Title: Kinematic Stratifications

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Abstract:

This lecture discusses the stratification of regions in the space of real symmetric matrices. The points of these regions are Mandelstam matrices for momentum vectors in particle physics. The kinematic strata are indexed by signs and rank two matroids. Matroid strata of Lorentzian polynomials arise when all signs are nonnegative. We describe the posets of strata, for massless and massive particles, with and without momentum conservation. This is joint work with Veronica Calvo and Hadleigh Frost.

Kinematic Stratifications

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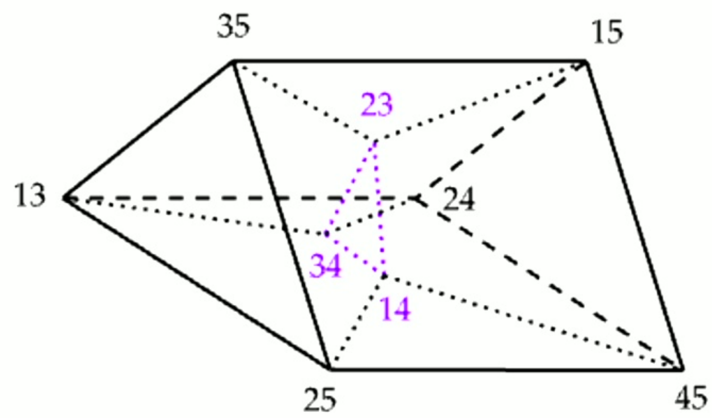


Joint work with Veronica Calvo Cortes and Hadleigh Frost

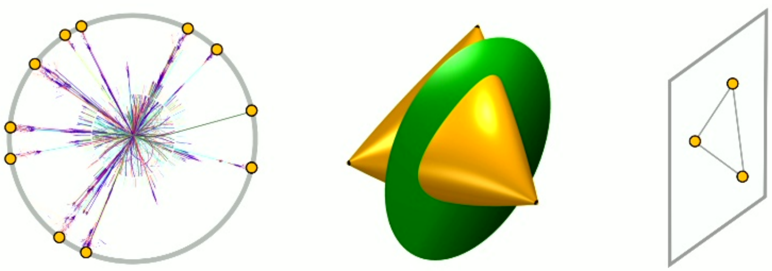
One Particle

In physics, particles are represented by momentum vectors p in *Minkowski space* $\mathbb{R}^{1+d}$, with Lorentzian inner product

$$p \cdot q \ = \ p_0 q_0 - p_1 q_1 - \cdots - p_d q_d.$$

The *universal speed limit* states that $p \cdot p \geq 0$ for each particle.

A particle is *massless* if the equality $p \cdot p = 0$ holds.



Massless: think photon
Massive: think proton

The Lightcone

Several Particles

Consider n particles, with momenta $p^{(1)}, p^{(2)}, \dots, p^{(n)} \in \mathbb{R}^{1+d}$.

The Lorentz group $\mathrm{SO}(1,d)$ acts on such configurations.
Kinematic data are invariant under this action.

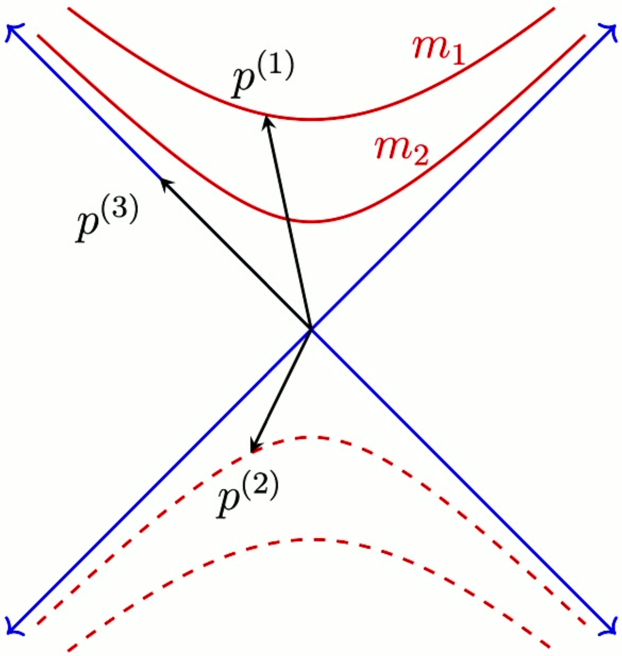
The *Mandelstam invariants* are the entries in the symmetric matrix

$$\begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{12} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & \cdots & s_{nn} \end{bmatrix} = \begin{bmatrix} - & p^{(1)} & - \\ - & p^{(2)} & - \\ & \vdots & \\ - & p^{(n)} & - \end{bmatrix} \begin{bmatrix} +1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ (p^{(1)})^T & (p^{(2)})^T & \cdots & (p^{(n)})^T \\ | & | & \cdots & | \end{bmatrix}$$

The *Mandelstam region* $\mathcal{M}_{n,r}$ is the semi-algebraic set of these matrices, for fixed rank $r \leq 1+d$. It has codimension $\binom{n-r-1}{2}$ in $\mathbb{R}^{\binom{n+1}{2}}$.

We examine the stratification of $\mathcal{M}_{n,r}$ by the signs of the s_{ij}.

The Lightcone



The light cone (blue) and the mass shells (red) for two given masses.

Being Lorentzian

The Mandelstam region $\mathcal{M}_{n,r}$ consists of matrices S that satisfy:

- the n diagonal entries s_{ii} of S are non-negative, and
- S has one positive eigenvalue and $r-1$ negative eigenvalues.

The *Lorentzian region* is the subset of nonnegative matrices

$$\mathcal{L}_{n,r} \;=\; \mathcal{M}_{n,r} \,\cap\, (\mathbb{R}_{\geq 0})^{\binom{n+1}{2}}.$$

Points in $\mathcal{L}_{n,\leq n} = \sqcup_{r=1}^{n} \mathcal{L}_{n,r}$ are *Lorentzian polynomials* of degree two.

Petter Brändén and June Huh: *Lorentzian polynomials*, Ann. Math. (2020).

Bränden proved that $\mathcal{L}_{n,\leq n}$ is a topological ball of dimension $\binom{n+1}{2}$.

P. Brändén: *Spaces of Lorentzian and real stable polynomials are* **Euclidean balls**, Forum Math. Sigma (2021)

Thus, $\mathcal{M}_{n,\leq n} = \sqcup_{r=1}^{n} \mathcal{M}_{n,r}$ is a disjoint union of 2^{n-1} such balls.

Our stratifications match those in

M. Baker, J. Huh, M. Kummer, O. Lorscheid: *Lorentzian polynomials and* **matroids** *over triangular hyperfields*.

Principal Minors

Lemma

$S \in \mathcal{M}_{n,r}$ if and only if the principal minors have alternating signs:

$$(-1)^{|I|-1} \cdot \det(S_I) \;\geq\; 0 \qquad \text{for all} \;\; I \subseteq [n].$$

For minors of size 2 and 3,

$$s_{ii}s_{jj} \leq s_{ij}^2 \quad \text{and} \quad 2s_{ij}s_{ik}s_{jk} + s_{ii}s_{jj}s_{kk} \geq s_{ii}s_{jk}^2 + s_{jj}s_{ik}^2 + s_{kk}s_{ij}^2.$$

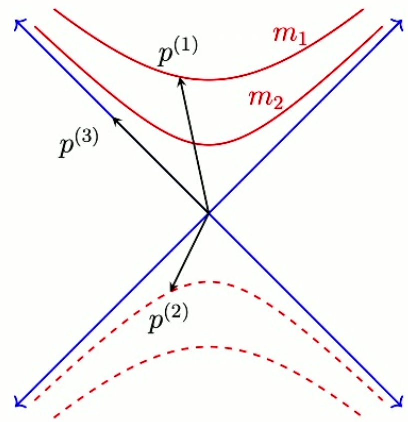
This implies

$$s_{ij}s_{ik}s_{jk} \;\geq\; 0.$$

Proposition

If $S \in \mathcal{M}_{n,r}$ has nonzero entries then there exists $\sigma \in \{-,+\}^n$ such that $\operatorname{sgn}(s_{ij}) = \sigma_i\sigma_j$ for all i,j.

Causality



Corollary

The region $\mathcal{M}_{n,r}$ is the union of the 2^{n-1} signed Mandelstam regions $\mathcal{M}_{n,\sigma,r}$. Their relative interiors are pairwise disjoint:

$$\mathcal{M}_{n,r} = \bigcup_{\sigma} \mathcal{M}_{n,\sigma,r}.$$

The sign vector σ distinguishes the **future** from the **past**.

The *Lorentzian region* $\mathcal{L}_{n,r}$ is the closure of the region $\mathcal{M}_{n,\sigma,r}$ with $\sigma = (+, +, \ldots, +)$. Everything lies in the future, or in the past.

Massless Particles

From now on, all particles are massless:

$$s_{11} = s_{22} = \cdots = s_{nn} = 0.$$

Principal 4×4 minors of $S \in \mathcal{M}^0_{n,r}$ satisfy $\det(S_{\{i,j,k,l\}}) =$

$$s_{ij}^2 s_{kl}^2 + s_{ik}^2 s_{jl}^2 + s_{il}^2 s_{jk}^2 - 2 \cdot \left(s_{ij} s_{ik} s_{jl} s_{kl} + s_{ij} s_{il} s_{jk} s_{kl} + s_{ik} s_{il} s_{jk} s_{jl} \right) \leq 0.$$

If we set $p_{ij} = \sqrt{s_{ij}}, \ldots, p_{kl} = \sqrt{s_{kl}}$, then this factors:

$$\begin{aligned} \det(S_{\{i,j,k,l\}}) \;=\; & (p_{ij}p_{kl} + p_{ik}p_{jl} + p_{il}p_{jk})(-p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}) \\ & (-p_{ij}p_{kl} + p_{ik}p_{jl} - p_{il}p_{jk})(p_{ij}p_{kl} - p_{ik}p_{jl} - p_{il}p_{jk}). \end{aligned}$$

Think: Plücker, Schouten, squared Grassmannian, ...

H. Friedman: *Likelihood geometry of the squared Grassmannian*, Proceedings AMS (2025+)

This guides us to **matroids**. In this talk, **all matroids have rank two**.

Matroids of rank two

A *matroid* is a partition $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_m$ of a subset of $[n] = \{1, \ldots, n\}$. The *bases* of P are pairs $\{u, v\}$ where $u \in P_i$ and $v \in P_j$ for $i \neq j$. Elements in $[n] \backslash P$ are *loops*.

The matroid P has $m \geq 2$ parts and $l = n - |P|$ loops.

Example

The *uniform matroid* U_n is the partition of $P = [n]$ into n singletons $P_i = \{i\}$.

For $\sigma \in \{-, +\}^P$, the pair (P, σ) is a *signed matroid*.

Definition

The stratum $\mathcal{M}^0_{P,\sigma,r}$ is the subset of the massless Mandelstam region $\mathcal{M}^0_{n,r}$ defined by

$$\mathrm{sign}(s_{ij}) = \sigma_i \sigma_j \text{ if } \{i, j\} \text{ is a basis of } P, \text{ and } s_{ij} = 0 \text{ otherwise.}$$

Stratification

Theorem

Fix $r \geq 1$. The massless Mandelstam region equals

$$\mathcal{M}^0_{n,r} = \bigsqcup_{(P,\sigma)} \mathcal{M}^0_{P,\sigma,r}.$$

The disjoint union is over all signed matroids (P,σ) on $[n]$.

The kinematic stratum *$\mathcal{M}^0_{P,\sigma,r}$ is non-empty if and only if $3 \leq r \leq m$ or $r = m = 2$. If this holds, then its dimension is*

$$\dim\left(\mathcal{M}^0_{P,\sigma,r}\right) = m(r-2) + n - l - \binom{r}{2}.$$

Recall: The matroid P has $m \geq 2$ parts and $l = n - |P|$ loops.

Enumerative Combinatorics

We write $\left\{ {n-l \atop m} \right\}$ for the *Stirling number of second kind*. This is the number of partitions of the set $[n-l]$ into exactly m parts.

Corollary

The number of kinematic strata $\mathcal{M}^0_{P,\sigma,r}$ of dimension d in the Mandelstam region $\mathcal{M}^0_{n,r}$ is given, for a fixed sign vector σ or for all possible sign vectors, respectively, by

$$\sum_{m \geq r} \binom{n}{l} \left\{ {n-l \atop m} \right\} \quad \textit{and} \quad \sum_{m \geq r} 2^{n-l-1} \binom{n}{l} \left\{ {n-l \atop m} \right\}.$$

d / r	2		3		4	
1	6	12				
2	12	48				
3	7	56	4	16		
4			6	48		
5			1	8		
6					1	8

(a) $n = 4$

d / r	2		3		4		5	
1	10	20						
2	30	120						
3	35	280	10	40				
4	15	240	30	240				
5			30	440				
6			10	160	5	40		
7			1	16	10	160		
8								
9					1	16		
10							1	16

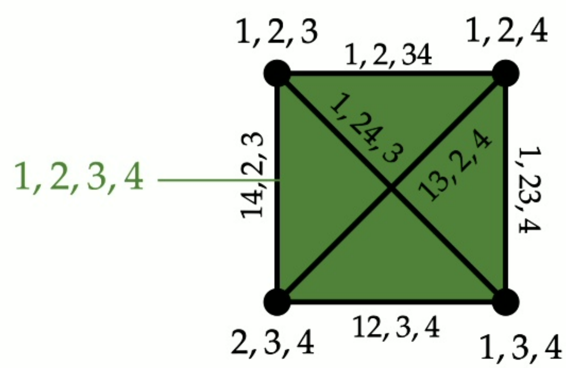
(b) $n = 5$

Posets of Matroids

The strata of $\mathcal{L}^0_{n,r}$ form a poset: $P \leq P'$ if every loop of P' is a loop in P, and the partition P' refines the partition P.

Same as containment of matroid polytopes.

For $n = 4, r = 3$, there are $11 = 1 + 6 + 4$ strata.
The top stratum $\mathcal{L}^0_{U_{4,3}}$ has three connected components:



The strata $\mathcal{M}^0_{P,\sigma,r}$ of $\mathcal{M}^0_{n,r}$ form a poset:
$(P, \sigma) \leq (P', \sigma')$ if $P \leq P'$ and $\sigma = \sigma'$ for all non-loops of P.

Inclusions and Topology

Our kinematic stratifications are nice: if a stratum intersects the closure of another stratum, then containment holds.

But, the topology of strata is quite interesting:

Proposition

$\mathcal{M}^0_{P,\sigma,3}$ has $(m-1)!/2$ connected components.

Theorem

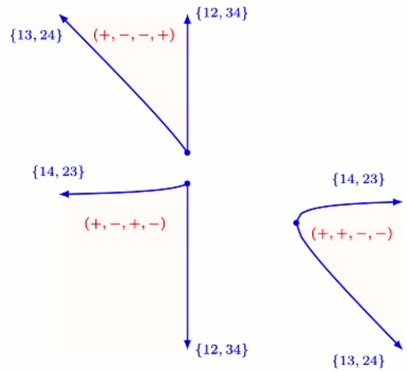
The kinematic stratum $\mathcal{M}^0_{P,\sigma,\leq r}$ is homotopic to the configuration space $F(\mathbb{S}^{r-2}, m)/\mathrm{SO}(r-1)$ for m points on the sphere $\mathbb{S}^{r-2}$.

Corollary

The stratum $\mathcal{M}^0_{P,\sigma,\leq 4}$ is homotopic to the moduli space $M_{0,m}(\mathbb{C})$, and hence to the complement of the affine braid arrangement.

E. Feichtner and G. Ziegler: *The integral cohomology algebras of ordered configuration spaces of spheres* (2000)

What matters for physics?



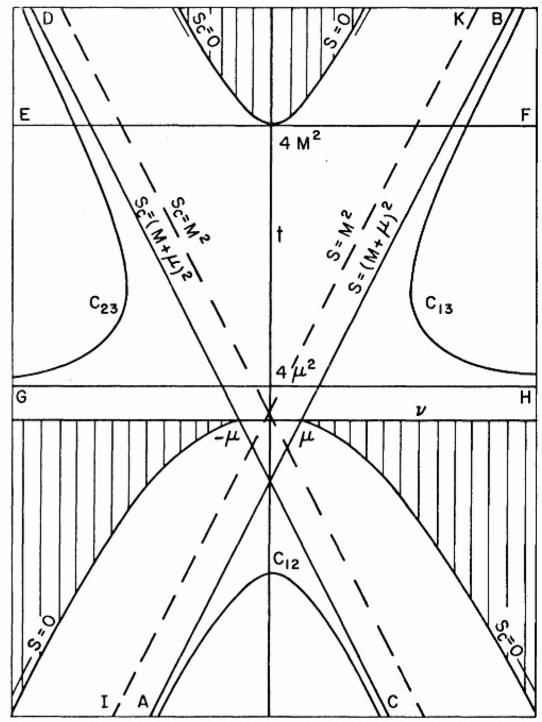


FIG. 1. Kinematics of the reactions I, II, and III.

Stanley Mandelstam: *Determination of the pion-nucleon scattering amplitude from dispersion relations and unitarity. General theory*, Physical Review (1958).

Momentum Conservation

The *massless momentum conserving (MMC) region* $\mathcal{C}^0_{n,r}$ is

$$\mathcal{C}^0_{n,r} = \bigsqcup_{(P,\sigma)} \mathcal{C}^0_{P,\sigma,r},$$

where $\mathcal{C}^0_{P,\sigma,r}$ is the intersection of $\mathcal{M}^0_{P,\sigma,r}$ with the subspace

$$\mathbb{R}^{n(n-3)/2} = \{S : s_{i1}+s_{i2}+\cdots+s_{in} = 0 \quad \text{for } i=1,2,\ldots,n\}.$$

Theorem

The stratum $\mathcal{C}^0_{P,\sigma,r}$ is non-empty if and only if

1. *For $3 \leq r < m$: there exist i,j,k,l in $[n]$, with $\sigma_i = \sigma_j = +$ and $\sigma_k = \sigma_l = -$, such that the restriction of the matroid P to $\{i,j,k,l\}$ is either U_4 or $\{ik, jl\}$,* **and**
2. *for $2 \leq r = m$: each part of P has elements with opposite signs.*

In this case, $\dim(\mathcal{C}^0_{P,\sigma,r}) = (m-1)(r-1) - \binom{r}{2} + (n-l-m) - 1.$

Four Particles

Use the rank 3 matrix

$$S = \begin{bmatrix} 0 & x & -x-y & y \\ x & 0 & y & -x-y \\ -x-y & y & 0 & x \\ y & -x-y & x & 0 \end{bmatrix}.$$

The principal 3×3 minors are $\det(S_{ijk}) = -2xy(x+y) \geq 0$.

The MMC region $\mathcal{C}^0_{4,\leq 3} = \mathcal{C}^0_{4,3} \cup \mathcal{C}^0_{4,2}$ has nine strata:

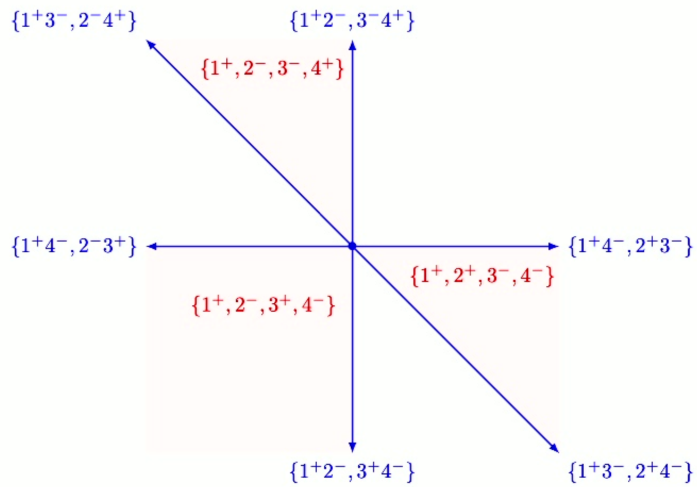


Figure 3: The $3+6$ MMC strata for $n=4$.

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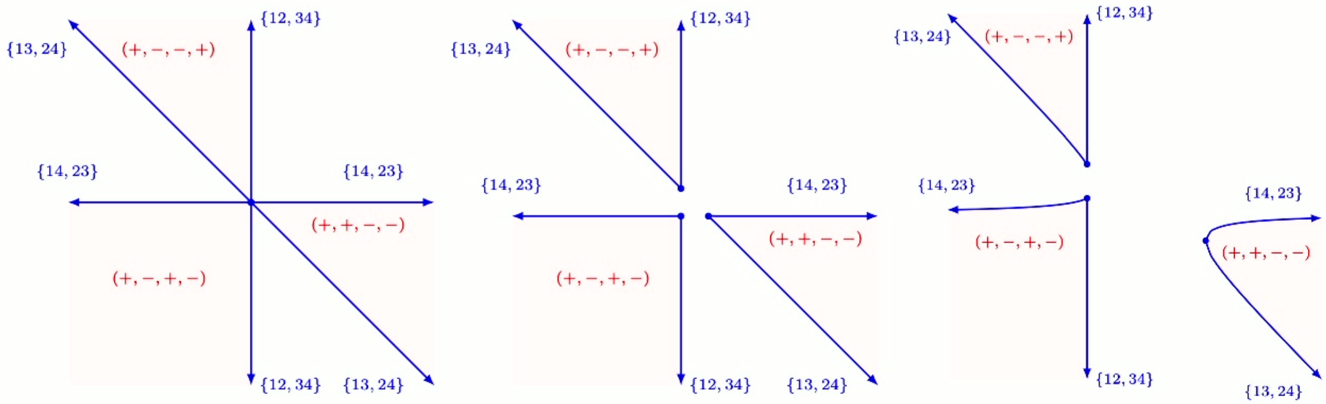
On-Shell

For $n = 4$ particles, fix masses $\mathbf{m} = (\mu, \mu, m, m)$ with $m > \mu > 0$.
We studied the MMC region $\mathcal{C}^{\mathbf{m}}_{4,3}$ by modifying our matrix:

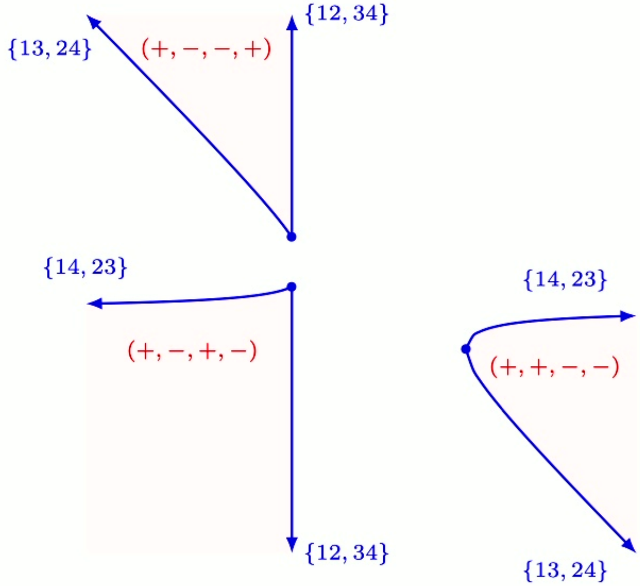
$$S = \begin{bmatrix} \mu^2 & x & -x-y-\mu^2 & y \\ x & \mu^2 & y & -x-y-\mu^2 \\ -x-y-\mu^2 & y & m^2 & \mu^2-m^2+x \\ y & -x-y-\mu^2 & \mu^2-m^2+x & m^2 \end{bmatrix}.$$



(a) (b) (c)

Figure 4: Regions for (a) massless, (b) equal masses, and (c) two unequal masses.

Back to 1958



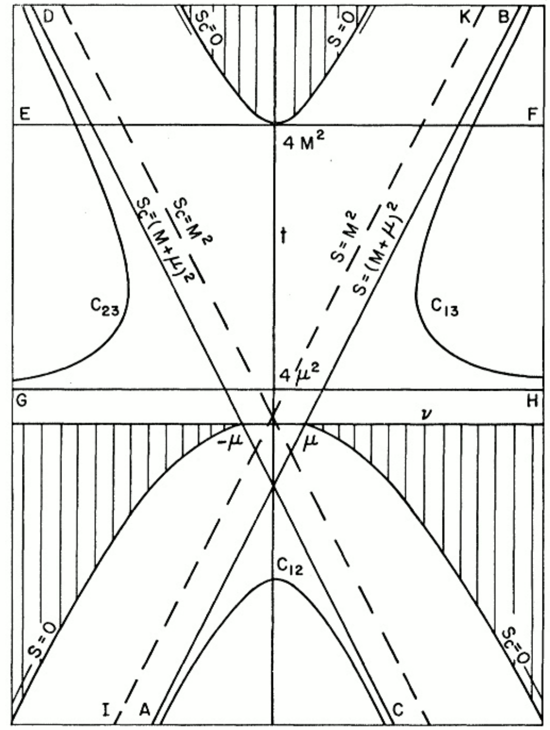


FIG. 1. Kinematics of the reactions I, II, and III.

Stanley Mandelstam: *Determination of the pion-nucleon scattering amplitude from dispersion relations and unitarity. General theory*, Physical Review (1958).

Five Particles

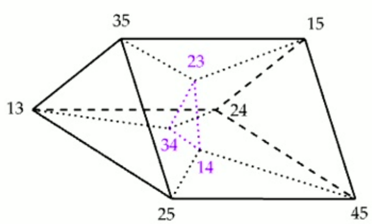
d / r	2		3		4	
1	6	30				
2	6	60	3	15		
3			9	90		
4			1	10		
5					1	10

10 cyclic polytopes $C(4,6)$
$f = (6, 15, 18, 9)$

$$S = \begin{bmatrix} 0 & a & -a-b+d & b-d-e & e \\ a & 0 & b & -b-c+e & -a+c-e \\ -a-b+d & b & 0 & c & a-c-d \\ b-d-e & -b-c+e & c & 0 & d \\ e & -a+c-e & a-c-d & d & 0 \end{bmatrix}.$$

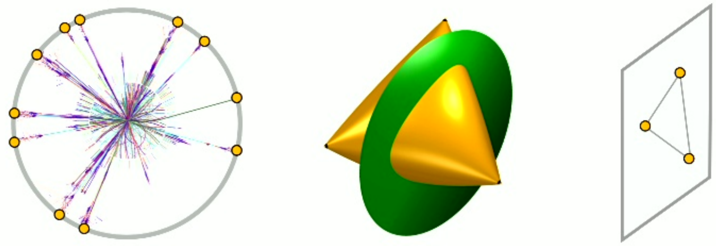
Igusa quartic

$$a^2b^2 + b^2c^2 + c^2d^2 + d^2e^2 + a^2e^2 + 2abcd + 2abce + 2abde + 2acde + 2bcde - 2ab^2c - 2bc^2d - 2cd^2e - 2ade^2 - 2a^2be < 0.$$



σ	s_{12}	s_{13}	s_{14}	s_{15}	s_{23}	s_{24}	s_{25}	s_{34}	s_{35}	s_{45}
$(-,-,+,+,+)$	+	−	−	−	−	−	−	+	+	+
$(-,+,-,+,+)$	−	+	−	−	−	+	+	−	−	+
$(-,+,+,-,+)$	−	−	+	−	+	−	+	−	+	−
$(-,+,+,+,-)$	−	−	−	+	+	+	−	+	−	−
$(+,-,-,+,+)$	−	−	+	+	+	−	−	−	−	+
$(+,-,+,-,+)$	−	+	−	+	−	+	−	−	+	−
$(+,-,-,+,+)$	−	+	+	−	−	−	+	+	−	−
$(+,+,-,-,+)$	+	−	−	+	−	−	+	+	−	−
$(+,+,-,+,-)$	+	−	+	−	−	+	−	−	+	−
$(+,+,+,-,-)$	+	+	−	−	+	−	−	−	−	

Conclusion



- We are part of the ERC Synergy project UNIVERSE+, titled

  Positive Geometry in Particle Physics and Cosmology.

- This talk discussed a collaboration between the nodes in Leipzig and Princeton. The other two are Amsterdam and Munich.
- Our paper was submitted to the journal

  Discrete and Computational Geometry

- **Thanks for listening!**

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