

Then, computation results imply the correspondence with the group cohomology: $T = H^{d+2}(B^{d-p}G, U(1))$

	G-particles with $G=\prod_i \mathbb{Z}_{N_i}$	G-loops with $G=\prod_i \mathbb{Z}_{N_i}$	G-membranes with $G=\prod_i \mathbb{Z}_{N_i}$
(1+1)D	$H^3(BG,U(1)) = \prod_i \mathbb{Z}_{N_i} \prod_{i<j} \mathbb{Z}_{(N_i,N_j)} \prod_{i<j<k} \mathbb{Z}_{(N_i,N_j,N_k)}$		
(2+1)D	$H^4(B^2G,U(1)) = \prod_i \mathbb{Z}_{(N_i,2)\times N_i} \prod_{i<j} \mathbb{Z}_{(N_i,N_j)}$	$H^4(BG,U(1)) = \prod_{i<j} \mathbb{Z}^2_{(N_i,N_j)} \prod_{i<j<k} \mathbb{Z}^2_{(N_i,N_j,N_k)} \prod_{i<j<k<l} \mathbb{Z}_{(N_i,N_j,N_k,N_l)}$	
(3+1)D	$H^5(B^3G,U(1)) = \prod_i \mathbb{Z}_{(N_i,2)}$	$H^5(B^2G,U(1)) = \prod_i \mathbb{Z}_{(N_i,2)} \prod_{i<j} \mathbb{Z}_{(N_i,N_j)}$	$H^5(BG,U(1)) = \prod_i \mathbb{Z}_{N_i} \prod_{i<j} \mathbb{Z}^2_{(N_i,N_j)} \prod_{i<j<k} \mathbb{Z}^4_{(N_i,N_j,N_k)} \prod_{i<j<k<l} \mathbb{Z}^3_{(N_i,N_j,N_k,N_l)} \prod_{i<j<k<l<m} \mathbb{Z}_{(N_i,N_j,N_k,N_l,N_m)}$

⇨ Verified for small groups G.

For instance, with d = 2, p = 0, G = ZN (anyons),

$$T = \mathbb{Z}_{2N} \quad \text{even } N$$
$$T = \mathbb{Z}_N \quad \text{odd } N$$

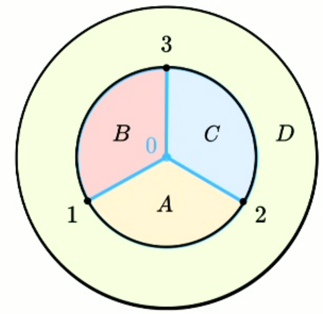
Spin quantization rule of anyons;
Checked up to N = 10 on laptop.

Examples of invariants

- 1+1D: 0-form ZN symmetry $Z_3(g) := [U(g)_{01}^{|g|}, U(g)_{02}]$

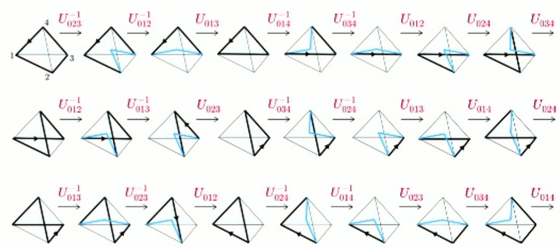
- 2+1D: 0-form ZN x ZN symmetry

$$Z_4^I(a,b) := (U(a)_{B+C})^{-N}\Big(U(a)_{B+C}\,[U(a)_B, [U(a)_A, U(b)_{A+B+C+D}]]\Big)^N,$$
$$Z_4^{II}(a,b) := (U(b)_{B+C})^{-N}\Big(U(b)_{B+C}\,[U(b)_B, [U(b)_A, U(a)_{A+B+C+D}]]\Big)^N.$$



- 3+1D: 1-form ZN symmetry

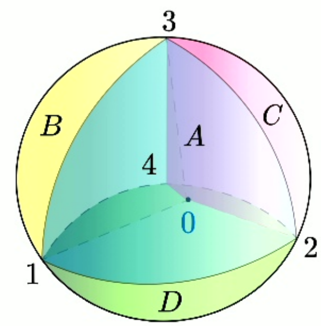
$$\begin{aligned}\mu_{24} := {} & U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1}\\ & \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1}\\ & \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1}\end{aligned}$$



"Fermionic loops" for N = 2

0-form ZN symmetry

$$Z_5(g) := (U(g)_{0234}U(g)_{0124})^{-N}\,\big(U(g)_{0234}[U(g)_{0134}, U(g)_{0123}^N]^{-1}U(g)_{0124}[U(g)_{0134}, U(g)_{0123}^N]\big)^N$$



Generalized statistics as anomalies: obstruction to gauging

The nontrivial invariant is directly regarded as obstruction to gauging the symmetry.

A take is that the product of unitaries $\langle a_0 | U(s_{n-1})^{\pm} \dots U(s_j)^{\pm} \dots U(s_0)^{\pm} | a_0 \rangle$ is the product of Gauss law operators.

$$G(\Delta) = 1, \quad U(s) = \prod_{\Delta \in s} G(\Delta)$$

Gauss law operator on local simplex Δ, and the unitary is product of Gauss laws

It means that the invariant obstructs commuting Gauss laws within the initial symmetric state.

⇨ Obstruction to gauging the symmetry = Microscopic definition of ‘t Hooft anomalies [Else=Nayak, Kawagoe=Levin...]

Generalized statistics as anomalies: dynamical consequences

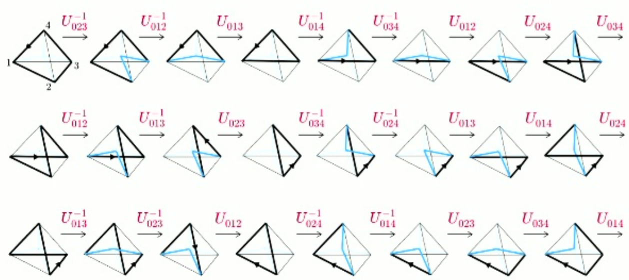
Generalized statistics is understood as the 't Hooft anomaly.

Indeed, generalized statistics has a direct dynamical consequence (similar to Lieb-Schultz-Mattis): [Lieb=Schultz=Mattis, Oshikawa=Hastings,...]

Generalized statistics $T \neq 1$ on the symmetric state $|\Psi\rangle$ implies that the state cannot be short-range entangled.

(i.e., cannot be connected to tensor product state by finite depth circuit)

For instance, Z2 1-form symmetry in (3+1)D:



$$\begin{aligned}\mu_{24} := \; & U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1} \\ & \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1} \\ & \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1}\end{aligned} = -1$$

$\Longrightarrow$ LRE state

Such result has been known for anyons in (2+1)D: T-junction must be trivial on SRE states [Bravyi=Hastings=Verstraete, Aharanov=Touati, Li=Lee=Yoshida]

Example: Fermionic loops imply long-range entanglement

Let's consider Z2 1-form symmetry in (3+1)D:

One can show that

$$\begin{aligned} \mu_{24} := \ & U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1} \\ & \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1} \\ & \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1} \end{aligned}$$

becomes trivial on symmetric SRE states.

Let's consider 3d SRE state $|\psi\rangle$ w/ Z2 1-form symmetry.

Then, each state $U|\psi\rangle$ can be taken to be a trivial product state away from excitations:

$$|\partial s\rangle := \quad U(s)|\psi\rangle = |a\rangle_{\partial s} \otimes |0\rangle_{\overline{\partial s}}$$

(up to finite depth circuit)

One can show that the generalized statistics becomes trivial for such effective 1d state (uses MPS rep of excitations).

Higher-form anomalies: Intrinsic long-range entanglement

For p-form symmetry with p >= 1, generalized statistics puts much tighter constraint on entanglement structure.

For symmetric gapped states $|\Psi\rangle$ one can show that

$$U_\Theta |\Psi\rangle = e^{i\Theta} |\Psi\rangle, \quad e^{i\Theta} \neq 1 \quad \Rightarrow \quad \langle \Psi | \mathrm{SRE} \rangle = O(L^{-\infty})$$

Generalized statistics

any SRE
Circuit depth < O(L)

[Hsin=RK=Prem, Li=Lee=Yoshida]

i.e., if generalized statistics on a symmetric state is nontrivial, overlap of $|\Psi\rangle$ with any SRE states decays exponentially.

"Intrinsic long-range entanglement"

This constraint is only valid for higher-form symmetry. (0-form anomalies are matched by symmetric cat state)

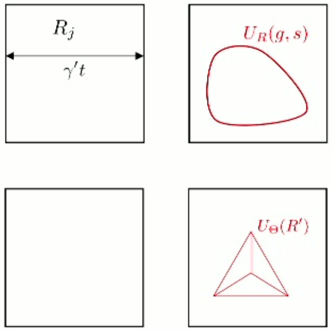
Proof of intrinsic long-range entanglement from higher-form anomalies

1. Separate the system into disjoint disks R_j. Each disk support closed symmetry operators.

   Higher-form symmetry is a strong symmetry of reduced density matrix ρ.

2. One can define generalized statistics invariant within each disk R_j.

   At each disk, the Schmidt state at R_j for each ensemble of ρ is not SRE.



3. The difference between SRE state can be said for each disk, and as a whole leads to exponential decay of overlap:

$$\langle \Psi | \mathrm{SRE} \rangle = O(L^{-\infty})$$

[Hsin=RK=Prem, Li=Lee=Yoshida]

Importance of higher-form symmetry:

1. 0-form symmetry doesn't generate symmetry at entangling surface
2. Even when it does, it is weak symmetry in general (e.g., SSB)

Higher-form anomalies: Intrinsic mixed state topological order

Intrinsic LRE leads to interesting mixed phases of matter

[Ellison=Cheng, Sohal=Prem, Wang=Wu=Wang, Lessa=Sang=Lu=Hsieh=Wang,...]

Phases can be classified through two-way finite depth local quantum channel between two mixed states

If a mixed state ρ has strong anomalous p-form symmetry w/ nontrivial generalized statistics $U\rho \propto \rho$,

$$\mathcal{F}(\rho, \sigma_{\text{SRE}}) = O(L^{-\infty})$$

$$\sigma_{\text{SRE}} = \sum_j \alpha_j \left|\text{SRE}\right\rangle_j \left\langle\text{SRE}\right|_j$$

i.e., fidelity between ρ and any mixed SRE state exponentially decays wrt system size.

Enforced long-range entanglement from higher-form anomalies: protects nontrivial mixed phases of matter

Intrinsic mixed state topological order in (3+1)D Z2 toric code

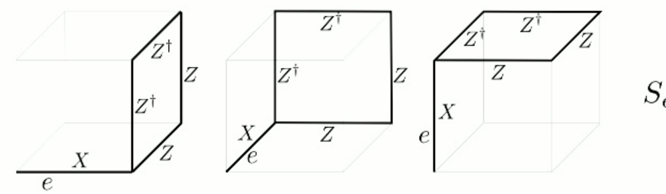
For instance, let's consider (3+1)D Z2 toric code. We define it with Z4 qudits for technical purpose:

$$H_{\mathrm{TC}} = -\sum_e X_e^2 - \sum_v (A_v + A_v^\dagger) - \sum_p B_p^2$$

(first term condenses m^2) [Hsin=RK=Prem]

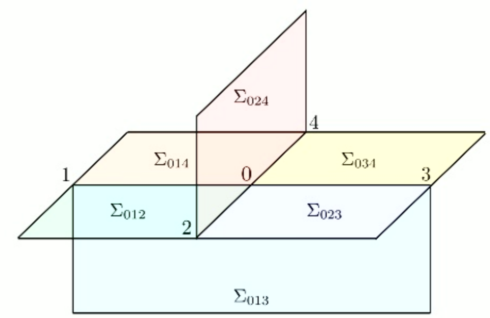
The toric code has anomalous Z2 1-form symmetry:

$$S_{\mathrm{f}}(\Sigma) = \prod_{e \subset \Sigma} S_e,$$



This symmetry carries nontrivial generalized statistics:

$$\begin{aligned} \mu_{24} := \; & U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1} \\ & \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1} \\ & \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1} \end{aligned} = -1$$



(Z4 presentation allows us to write anomalous symmetry in terms of Pauli)

Intrinsic mixed state topological order in (3+1)D Z2 toric code

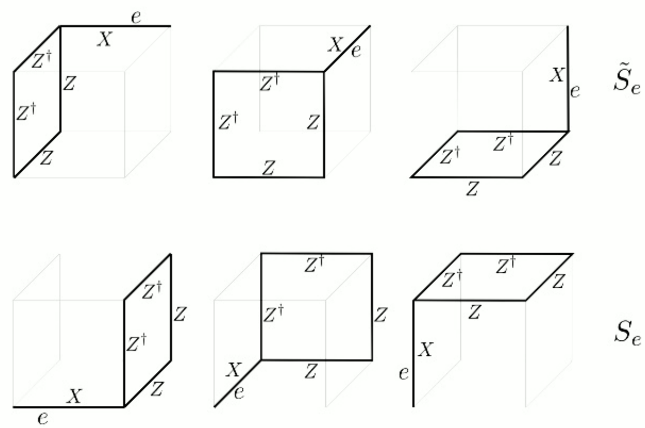
For instance, let's consider (3+1)D Z2 toric code. We define it with Z4 qudits for technical purpose:

$$H_{\mathrm{TC}} = -\sum_e X_e^2 - \sum_v (A_v + A_v^\dagger) - \sum_p B_p^2$$

(first term condenses m²) [Hsin=RK=Prem]

Let's consider the error channel of (3+1)D Z2 toric code:

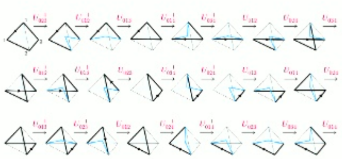
$$\mathcal{N} = \prod_e \mathcal{N}_e, \quad \mathcal{N}_e(\rho) = p\rho + (1-p)\tilde{S}_e \rho \tilde{S}_e^\dagger$$



This preserves strong anomalous (emergent) Z2 1-form symmetry generated by:

$$S_{\mathrm{f}}(\Sigma) = \prod_{e\subset\Sigma} S_e,$$

Generalized statistics enforces LRE and intrinsic mixed TO in decohered phase:



$$\mu_{24} := U_{014}U_{034}U_{023}U_{014}^{-1}U_{024}^{-1}U_{012}U_{023}^{-1}U_{013}^{-1} \times U_{024}U_{014}U_{013}U_{024}^{-1}U_{034}^{-1}U_{023}U_{013}^{-1}U_{012}^{-1} \times U_{034}U_{024}U_{012}U_{034}^{-1}U_{014}^{-1}U_{013}U_{012}^{-1}U_{023}^{-1} = -1 \Longrightarrow$$

Intrinsic LRE in mixed phases

Intrinsic mixed state topological order in (3+1)D Z2 toric code

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Maximally decohered phase has the following property:

- Maximally decohered phase is the nontrivial mixed phase, protected by anomalous 1-form symmetry

- Strong symmetry is a single surface operator generating anomalous Z2 1-form symmetry.

  Forms an algebra (braided fusion 2-category) that violates remote detectability, which cannot be found in pure phases

  (In pure phases, found in boundary of Walker-Wang type model, i.e., 2-form Z2 gauge theory in 4+1 spacetime dim)

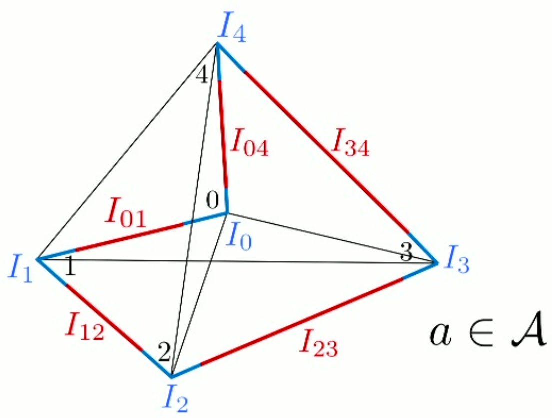
Summary

- Universal microscopic descriptions for statistics of invertible deconfined excitations
- Generalized statistics is quantized, and systematically computed using Smith normal form
- Generalized statistics gives microscopic definition of anomalies, and constrains low-energy spectrum
- Generalized statistics enforces intrinsic long-range entanglement, and leads to new mixed phases of matter

Future directions

- Gapless systems? We assumed gapped system, but hopefully one can formulate invariants w/o reference to states.
- Non-invertible symmetries / non-Abelian anyons? Is there analogue of higher commutators of unitaries?
- Proof for the correspondence between generalized statistics and cohomology? $T = H^{d+2}(B^{d-p}G, U(1))$
- Comprehensive understanding of mixed phases using theories without remote detectability?

Fermionic loops imply long-range entanglement



Each excited state in SRE is the 1d MPS state along excitations.

Let's consider a "patchwork" of MPS:

For instance, $$|a\rangle = \mathrm{Tr}\left[V^0 E^{01} V^1 E^{12} V^2 E^{23} V^3 E^{34} V^4 E^{40}\right]$$

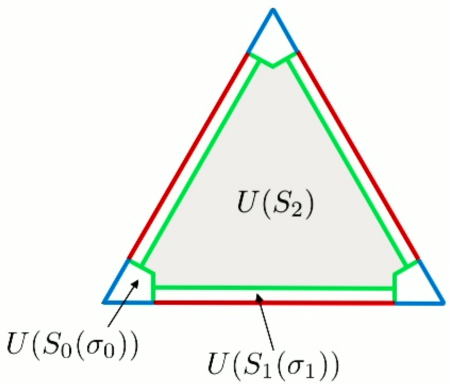
MPS V only depends on excitation configuration near a vertex, and E only depends on those near an edge.

This patchwork representation allows us to construct a canonical choice of excited state $|a\rangle$ for generic configuration.

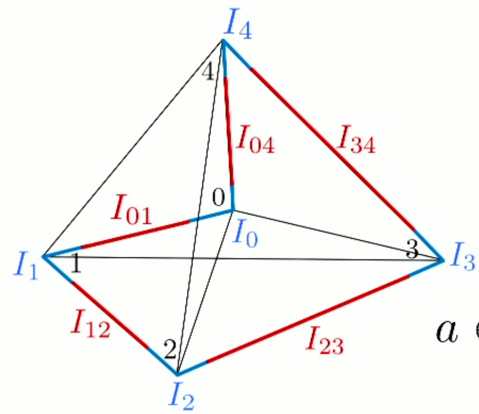
This specific structure of an excited state again greatly constrains the Berry phase $U(s)\,|a\rangle = \exp(i\theta(s,a))\,|a+\partial s\rangle$

Fermionic loops imply long-range entanglement

The symmetry operator also decomposes into circuits near vertex, edge, bulk.



$$U_{jkl} = U_j^{(0)} U_k^{(0)} U_l^{(0)} U_{jk}^{(1)} U_{kl}^{(1)} U_{jl}^{(1)} U_{jkl}^{(2)}$$



$a \in \mathcal{A}$

Berry phase decomposes into smaller part, and each phase only depends on MPS on specific j-simplex:

$$\theta(U_{jkl}, a) = \theta(U_{j;jkl}^{(0)}, a) + \theta(U_{k;jkl}^{(0)}, a) + \theta(U_{l;jkl}^{(0)}, a) + \theta(U_{jk}^{(1)}, a) + \theta(U_{kl}^{(1)}, a) + \theta(U_{jl}^{(1)}, a) + \theta(U_{jkl}^{(2)}, a)$$

Then, invariance under local perturbations at j-simplex enforces the Berry phase on each j-simplex to cancel out.

One can then show $e \in E_{\text{inv}}$ has trivial invariant on SRE.