Title: Numerical Methods in (Loop) Quantum Gravity

Speakers: Dongxue Qu

Collection/Series: Quantum Gravity

Subject: Quantum Gravity

Date: May 08, 2025 - 2:30 PM

URL: https://pirsa.org/25050022

Abstract:

Numerical methods are powerful tools for advancing our understanding of quantum gravity. In this talk, I will introduce two complementary numerical approaches. The first focuses on solving nonlinear partial differential equations that arise in Loop Quantum Gravity (LQG)-inspired effective models. This framework enables us to investigate the formation and evolution of shock waves in spherically symmetric gravitational collapse. The second approach involves the use of complex critical points, Lefschetz thimble techniques, and the Metropolis Monte Carlo algorithm to study the Lorentzian path integral in Spinfoam models and Quantum Regge Calculus. These methods offer new insights into quantum cosmology and black-to-white hole transitions.

Numerical Methods in (Loop) Quantum Gravity

Dongxue Qu

May. 8th @ Perimeter Institute QG Seminar

& Computing Quantum Gravity Workshop (CQGW), 2025

Motivation & Outline

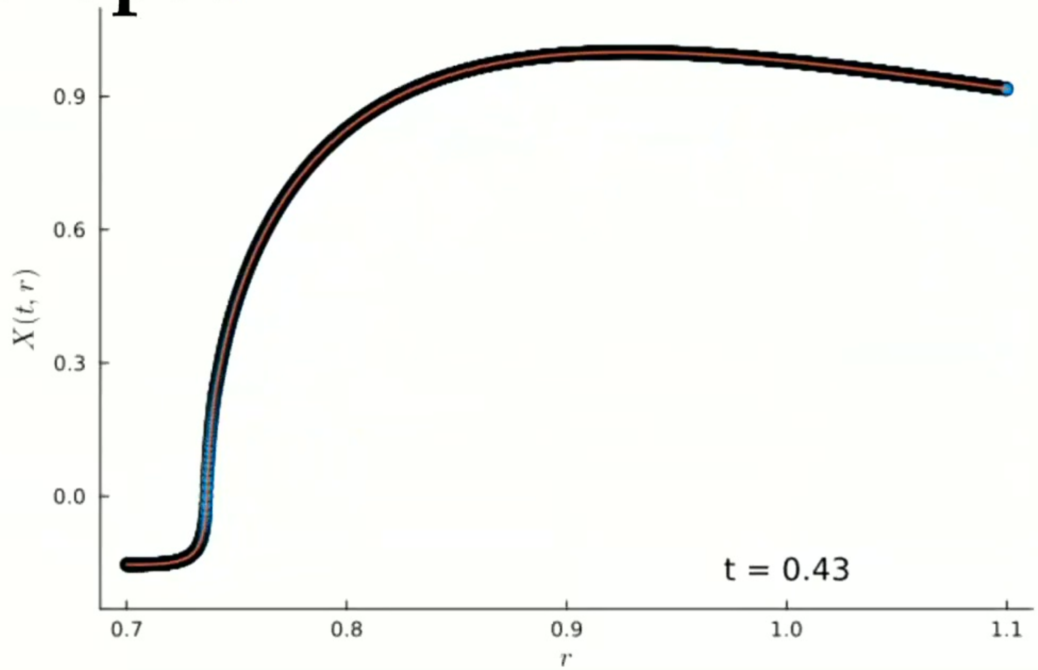
Solving problems beyond analytic reach

- **Numerical algorithms to solve quasi-linear PDEs**
- $\Longrightarrow$ Quantum induced shock dynamics in gravitational collapse
- **Numerical algorithms to compute the Lorentzian path integral**
- The method of complex critical points $\Longrightarrow$ Cosmological Dynamics from Covariant LQG with Scalar Matter
- Markov Chain Monte Carlo computation on a Lefschetz thimble $\Longrightarrow$ Regge Calculus [H. Liu, DQ@ CQGW]
- Wynn-Epsilon algorithm $\Longrightarrow$ Effective Spinfoams and Area Regge calculus [B. Dittrich, J. Padua-Argüelles, S. Asante @ CQGW]

⋮

[W. Sherif, H. Sahlmann, J. Duin, M. Schiffer@ CQGW]

Quantum induced shock dynamics in gravitational collapse



arXiv: 2504.18462(2025.04)

Hongguang Liu and D.Q.

Inspired by a panel discussion with Carlo Rovelli, Viqar Husain, and Edward Wilson-Ewing at the FAU² Workshop, 2024, in Erlangen, Germany

Preliminary

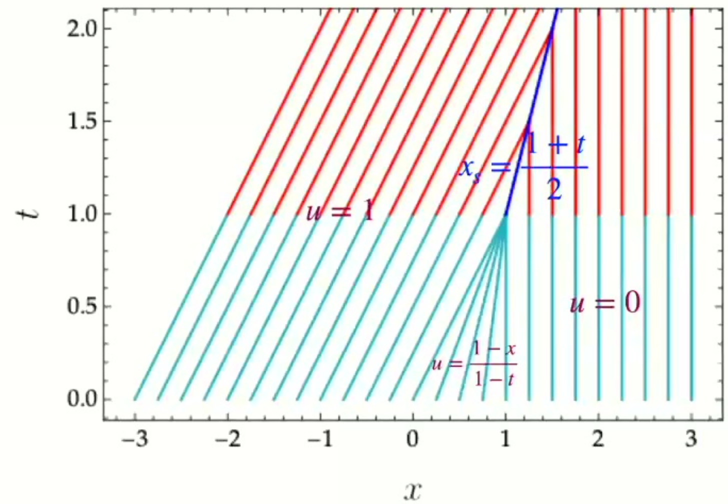
Quasi-linear PDE: $\partial_t u + \partial_x F(u,x) = J(u,x), \quad u(t,x) \in \mathbb{R}$

A toy model (Burger's equation):

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0,$$

Initial Data: $u(t=0,x) = \begin{cases} 1, & x < 0 \\ 1 - x, & 0 \le x < 1, \\ 0, & x > 1 \end{cases}$

Boundary data: $u(t, x_l) = 1, \quad u(t, x_r) = 0$



The method of characteristics(PDE $\Longrightarrow$ ODEs):

$$\begin{cases} \frac{\mathrm{d}t}{\mathrm{d}s} = 1 \\ \frac{\mathrm{d}x}{\mathrm{d}s} = u, \text{ gives the evolution of } u(t,x) \text{ along the trajectories } x(t). \\ \frac{\mathrm{d}u}{\mathrm{d}s} = 0 \end{cases}$$

Characteristics intersect $\Longrightarrow$ No strong (smooth) solutions

Weak solutions: **it may not be continuous but satisfy the integral form of the PDEs.**

u **field develops a moving discontinuity (shock surface x_s) $\Longrightarrow$ shock wave**

[R. P. Newman, P. S. Joshi]

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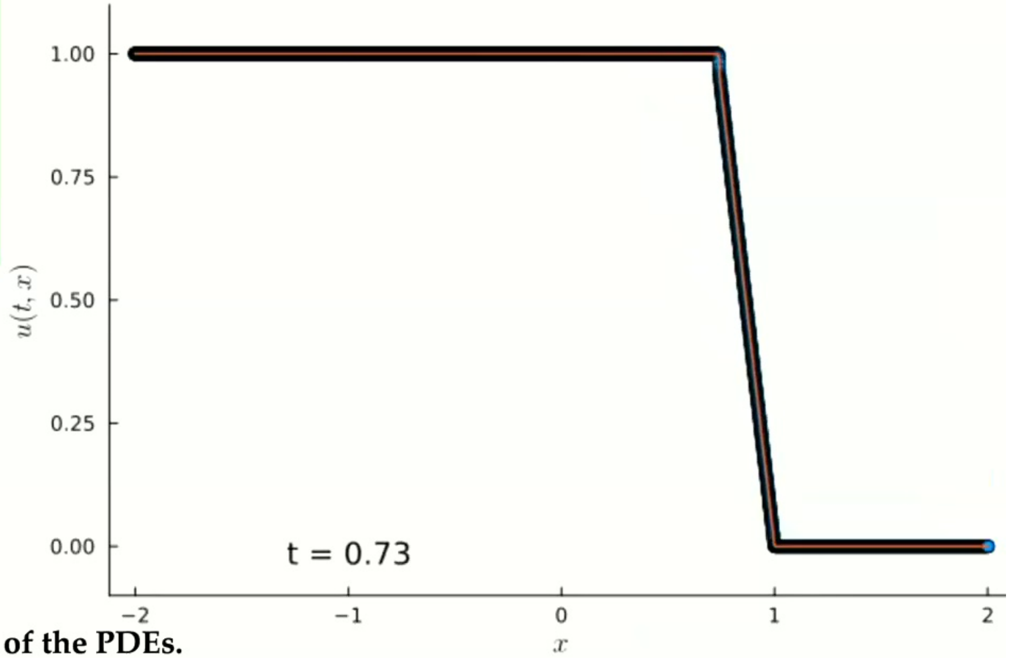
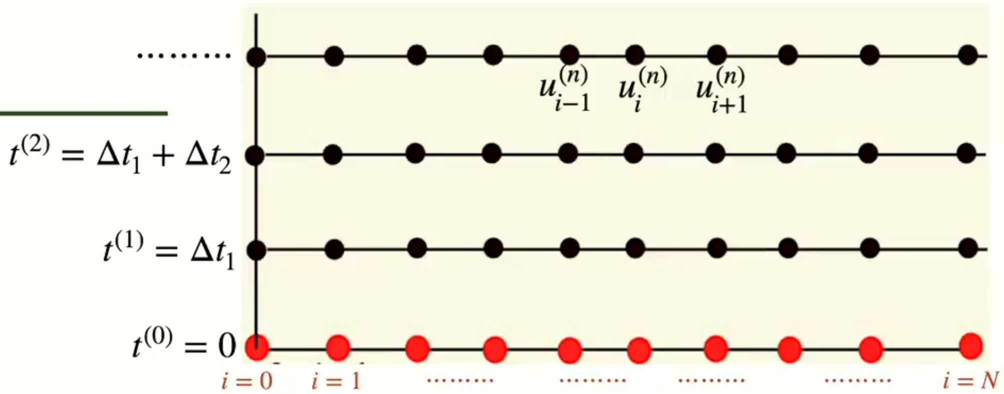


[R. P. Newman, P. S. Joshi]

The general problem of approximating solutions to the form:

$$\partial_t u + \partial_x F(u,x) = J(u,x), \quad x \in \mathbb{R}, \quad u(t,x) \in \mathbb{R}$$

with the given initial data and boundary data.



Spatial Discretiztion

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$$\partial_t u + \partial_x F(u,x) = J(u,x), \quad x \in \mathbb{R}, \quad u(t,x) \in \mathbb{R}$$

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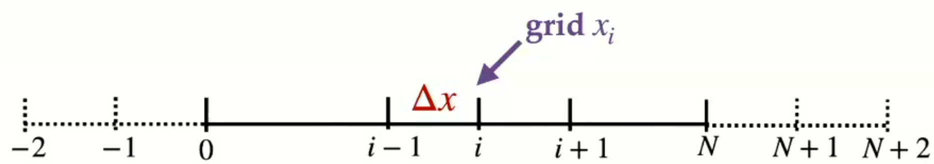
Central-Upwind Schemes: $\left(\partial_x u\right)_i \approx \dfrac{u_{i+1} - u_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$



Spatial Discretiztion

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$$\partial_t u + \partial_x F(u,x) = J(u,x), \quad x \in \mathbb{R}, \quad u(t,x) \in \mathbb{R}$$

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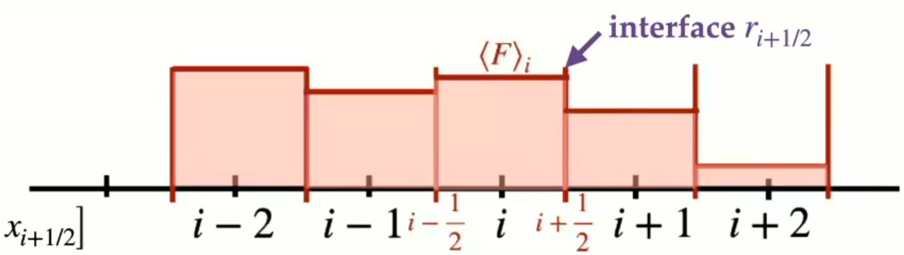
Finite-Volume Method:



Solving conservation laws: $\partial_t u + \partial_x F(u,x) = 0$

Averaging it over a cell $[x_{i-1/2}, x_{i+1/2}]$

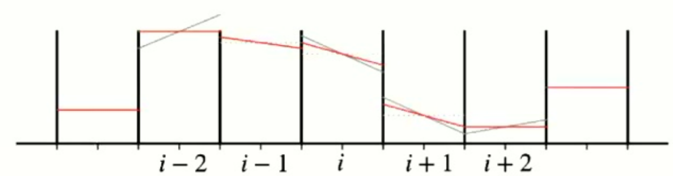
$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}x = -\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial F}{\partial x}\mathrm{d}x\,.$$

$$\frac{\mathrm{d}\langle u \rangle_i}{\mathrm{d}t} = -\frac{1}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right]$$

$$F_{i+1/2} = F(u_{i+1/2}) = \,?$$

$$\mathrm{minmod}(a,b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } a \cdot b > 0 \\ b & \text{if } |b| < |a| \text{ and } a \cdot b > 0 \\ 0 & \text{otherwise} \end{cases}$$

Piecewise linear reconstruction with the minmod slope limiter:

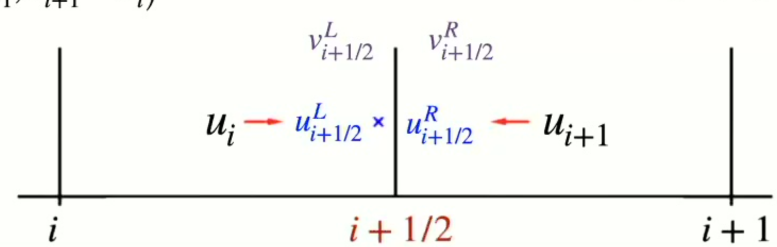


$$u(x) = \frac{\Delta u_i}{\Delta x}\left(x - x_i\right) + u_i, \quad \textbf{with } \Delta u_i = \mathrm{minmod}\left(u_i - u_{i-1}, u_{i+1} - u_i\right)$$

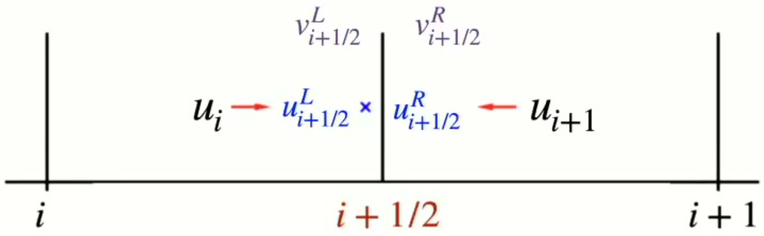
Interface states:

$$u^L_{i+1/2} = u(x_i + \Delta x/2) = u_i + \Delta u_i/2,$$

$$u^R_{i+1/2} = u(x_{i+1} - \Delta x/2) = u_{i+1} - \Delta u_{i+1}/2.$$



Riemann Problem and Shocks



Interface states:

$$u^L_{i+1/2} = u(x_i + \Delta x/2) = u_i + \Delta u_i/2,$$
$$u^R_{i+1/2} = u(x_{i+1} - \Delta x/2) = u_{i+1} - \Delta u_{i+1}/2.$$

Resolve the degeneracy, solve the Riemann problem

1. **No shock or rarefactions:** $u_{i+1/2} = \mathscr{R}\left(u^L_{i+1/2}, u^R_{i+1/2}\right) = \begin{cases} u^L_{i+1/2}, & \textbf{for } v^L_{i+1/2}, v^R_{i+1/2} > 0 \\ u^R_{i+1/2}, & \textbf{for } v^L_{i+1/2}, v^R_{i+1/2} < 0 \end{cases}$

1. **Shock forms** $(v^L_{i+1/2} > v^R_{i+1/2})$: $u_{i+1/2} = \begin{cases} u_s & \textbf{for } v^L_{i+1/2} > v_s > v^R_{i+1/2} \\ u_{\text{other}} & \textbf{otherwise} \end{cases}$, **where** $u_s = \begin{cases} u^L_{i+1/2} & \textbf{if } v_s > 0 \\ u^R_{i+1/2} & \textbf{if } v_s < 0 \end{cases}$ **and** $u_{\text{other}} = \begin{cases} u^L_{i+1/2} & \textbf{if } v^L_{i+1/2} > 0 \\ u^R_{i+1/2} & \textbf{if } v^R_{i+1/2} < 0 \\ 0 & \textbf{otherwise} \end{cases}$

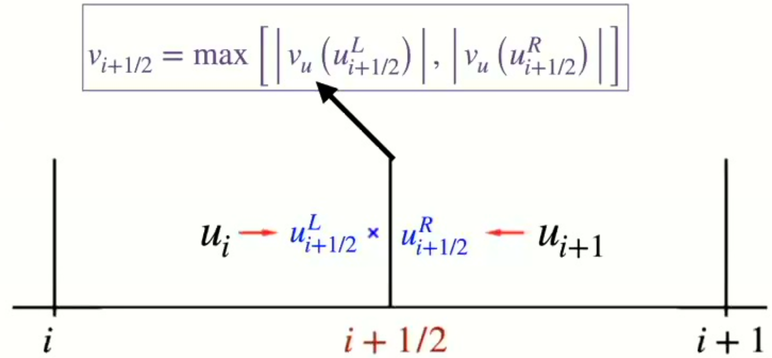
Shock speed (jump condition): $v_s = \dot{x}_s = \dfrac{F(u^R_{i+1/2}) - F(u^L_{i+1/2})}{u^R_{i+1/2} - u^L_{i+1/2}}$

Spatial Discretiztion

Kurganov and Tadmor (KT) Central Scheme:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u\rangle_i = -\frac{1}{\Delta r}\left[F^*_{i+\frac{1}{2}} - F^*_{i-\frac{1}{2}}\right],$$

$$F^*_{i+\frac{1}{2}} = \frac{1}{2}\left[F\left(u^R_{i+\frac{1}{2}}\right) + F\left(u^L_{i+\frac{1}{2}}\right) - v_{i+\frac{1}{2}}\left(u^R_{i+\frac{1}{2}} - u^L_{i+\frac{1}{2}}\right)\right].$$



In our computation, Riemann solver + KT central scheme:

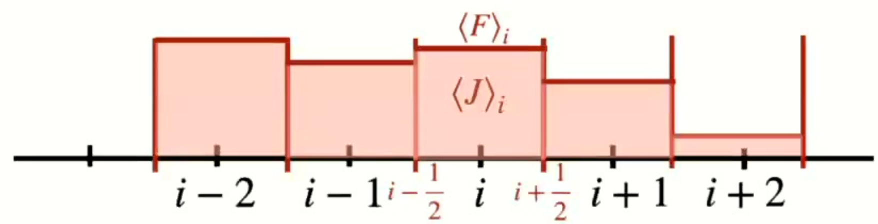
Locations where shocks occur (Riemann solver) — **Smooth regions** (KT central scheme)

Discretization of the Source Term:

Averaging the PDE over a cell $[r_{i-1/2}, r_{i+1/2}]$:

$$\frac{\mathrm{d}\langle u\rangle_i}{\mathrm{d}t} = -\frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} + \langle J\rangle_i$$

$$\langle J\rangle_i := \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} J(t,x)\mathrm{d}x = \frac{1}{2}\left[J(u^L_{i+1/2}, x_{i+1/2}) + J(u^R_{i-1/2}, x_{i-1/2})\right] + \mathcal{O}(\Delta x^3)$$

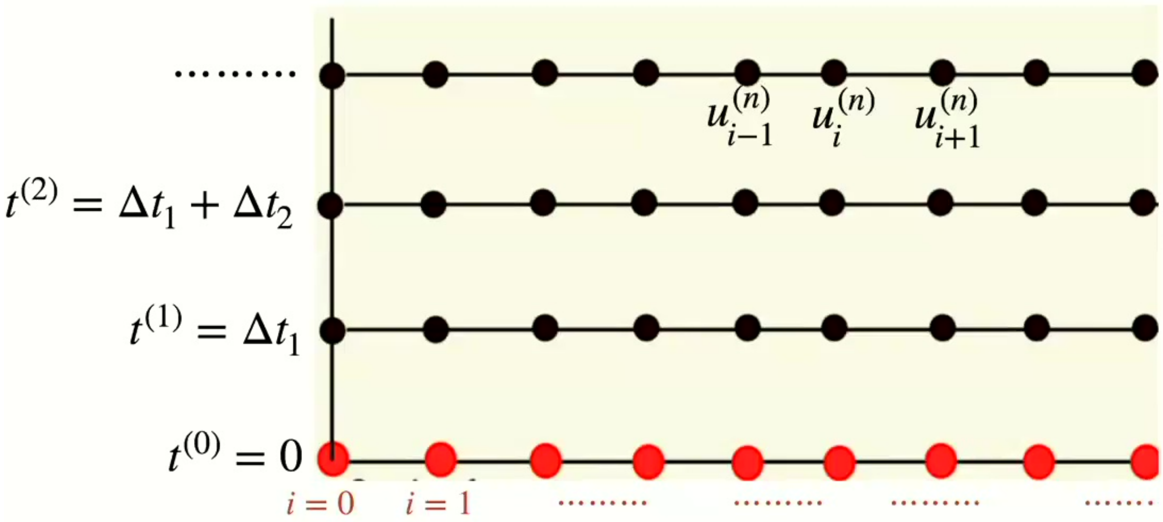


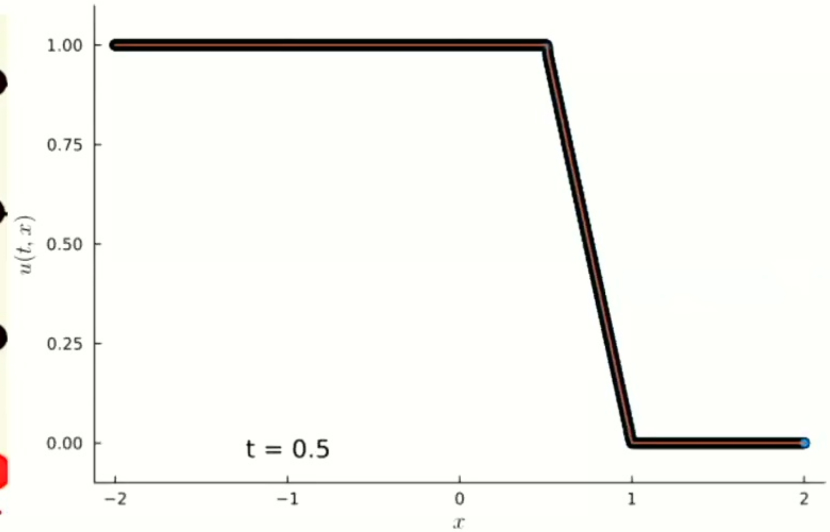
Time update

The second-order Runge–Kutta integrator to update the solution:

$$u_i^{(0)} = u_i^{(n)},$$
$$u_i^{(1)} = u_i^{(0)} + \frac{1}{2} R^j \left(u_i^{(0)} \right) \Delta t,$$
$$u_i^{(n+1)} = u_i^{(0)} + R^j \left(u_i^{(1)} \right) \Delta t$$

R^j **is the spatial operator representing the right-hand side of** $\frac{\mathrm{d}}{\mathrm{d}t} \langle u \rangle_i = - \frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} + \langle J \rangle_i .$





Spherically symmetric spacetime in LTB coordinates

General metric for a spherically symmetric spacetime:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \frac{(E^\phi)^2}{|E^x|}\left(\mathrm{d}x + N^x \mathrm{d}t\right)^2 + |E^x|\,\mathrm{d}\Omega^2 .$$

Lemaître-Tolman-Bondi (LTB) conditions: [M. Bojowald, T. Harada, R. Tibrewala]

$$E^\phi = \frac{E^{x'}}{2\sqrt{1+\epsilon(x)}}, \quad K_x = \frac{K_\phi{}'}{2\sqrt{1+\epsilon(x)}}$$

The effective dynamics of an LQG-inspired model: [K. Giesel, H. Liu, E. Rullit, P. Singh, S. A. Weigl]

$$\mathcal{H}^\Delta_{pri} = \int \mathrm{d}x\ \left(C^\Delta + N^x C_x\right),$$

where $C_x = \frac{1}{G}\left(E^\phi K_\phi' - K_x E^{x\prime}\right)$,

$$C^\Delta = -\frac{E^\phi\sqrt{E^x}}{2G}\left[\frac{3}{\zeta^2}\sin^2\left(\frac{\zeta K_\phi}{\sqrt{E^x}}\right) + \frac{(2E^x K_x - E^\phi K_\phi)}{\zeta\sqrt{E^x}E^\phi}\sin\left(\frac{2\zeta K_\phi}{\sqrt{E^x}}\right) + \frac{1-(\frac{E^{x'}}{2E^\phi})^2}{E^x} - \frac{2}{E^\phi}\left(\frac{E^{x'}}{2E^\phi}\right)'\right]$$

[A. Ashtekar, C. G. Boehmer, M. Bojowald, D.-W. Chiou, A. Corichi, R. Gambini, N. Dadhich, M. Han, A. Joe, L. Modesto, W.-T. Ni, J. Olmedo, J. Pullin, S. Sain, P. Singh, A. Tang, K. Vandersloot]

Spherically symmetric spacetime in LTB coordinates

General metric for a spherically symmetric spacetime:

$$ds^2 = -\,dt^2 + \frac{(E^\phi)^2}{|E^x|}\left(dx + N^x dt\right)^2 + |E^x|\,d\Omega^2 .$$

Spherically symmetric spacetime in LTB coordinates:

$$ds^2 = -\,dt^2 + \frac{(R')^2}{1+\epsilon(x)}dx^2 + R^2 d\Omega^2,$$

Where $R(t,x) \equiv \sqrt{E^x}$ is the areal radius of the shell, and $\epsilon(x) = \begin{cases} < 0, & \text{Spactime is bound} \\ = 0, & \text{Spactime is marginally bound} \\ > 0, & \text{Spactime is unbound} \end{cases}$

[A. Ashtekar, C. G. Boehmer, M. Bojowald, D.-W. Chiou, A. Corichi, R. Gambini, N. Dadhich, M. Han, A. Joe, L. Modesto, W.-T. Ni, J. Olmedo, J. Pullin, S. Sain, P. Singh, A. Tang, K. Vandersloot]

Spherically symmetric spacetime in LTB coordinates

Spherically symmetric spacetime in LTB coordinates:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + (R')^2\mathrm{d}x^2 + R^2\mathrm{d}\Omega^2,$$

The dynamics are given by:

$$\frac{\dot{R}}{R} = \frac{\sin(2\zeta b)}{2\zeta}, \qquad \dot{K}_\phi = -\frac{\frac{3\sin(\zeta b)^2}{\zeta^2} - 2b\frac{\sin(2\zeta b)}{2\zeta}}{2R},$$

Modified Friedmann equation for each x

$$\frac{\dot{R}^2}{R^2}(x) = \frac{\kappa\rho}{6}\left(1 - \frac{\kappa\rho\zeta^2}{6}\right)(x)$$

The energy density of the dust field is $\rho \equiv \frac{3}{4\pi R^3}M'(x)$,

$M(x)$ is the mass function: $2GM(x) \equiv \partial_x C^\Delta = R^3\frac{\sin(\zeta b)^2}{\zeta^2}$.

The general solution is:

$$R(t,x) = \left(2GM(x)\right)^{\frac{1}{3}}\left(\zeta^2 + \frac{9}{4}z^2\right)^{\frac{1}{3}}, \quad z = s(x) - t$$

$$s(x) = \begin{cases} x, & M(x) = const \\ 0 & \text{homogeneous dust case} \end{cases}$$

[A. Ashtekar, C. G. Boehmer, M. Bojowald, D.-W. Chiou, A. Corichi, R. Gambini, N. Dadhich, M. Han, A. Joe, L. Modesto, W.-T. Ni, J. Olmedo, J. Pullin, S. Sain, P. Singh, A. Tang, K. Vandersloot]

Shell-crossing singularities

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Shell-crossing singularities occur in systems with a inhomogeneous dust density distribution. [F. Fazzini, V. Husain, E. Wilson-Ewing]

A quadratic divergence of the Kretschmann scalar:

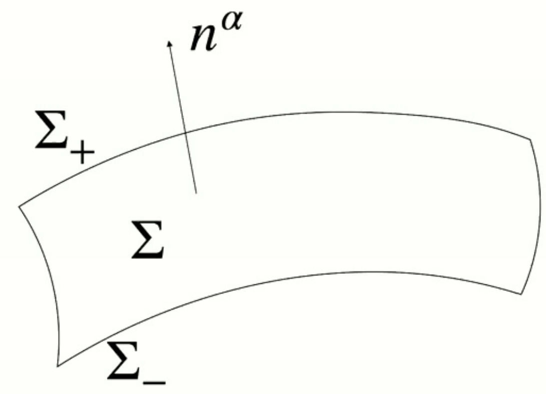
$$\mathscr{K} = R_{abcd}R^{abcd} = \frac{\mathscr{A}}{\left(9(s(x)-t)^2 + 4\zeta^2\right)^4 \mathscr{S}^2},$$ [K. Giesel, H. Liu, P. Singh, and S. A. Weigl]

where $\mathscr{S} = M'(x)\left[9(s(x)-t)^2 + 4\zeta^2\right] + 18M(x)s'(x)\left[s(x)-t\right]$.

$\mathscr{S} = 0$ is **unavoidable if** $9M(x)^2 s'(x)^2 - 4\zeta^2 M'(x)^2 \geq 0$

Shell-crossing singularities occur at $\mathscr{S} = 0$

Spherically symmetric spacetime in PG coordinates



The spherically symmetric spacetime in LTB coordinates:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + (R')^2\mathrm{d}x^2 + R^2\mathrm{d}\Omega^2,$$

To investigate the extension of the metric across singularities:
Junction conditions $R|_{\Sigma_+} = R|_{\Sigma_-}$

The spherically symmetric spacetime in Painlevé-Gullstrand (PG) coordinates $(t, r = R)$**:**

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \left(\mathrm{d}r + N^x\mathrm{d}t\right)^2 + r^2\mathrm{d}\Omega^2,$$

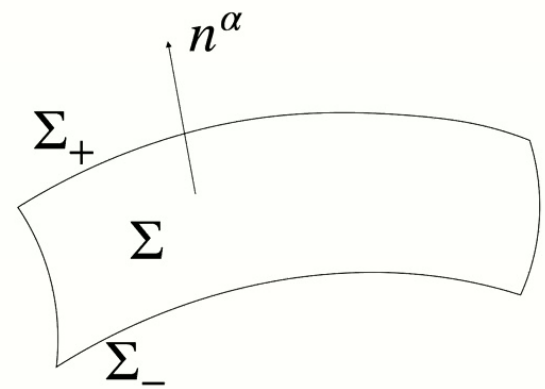
$$N^x = -\,\partial_t R(t,x) = -\mathrm{sgn}_b\sqrt{\frac{2GM}{r}\left(1 - \frac{\zeta^2}{r^2}\frac{2GM}{r}\right)}$$

where the sign function $\mathrm{sgn}_b \equiv \mathrm{sign}(\partial_t R(t,x))$ defines a lift of the square root function to its covering space

The induced metric on Σ is:

$$\mathrm{d}S_\Sigma^2 = \left[-1 + (N^x + r'(t))^2\right]\mathrm{d}t^2 + r^2 d\Omega^2$$

Junction condition and Jump condition



The induced metric on Σ is:

$$dS_\Sigma^2 = \left[-1 + (N^x + r'(t))^2\right] dt^2 + r^2 d\Omega^2$$

The continuity of the induced metric across the junction surface requires:

$[r] = 0, \quad \left[\left(N^x + r'(t)\right)^2\right] = 0.$ **(Junction Condition)** $\Longrightarrow r'(t) = -\dfrac{N_-^x + N_+^x}{2}$

First-order ODE $\rightarrow r = r(t)$ can be uniquely determined

A jump of the quantity A across Σ: $[A] = A|_{\Sigma_+} - A|_{\Sigma_-}$

Jump condition:

For a first-order quasi-linear PDE: $\partial_t u + \partial_r A(u, r) = J(u, r) \Longrightarrow r'(t) = \dfrac{[A]}{[u]}$ **(Jump condition)**

[L. C. Evans]

Quantum gravity-inspired models in PG coordinates : $\partial_t M + N^x \partial_r M = 0 \Longrightarrow r'(t) = \dfrac{\left[\int dM N^x\right]}{[M]}$

[K. Giesel, H. Liu, P. Singh, S. A. Weigl, F. Fazzini, V. Husain, E. Wilson Ewing]

Change of variables N^x as the variable $\Downarrow$

$$\partial_t N^x - \frac{1}{2}\partial_r (N^x)^2 = J(N^x, r),$$

$$J(N^x, r) = \frac{(N^x)^2}{r} + \frac{3\left(\text{sgn}\sqrt{r^2 - 4\zeta^2 (N^x)^2} - r\right)}{4\zeta^2}$$

$\Longrightarrow r'(t) = -\dfrac{N_-^x + N_+^x}{2}$

New variables in the Quantum gravity-inspired models

Junction condition $\Longleftrightarrow$ Jump condition

$$\partial_t N^x - \frac{1}{2}\partial_r (N^x)^2 = J(N^x, r),$$

$$J(N^x, r) = \frac{(N^x)^2}{r} + \frac{3\left(\text{sgn}\sqrt{r^2 - 4\zeta^2 (N^x)^2} - r\right)}{4\zeta^2}$$

Introduce a new variable: $X = N^x r^l$

Numerical computation:
$l = -1, \zeta = 0.5$

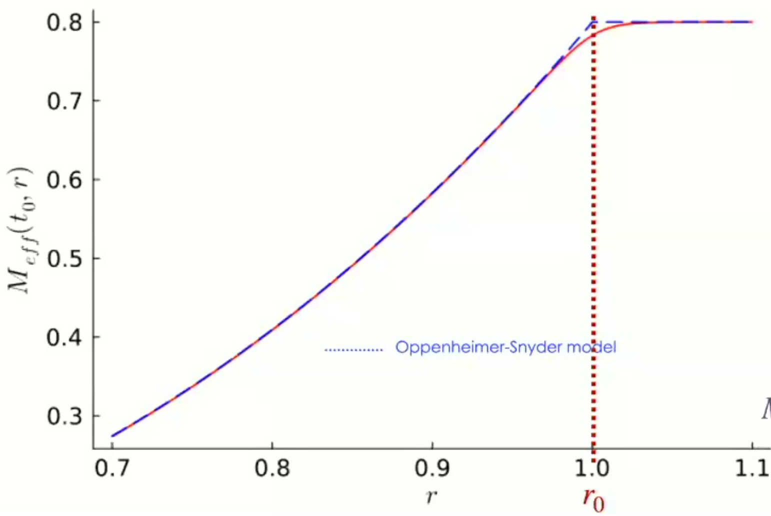
$$\partial_t X + \partial_r F(X, r) = J(X, r),$$

Flux term: $F(X, r) = -\dfrac{X^2}{2r^l}$,

Source term:

$$J(X, r) = \frac{X^2(4 - l)}{2r^{l+1}} + \frac{3\left(\text{sgn}\sqrt{r^{2l+2} - 4\zeta^2 X^2} - r^{l+1}\right)}{4\zeta^2},$$

Velocity of X field: $v_X = -\dfrac{X}{r^l}$.



$$M = \frac{r^2\left(r + \text{sgn}\sqrt{r^2 - 4\zeta^2 r^{-2l} X^2}\right)}{4\zeta^2} \Longrightarrow X(t_0, r) = \sqrt{2Mr^{2l-1} - 4M^2\zeta^2 r^{2l-4}}.$$

New variables in the Quantum gravity-inspired models

Junction condition $\Longleftrightarrow$ Jump condition

$$\partial_t N^x - \frac{1}{2}\partial_r (N^x)^2 = J(N^x, r),$$

$$J(N^x, r) = \frac{(N^x)^2}{r} + \frac{3\left(\text{sgn}\sqrt{r^2 - 4\zeta^2 (N^x)^2} - r\right)}{4\zeta^2}$$

Introduce a new variable: $X = N^x r^l$

$$\partial_t X + \partial_r F(X, r) = J(X, r),$$

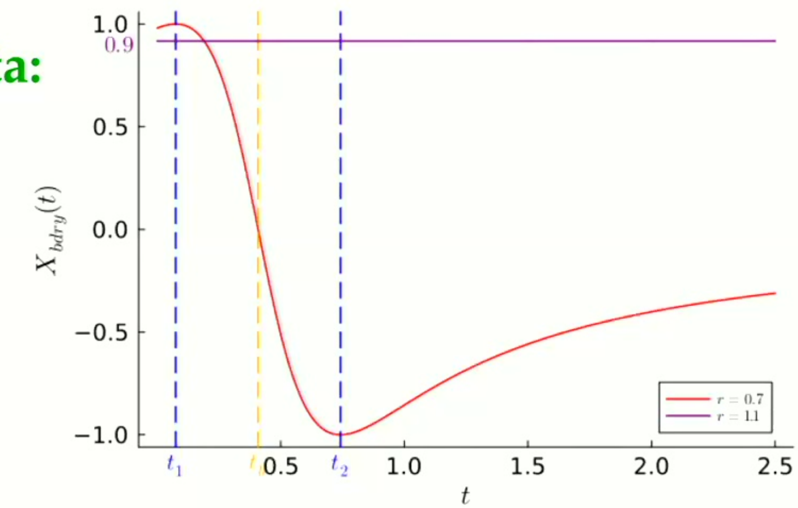
Flux term: $F(X, r) = -\dfrac{X^2}{2r^l}$,

Source term:

$$J(X, r) = \frac{X^2(4-l)}{2r^{l+1}} + \frac{3\left(\text{sgn}\sqrt{r^{2l+2} - 4\zeta^2 X^2} - r^{l+1}\right)}{4\zeta^2},$$

Velocity of X field: $v_X = -\dfrac{X}{r^l}$.

Boundary Data:

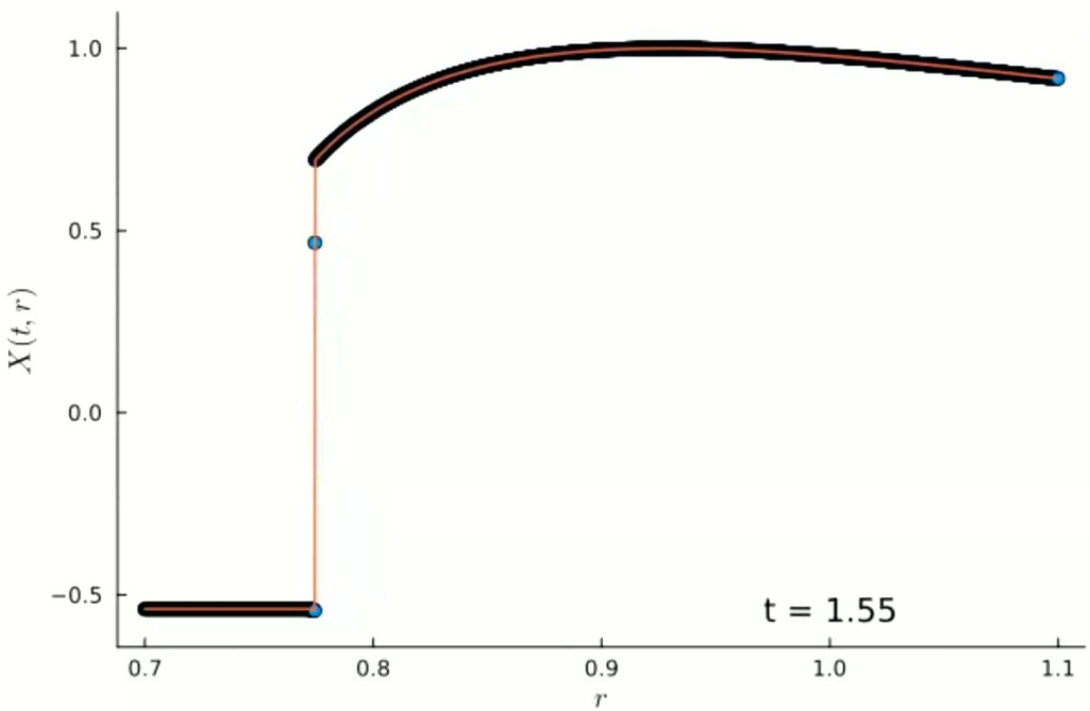


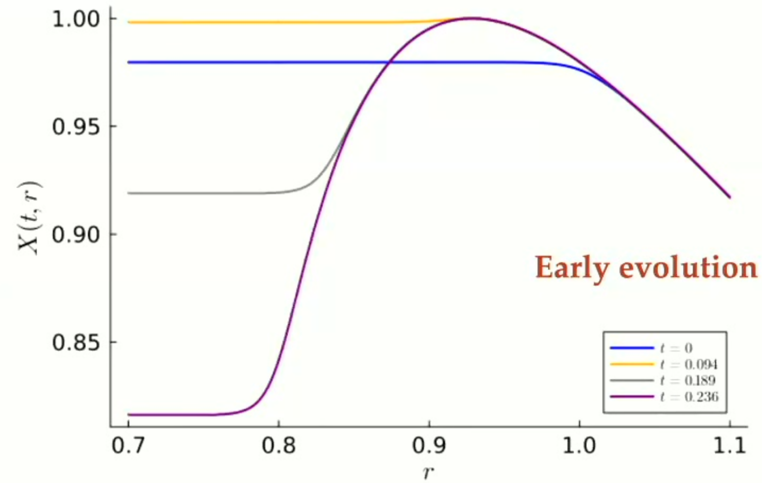
Bounce occurs at $t_b = \dfrac{t_1 + t_2}{2} \approx 0.409$ when $X = 0$

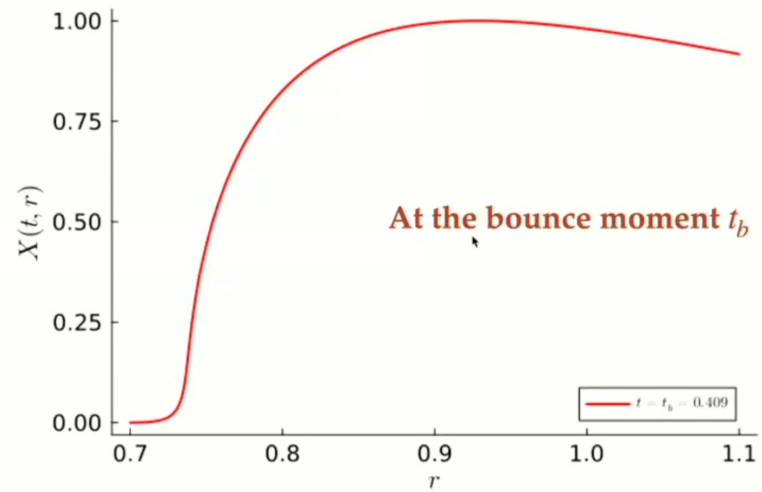
$\Longrightarrow r_b \approx 0.737$

Numerical Results

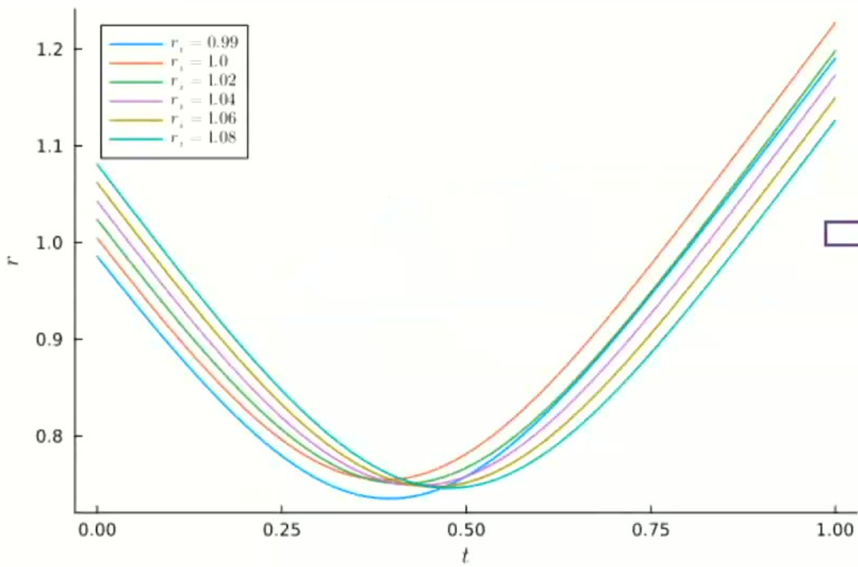
Case $M = 0.8$

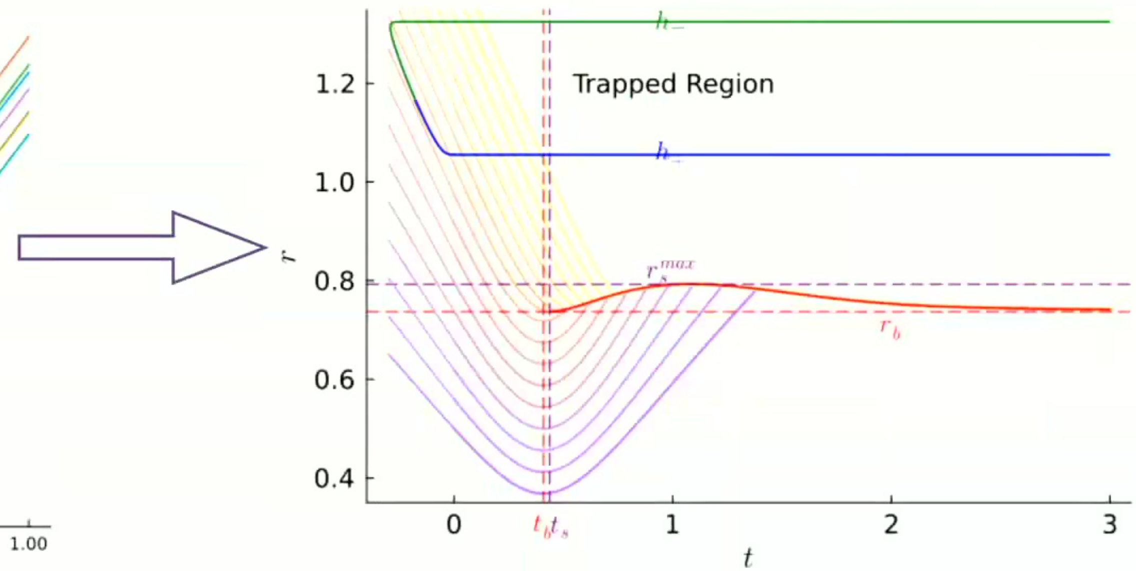




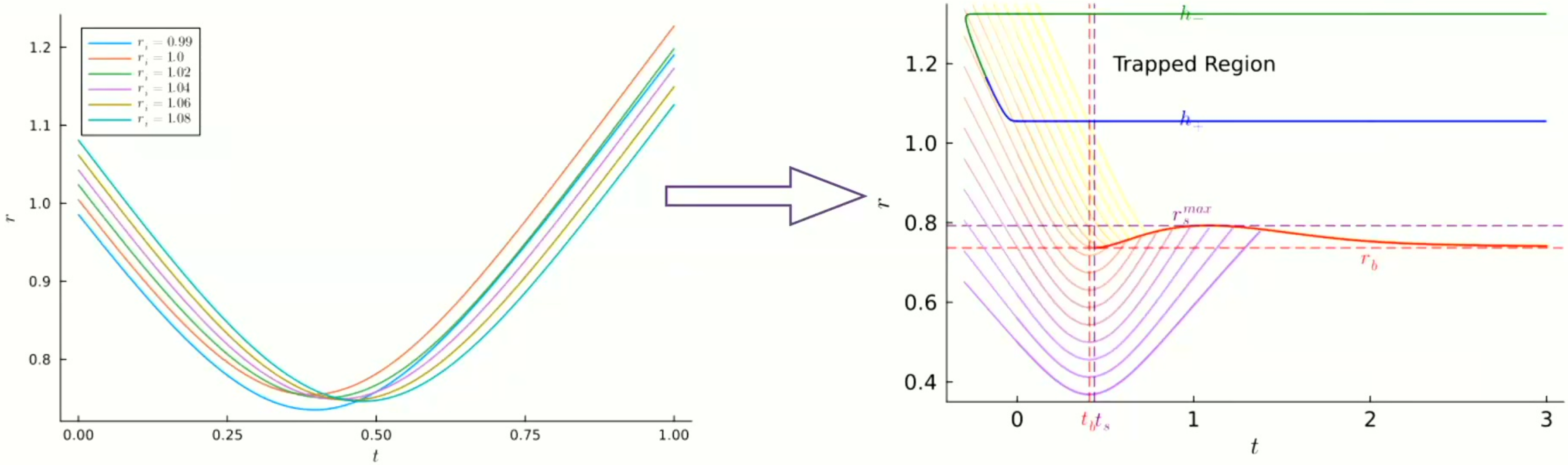


Characteristic curves





Characteristic curves

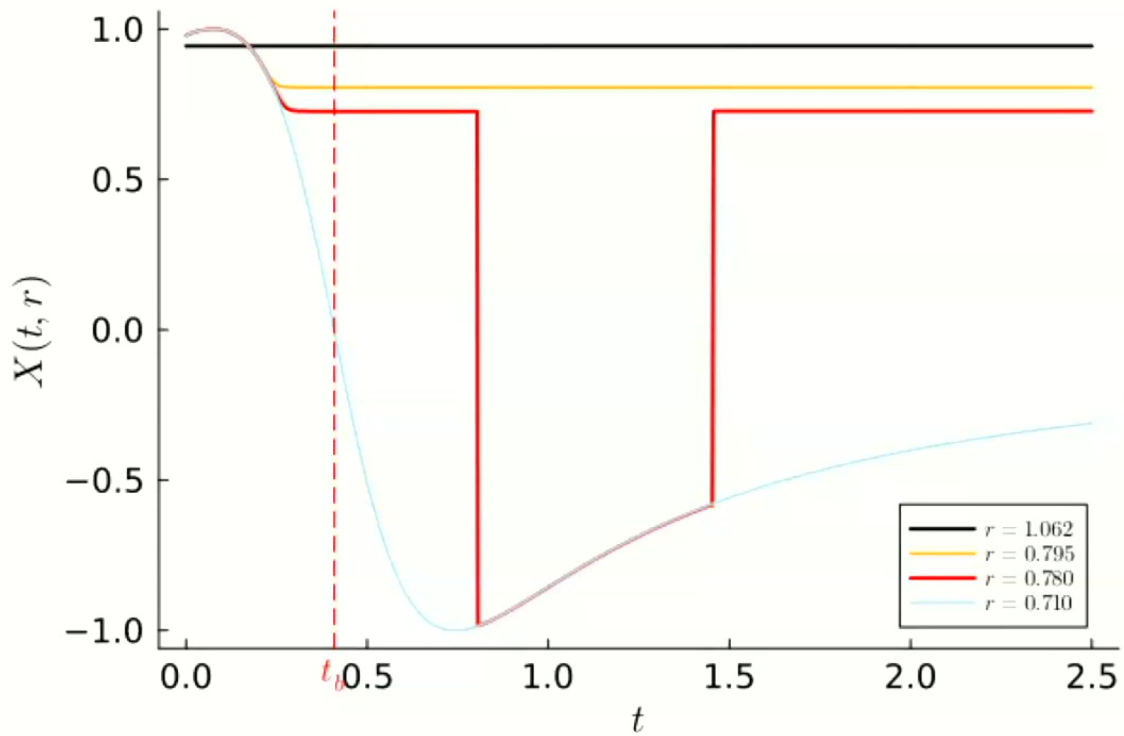


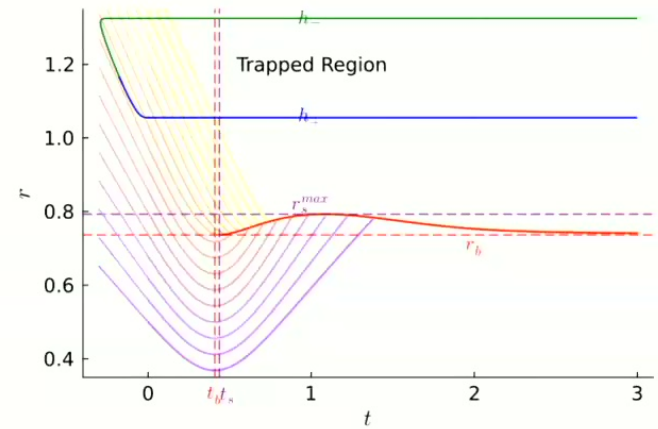
The shock position at different times:

$$r_s(t) = \left(1.31 - 4.22t + 12.41t^2 - 19.47t^3 + 19.04t^4 - 12.45t^5 + 5.56t^6 - 1.67t^7 + 0.32t^8 - 0.04t^9 + 0.002t^{10}\right)_{\pm 4.74\times 10^{-4}}$$

Numerical Results

The evolution of $X(t,r)$ at several fixed radii r_{const}





Four main regimes:

1.$r_{\text{const}} > r_0$ (black line)

2.$r_s^{\max} < r_{\text{const}} < r_0$ (orange line)

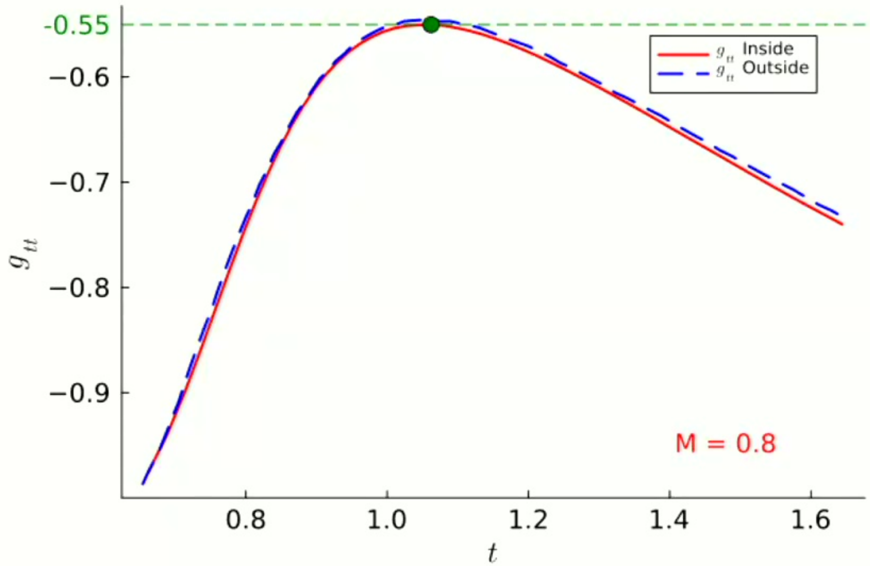
3.$r_b < r_{\text{const}} < r_s^{\max}$(red line)

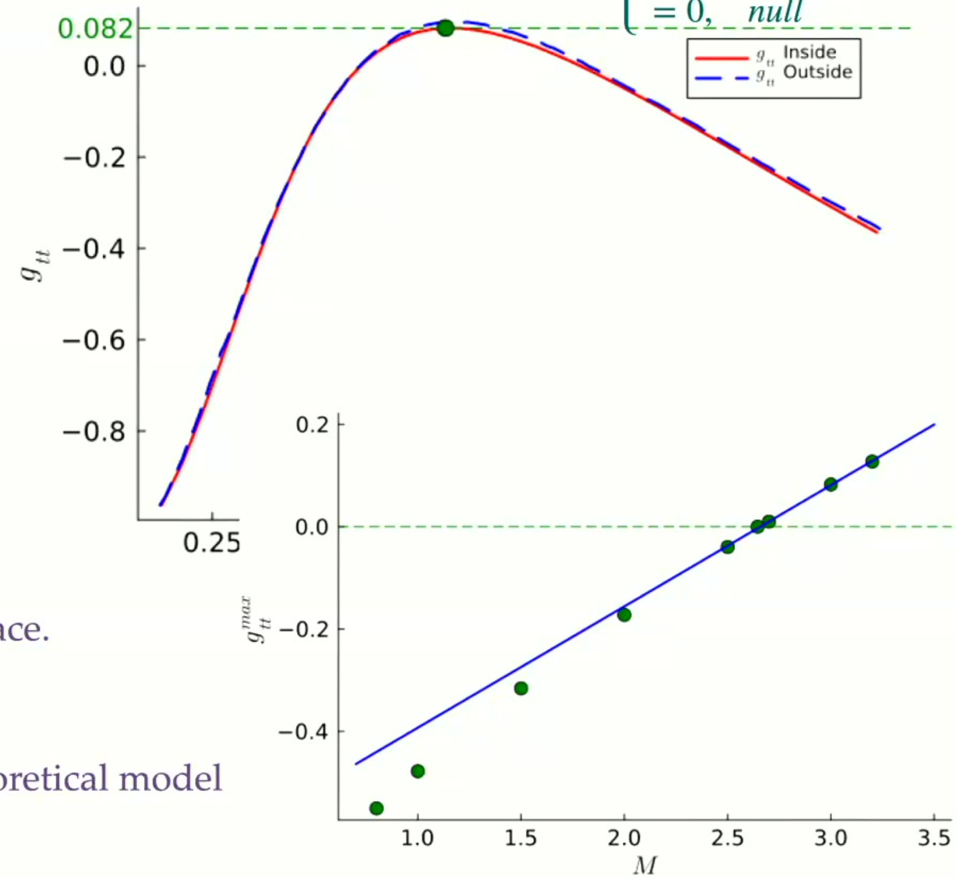
4.$r < r_b$ (light-blue line)

Signature of the shock surface

The induced metric on Σ is:

$$\mathrm{d}S_\Sigma^2 = \left[-1 + (N^x + r'(t))^2\right] \mathrm{d}t^2 + r^2 d\Omega^2 \qquad g_{tt} = -1 + (N^x + r'(t))^2 \begin{cases} < 0, & \textit{timelike} \\ > 0, & \textit{spacelike} \\ = 0, & \textit{null} \end{cases}$$





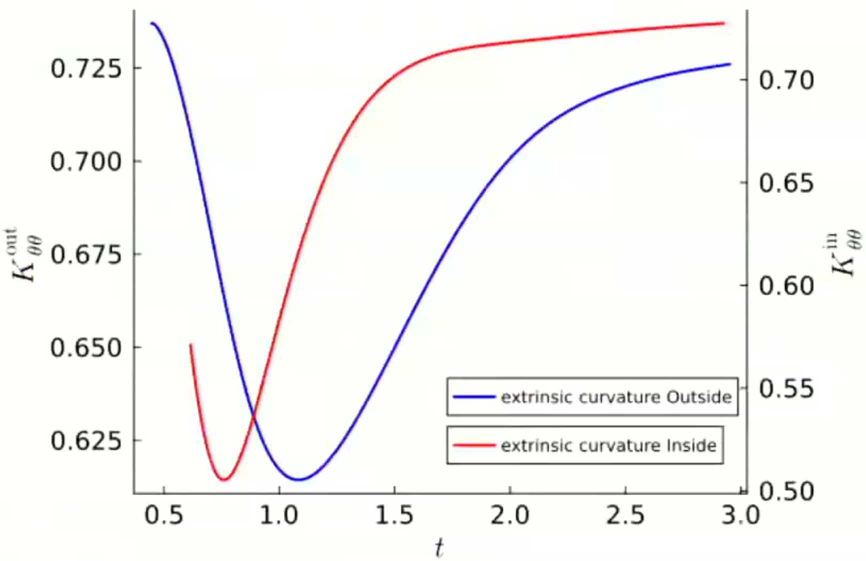
1.Verify the induced metric is continuous on the shock surface.

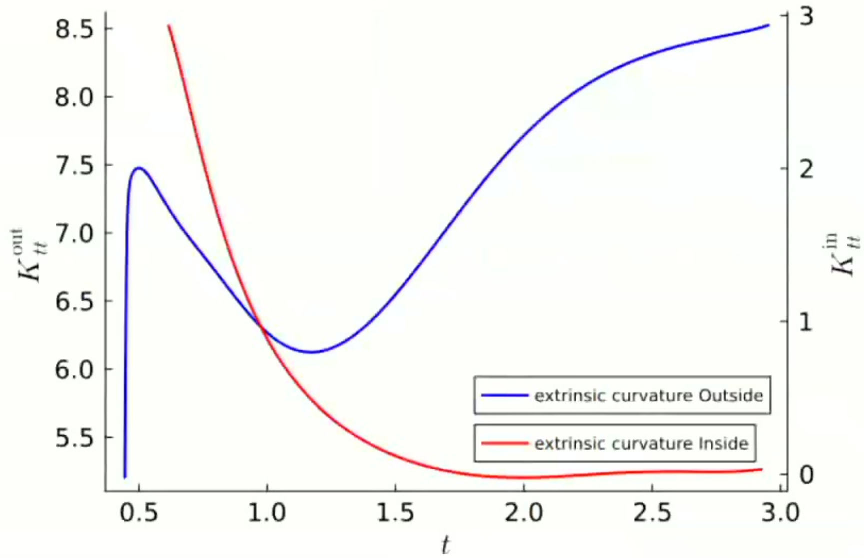
2. Physical shock for small masses ($M < 2.645$);

3. Unphysical shock appear for large masses $\Longrightarrow$ better theoretical model with evaporation need to be considered.

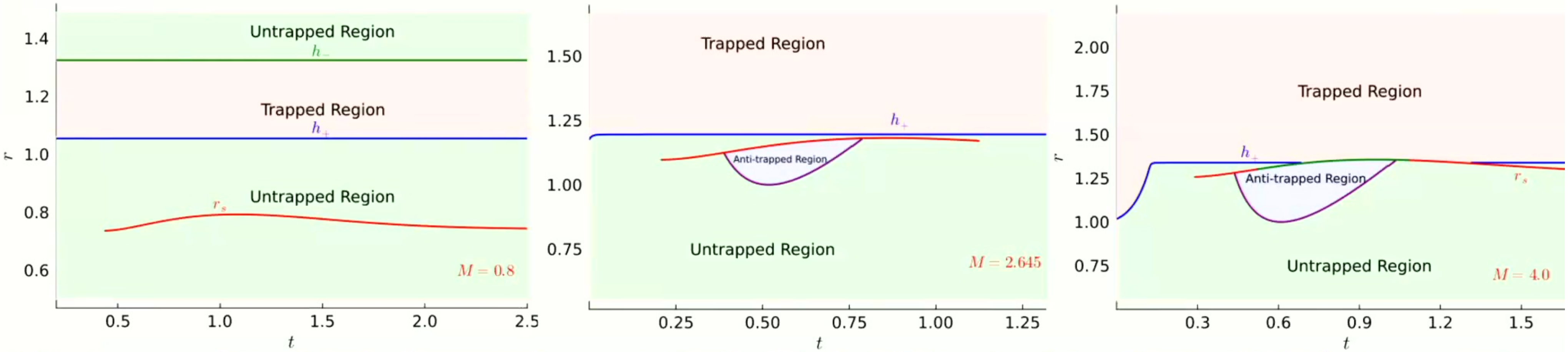
Curvatures inside and outside along the shock surface

The extrinsic curvature components:





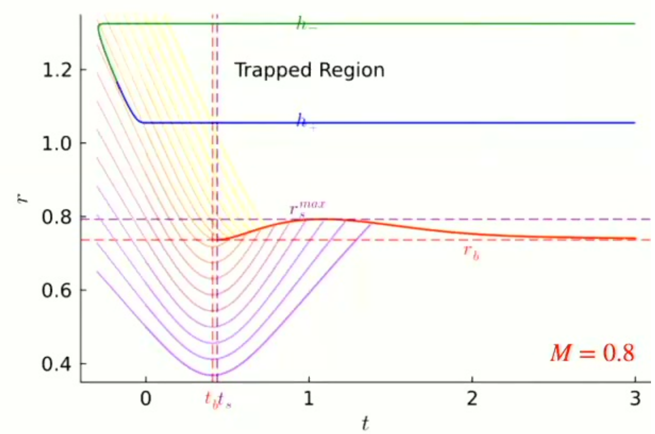
The formation of trapped and anti-trapped regions



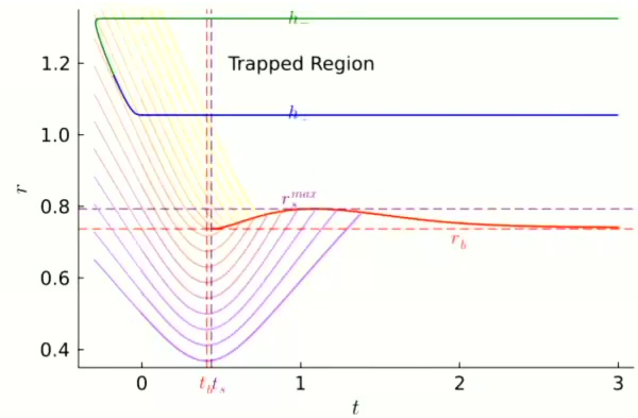
1. Horizons are determined by the expansion parameter $\theta_\pm$ of the two future-directed null vector fields that are normal to a constant-radius shell $\Longrightarrow \begin{cases} \dot{R} = \pm 1, & \text{LTB coords} \\ N^x = \mp 1, & \text{PG coords} \end{cases}$.

2. $\begin{cases} \theta_\pm < 0, & \text{Trapped Region} \\ \theta_\pm > 0, & \text{Anti-trapped Region} \\ \theta_+\theta_- < 0, & \text{Antrapped Region} \end{cases}$.

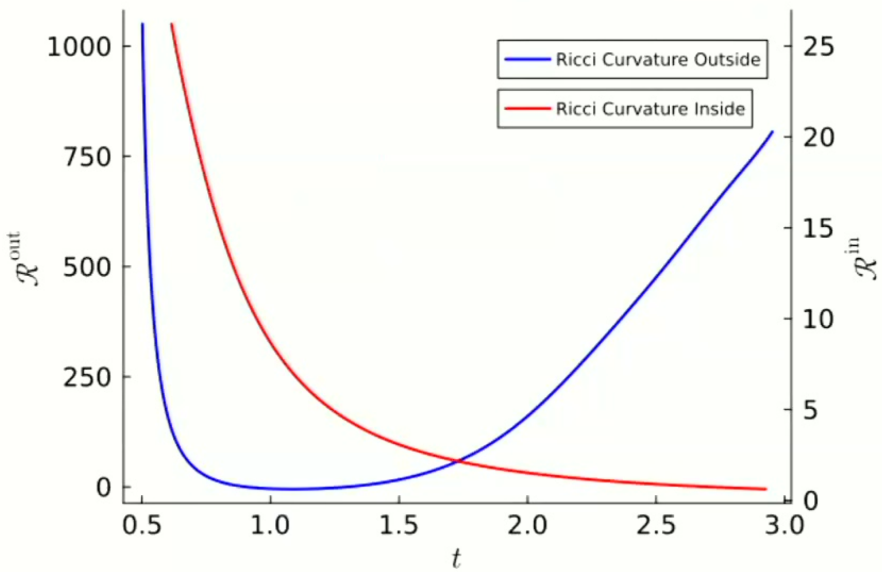
3. Critical Mass for Black Hole formation is $M_c = \dfrac{8\zeta}{3\sqrt{3}G} \simeq 0.77$.

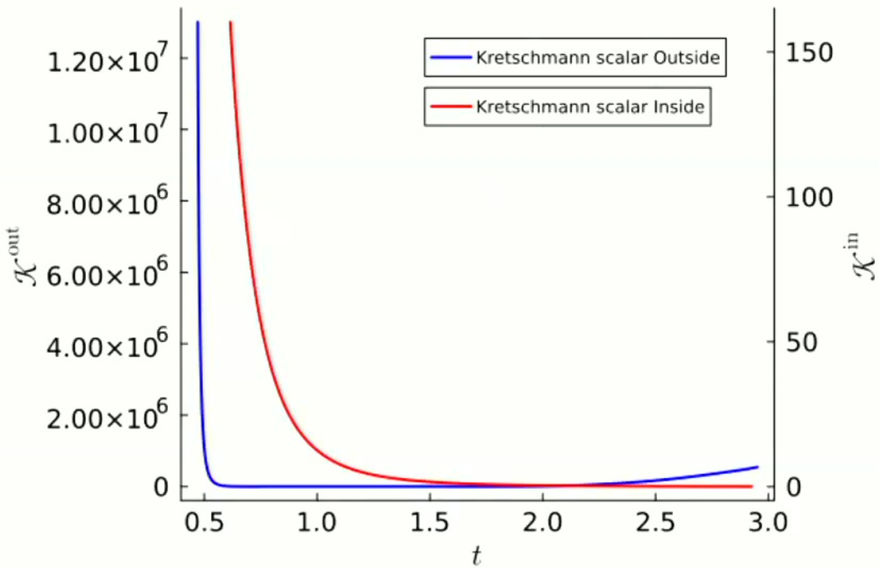


Curvatures inside and outside along the shock surface



The Ricci curvature evolution and the Kretschmann scalar:





Summary

- **Key Contributions:**
  - Developed a numerical framework using generalized PG coordinates.
  - Derived a first-order PDE for shock surface via junction conditions.
  - Handles complex source terms and square-root structures.
  - Applicable to various black hole models.
- **Main results**
  - Induced metric remains continuous across shock surface
  - Causal nature depends on mass
  - Shock asymptotically approaches bounce radius

Summary

- **Implications:**

- Limitations of semiclassical models for macroscopic black holes.
- Suggests mass-driven quantum phase transitions.
- Requires modeling black hole evaporation and tunneling.
- Motivates fully quantum treatment in LQG.

- **Future Work :**

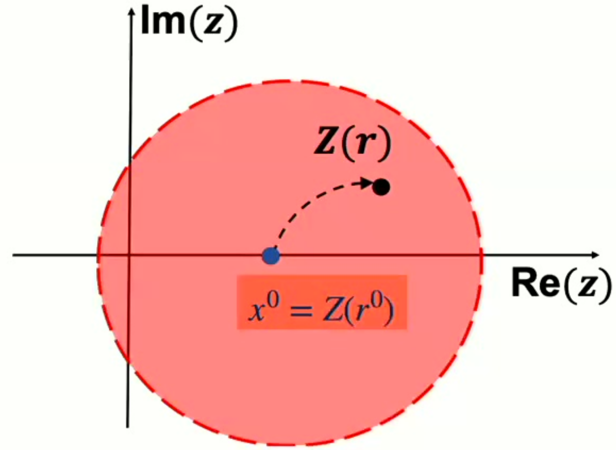
- Extend to non-marginally bound and non-dust collapse.
- Apply to asymmetric bounce and evaporation models
- Requires modeling black hole evaporation and tunneling.
- Move beyond effective theory to full quantum regimes.

Complex critical points

We consider the large-λ integral:

$$\int_K \mathrm{d}^N x\, \mu(x)\, e^{\lambda S(r,x)},$$

- *$S(r,x)$ and $\mu(x)$ are analytic functions for $r \in U \subset \mathbb{R}^k, x \in K \subset \mathbb{R}^N$.*
- $U \times K$ **is a compact neighborhood of** (r^0, x^0), x^0 **is a** ***(real) critical point***: $\delta_x S = 0, \quad \mathrm{Re}(S) = 0.$



Analytic Extension:

$$x \to z \in \mathbb{C}^N, \quad S(r,x) \to \mathcal{S}(r,z)$$

Complex critical points: $z = Z(r)$ are the solutions of the complex critical equation

$$\partial_z \mathcal{S} = 0$$

Large-λ asymptotic expansion for the integral:

$$\int_K \mathrm{d}^N x\, \mu(x) e^{\lambda S(r,x)} = \left(\frac{1}{\lambda}\right)^{\frac{N}{2}} \frac{e^{\lambda \mathcal{S}(r,Z(r))}\mu(Z(r))}{\sqrt{\det\left(-\delta^2_{z,z}\mathcal{S}(r,Z(r))/2\pi\right)}} \left[1 + O(1/\lambda)\right]$$

- **There exists constant** $C > 0$ **such that** $\mathrm{Re}(\mathcal{S}) \leq -C\,|\,\mathrm{Im}(Z)\,|^2$.
- **Interpolating two regimes:**

$$\begin{cases} \mathrm{Re}(\mathcal{S}(r^0, Z(r^0)) = 0, & r = r^0, \quad \textbf{oscillatory phase} \\ \mathrm{Re}(\mathcal{S}(r, Z(r)) < 0, & r \neq r^0, \quad \textbf{exponentially decaying amplitude } e^{\lambda \mathrm{Re}(\mathcal{S})}. \end{cases}$$

It gives a smooth description of the asymptotic in the parameter space r

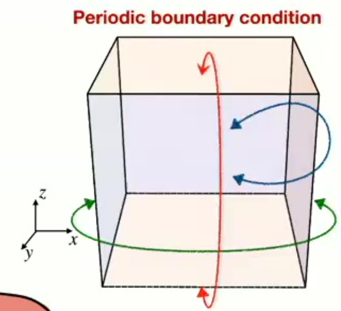
[Hörmander, 1983, Melin, Sjöstrand, 1975]

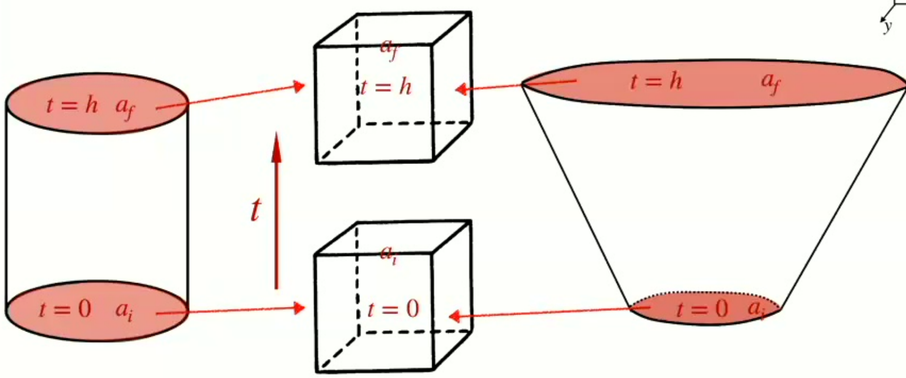
Cosmological Dynamics from Covariant LQG with Scalar Matter

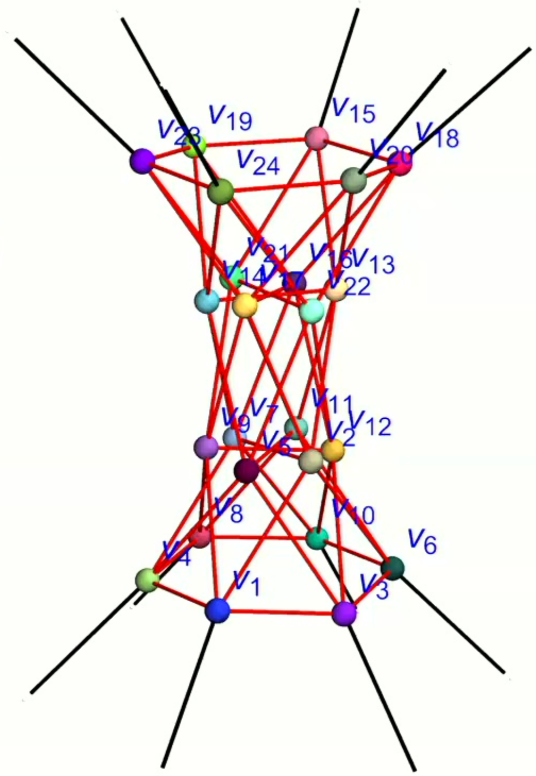
Spinfoam cosmology

A hypercube consists of 24 4-simplices: $(v_1, v_2, \cdots, v_{24})$.

- **Flat hypercube:** $a_f = a_i$.
- **Curved hypercube:** $a_f = a_i - 2\delta a, \quad \delta a \neq 0$







- **The spinfoam action with a coherent spin-network boundary state is**

$$S_{\rm SF} = S[j_h, X; j_b, \xi_{eb}] + \left[i \sum_{b_f} \gamma \vartheta^0_{b_f} \left(j_{b_f} - j^0_{b_f} \right) - i \sum_{b_i} \gamma \vartheta^0_{b_i} \left(j_{b_i} - j^0_{b_i} \right) - \sum_b \frac{1}{2 j^0_b} \left(j_b - j^0_b \right)^2 \right].$$

- **The scalar field action with the coherent state as the boundary state**

$$S_{\rm Scalar}(g, \varphi_v; \phi_{v_{b_{i(f)}}}, \pi_{v_{b_{i(f)}}}) = \frac{i}{2} \sum_{b_{vv'}} \rho_{vv'} \left(\varphi_v - \varphi_{v'} \right)^2 + \frac{1}{4\hbar} \sum_{v_{b_i}} \left(z^2_{v_{b_i}} - 2 \left(\varphi_{v_{b_i}} - z_{v_{b_i}} \right)^2 - z_{v_{b_i}} \bar{z}_{v_{b_i}} \right) + \frac{1}{4\hbar} \sum_{v_{b_f}} \left(\bar{z}^2_{v_{b_f}} - 2 \left(\varphi_{v_{b_f}} - \bar{z}_{v_{b_f}} \right)^2 - z_{v_{b_f}} \bar{z}_{v_{b_f}} \right)$$

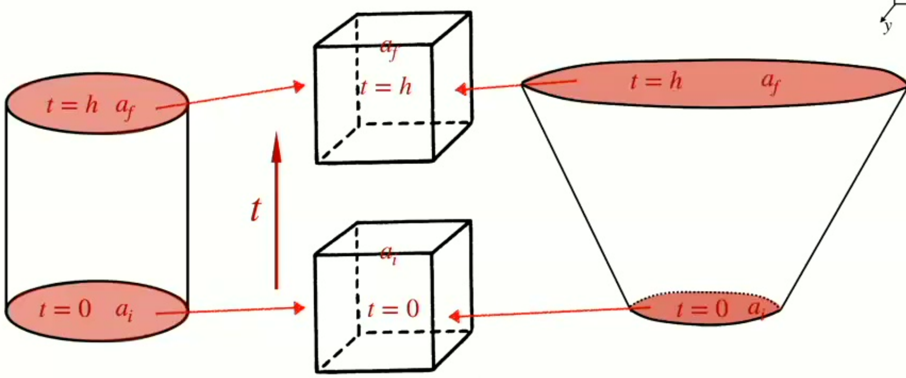
where the initial and final scalar data are $z_{v_{b_i}} = \phi_{v_{b_i}} + i\pi_{v_{b_i}}, \quad z_{v_{b_f}} = \phi_{v_{b_f}} + i\pi_{v_{b_f}}$.

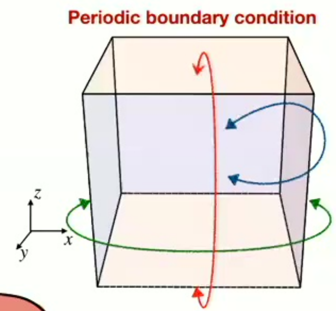
Cosmological Dynamics from Covariant LQG with Scalar Matter

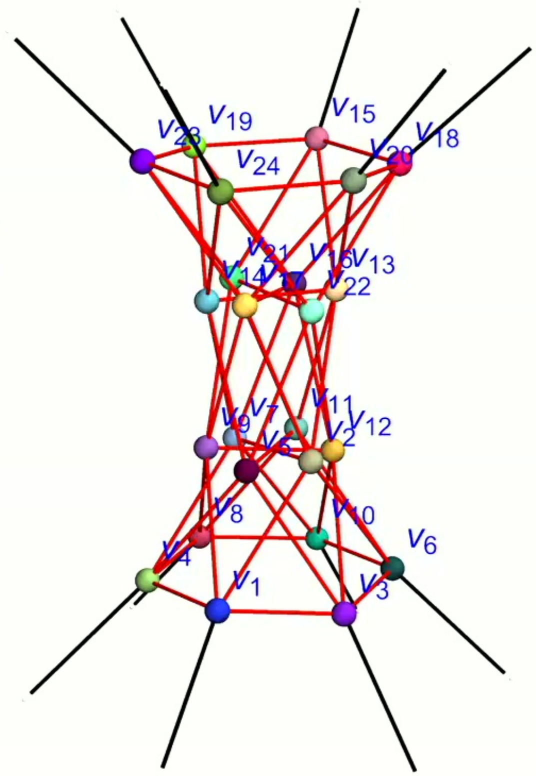
Spinfoam cosmology

A hypercube consists of 24 4-simplices: $(v_1, v_2, \cdots, v_{24})$.

- **Flat hypercube:** $a_f = a_i$.
- **Curved hypercube:** $a_f = a_i - 2\delta a, \quad \delta a \neq 0$







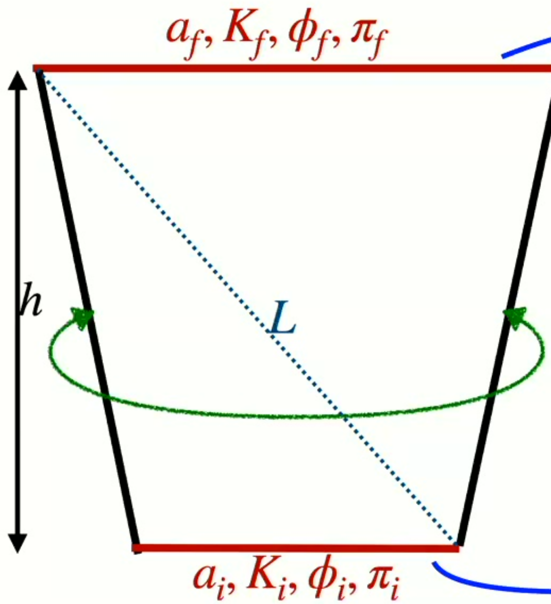
The spinfoam amplitude coupled with scalar matter:

$$\int \prod_{I=1}^{N_{\text{out}}} \mathrm{d}j_I^{\text{out}} \mathcal{Z}_{\mathcal{K}} \left(j_I^{\text{out}}, \xi_{eb}, K_{i(f)}, \phi_{i(f)}, \pi_{i(f)} \right), \qquad \mathcal{Z}_{\mathcal{K}} = \int \mathrm{d}^N \mathbf{x}\, \mu(\mathbf{x})\, e^{S_{\text{tot}}(r,\mathbf{x})}, \qquad S_{\text{tot}}(r,\mathbf{x}) = S_{\text{SF}} + S_{\text{Scalar}}.$$

- **External data** $r = (a_{i(f)}, K_{i(f)}, \phi_{i(f)}, \pi_{i(f)})$.
- **The integration variables** $\mathbf{x} = \{g_{ve}, z_{vf}, \xi^{\pm}_{eh}, l^{+}_{eh}, j_{\bar{h}}, \varphi_v\} \Longrightarrow$ **1192 real variables**

Numerical Result of Hypercube Complex

- **When r^0 is determined by $a_i = a_f = 1$, fixed h value, and $K_b = \phi_{v_b} = \pi_{v_b} = 0 \Longrightarrow$ Real critical point.**
- **When $r = r^0 + \delta r \Longrightarrow$ Complex critical point.**



On $t = h$ slice, the final data $(a_f, K_f, \phi_f, \pi_f)$ are given with the condition:

$$a_f > a_i, \quad \phi_f > \phi_i, \quad K_f > 0, \quad \pi_f > 0$$

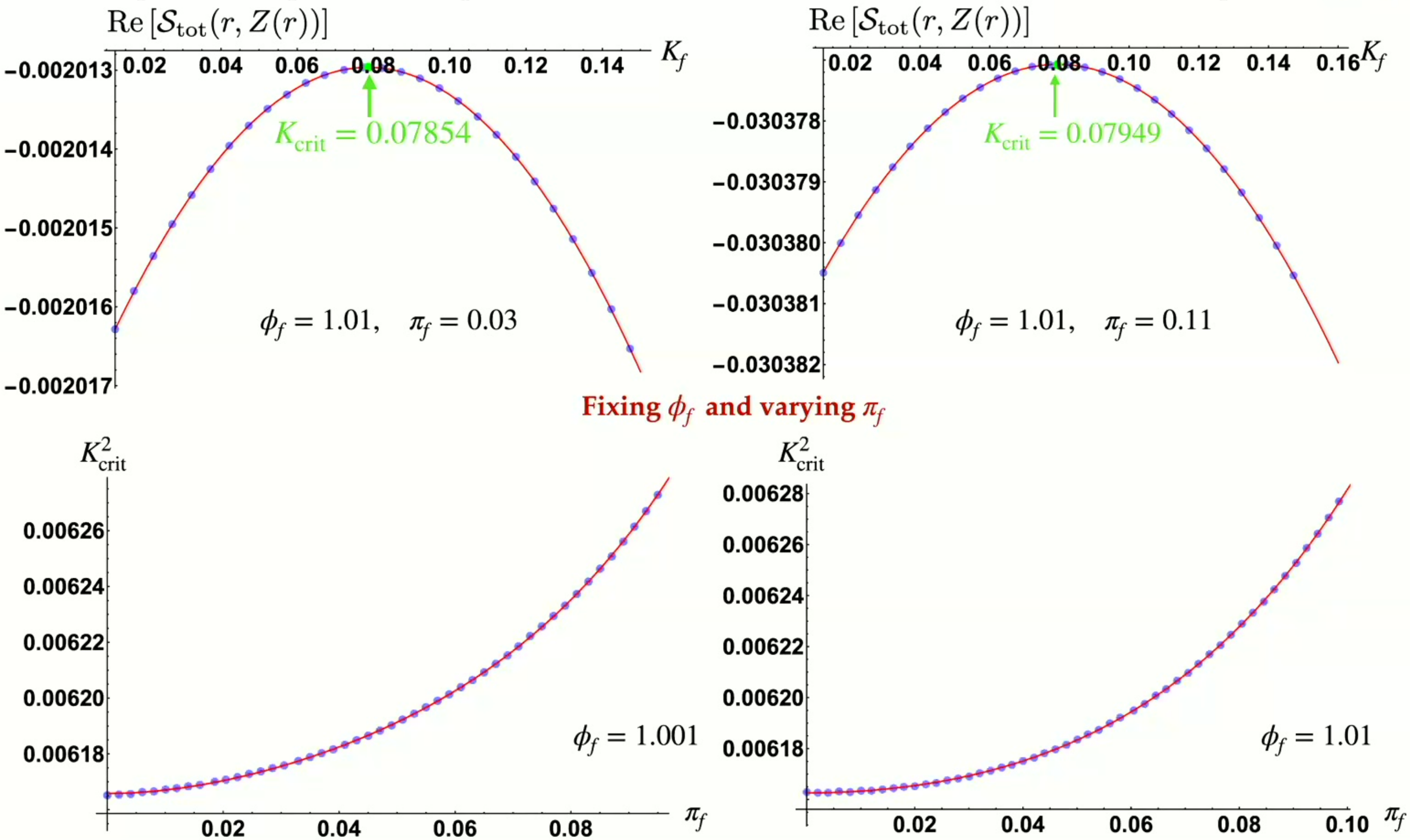
On $t = 0$ slice, initial data $(a_i, K_i, \phi_i, \pi_i)$ satisfy the Friedmann equation:

$$\left(\frac{\dot{a}_i}{a_i}\right)^2 = \frac{8\pi G\rho_i}{3} \Longrightarrow K_i^2 = \frac{3\pi_i^2}{2a_i^6}$$

Each sample (K_f, ϕ_f, π_f), we can find the numerical solutions to $\partial_{\mathbf{z}}\mathcal{S}_{\text{tot}}(r, \mathbf{z}) = 0 \Longrightarrow Z(r)$ is the complex critical point

Numerical Result of Hypercube Complex

The maximum of spinfoam amplitude corresponds to the Hamiltonian constraint (modified Friedmann equation) [E. Bianchi, C. Rovelli, F. Vidotto, 2010]



Numerical Result of Hypercube Complex

The constraint equation of K_f and π_f is given by

$$K_{\rm crit}^2 = \alpha_0 + \alpha_2 \pi_f^2 + \alpha_3 \pi_f^3 + \alpha_4 \pi_f^4 + O(\pi_f^5)$$

For $\phi_f = 1.001$:

$\alpha_0 = 0.00617_{\pm 1.08\times 10^{-8}}, \quad \alpha_2 = 0.0133_{\pm 6.32\times 10^{-5}},$
$\alpha_3 = -0.113_{\pm 1.52\times 10^{-3}}, \quad \alpha_4 = 1.034_{\pm 9.39\times 10^{-3}}.$

For $\phi_f = 1.01$:

$\alpha_0 = 0.00616_{\pm 1.56\times 10^{-8}}, \quad \alpha_2 = 0.00690_{\pm 4.38\times 10^{-5}},$
$\alpha_3 = 0.00518_{\pm 1.05\times 10^{-3}}, \quad \alpha_4 = 0.447_{\pm 6.50\times 10^{-3}}.$

Compare to classical Friedmann equation $K_i^2 = 8\pi G \dfrac{3\pi_i^2}{2a_i^6}$:

- **An effective scalar density $\rho_{\rm eff}$: π_f^2 terms and higher derivative terms with π_f^3 and π_f^4.**
- **$\alpha_0(\phi_f)$ is understood as an effective scalar potential.**
- **$\alpha_0 > 0$ plays a role similar to an effective positive cosmological constant.**
- **$\alpha_0 \neq 0$ indicates that on the final slice $K_{\rm crit} > K_i$, implying the accelerating expansion of the universe.**

Summary

- **Pros:**

- Generalizability: Techniques for quasi-linear PDEs apply to various gravitational collapse models.
- Computational Power: Complex critical point methods + Picard–Lefschetz thimbles + Monte Carlo algorithms = effective tools for Lorentzian path integrals.
- Broad Applicability: Well-suited for exploring covariant quantum gravity frameworks (e.g., spinfoams, quantum Regge calculus ...).

- **Cons/ Limitations**

- A more complete and robust theoretical framework is still needed to guide and justify these numerical computations.