Title: Regular black holes: from non-linear electrodynamics to pure gravity models

Speakers: David Kubiznak

Collection/Series: Strong Gravity

Subject: Strong Gravity

Date: May 01, 2025 - 1:00 PM

URL: https://pirsa.org/25050014

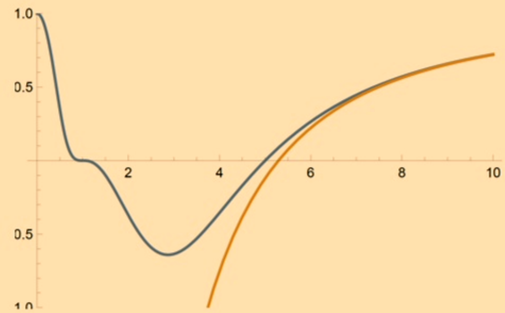
Abstract:

It is well known that (static) regular black hole spacetimes can be sourced by appropriately chosen theories of non-linear electrodynamics. More recently, it was shown that many such models can also be obtained as solutions of vacuum gravity equations, upon considering an infinite series of quasi-topological higher-curvature corrections. After reviewing both these approaches, I will show that the latter construction can be upgraded to yield regular black holes with vanishing inner horizon surface gravity -- a necessary condition for the absence of classical instabilities associated with mass inflation on the inner horizon. I will also comment on singular charged black holes in theories with finite electromagnetic self-energy.

Regular black holes: from non-linear electrodynamics to pure gravity models

David Kubizňák

(ITP, Charles University)



Strong Gravity Seminar

Perimeter Institute, Waterloo, Canada

May 1, 2025

And God said...

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_e$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

...and there was light.

Plan for the talk

I. A few words about **regular black holes**

II. Non-linear electrodynamics & Regular BHs

III. Pure gravity models

  I. Regular BHs from **quasitop gravities**

  II. Inner-extremal BH models

IV. Singular BH models from **NLE & COMBO!**

V. Summary

Penrose singularity theorem:

- Pseudo-Riemannian geometry provides an adequate description of spacetime
- Trapping surface is formed
- The spacetime is globally hyperbolic
- Null convergence condition holds:

$$R_{ab}K^aK^b \geq 0 \quad \Rightarrow$$

Spacetime is **geodesically incomplete!**

Instead: Regular black holes

- Pseudo-Riemannian geometry provides an adequate description of spacetime
- Trapping surface is formed
- Spacetime is geodesically complete
- There are no curvature singularities

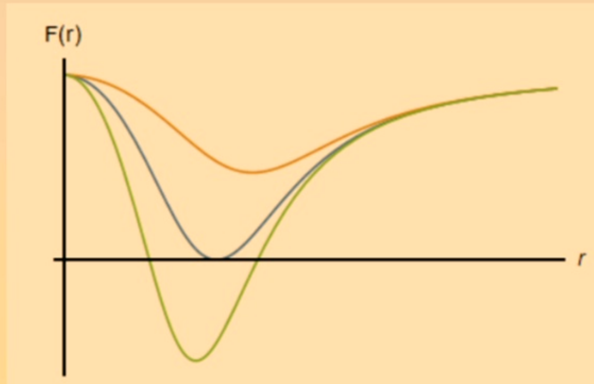
Regular black holes

$$ds^2 = -\, e^{-2\phi(r)} F(r) dv^2 + 2e^{-\phi(r)} dv dr + r^2 d\Omega^2$$

- Horizon condition: $F(r) = 0$
- Moreover

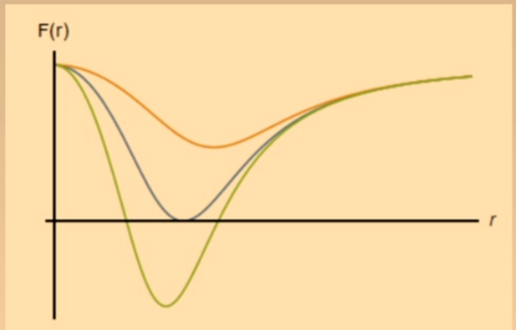
$$\lim_{r->\infty} F(r) = 1 \qquad \lim_{r->0} F(r) = 1$$

- Even number of horizons

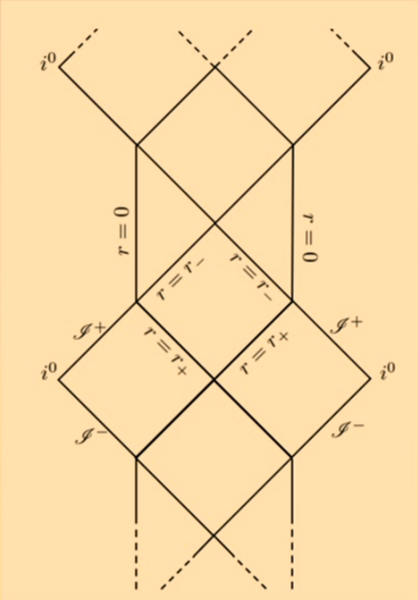


Regular black holes

$$ds^2 = -\,e^{-2\phi(r)}F(r)dv^2 + 2e^{-\phi(r)}dvdr + r^2d\Omega^2$$



- Reissner-Nordstrom like causal structure



- Surface gravities

$$\kappa_\pm = \frac{1}{2}e^{-\phi(r_\pm)}\frac{dF}{dr}\bigg|_{r=r_\pm}$$

$$\Rightarrow \quad \kappa_- < 0\,, \kappa_+ > 0$$

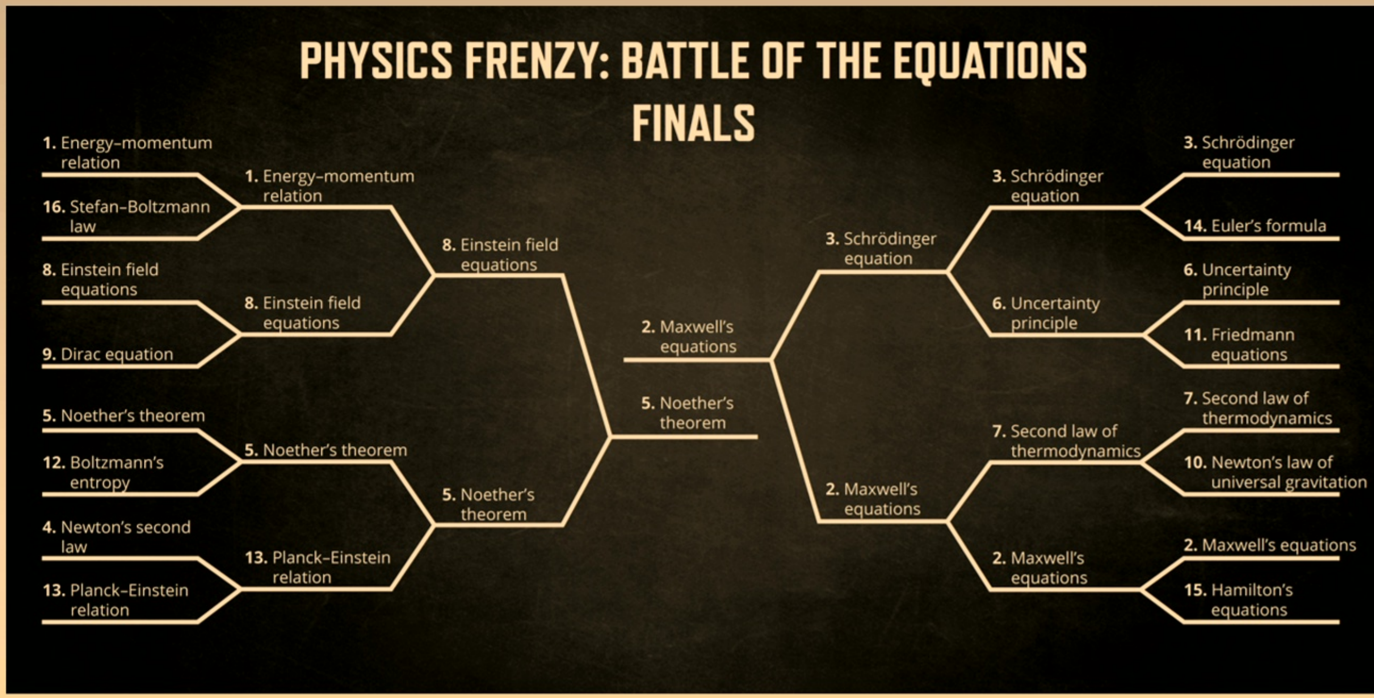
Examples of regular black holes

$$ds^2 = -e^{-2\phi(r)}F(r)dv^2 + 2e^{-\phi(r)}dvdr + r^2d\Omega^2$$

$$F(r) = 1 - \frac{2m(r)}{r} \qquad \phi = 0$$

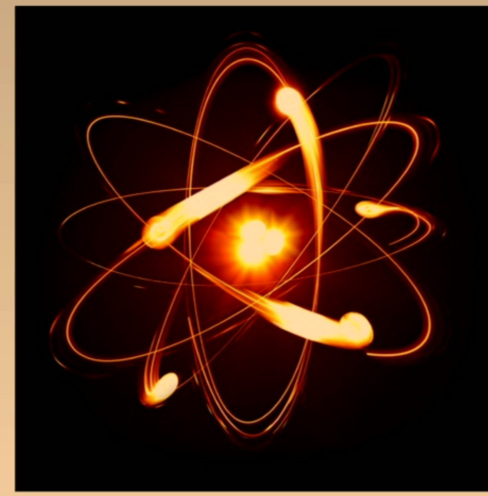
- Bardeen BH: $$F = 1 - \frac{2Mr^2}{\left(r^2 + l^2\right)^{3/2}}$$
- Hayward BH: $$F = 1 - \frac{2Mr^2}{r^3 + 2Ml^2}$$
- Dymnikova BH: $$F = 1 - 2Mr^2\left(1 - e^{-r^3/2Ml^2}\right)$$

Perimeter's battle of equations (2022)



What is non-linear electrodynamics (NLE)?

- Problem: In Maxell theory, field of a point-like electron **diverges** and has an **infinite self-energy**
- Idea: **Modify Maxwell equations** in strong field regime to get finite field and self-energy (Mie 1912)



NLE framework:

$$\mathcal{S} = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Maxwell: $\mathcal{L}^{(\mathrm{M})} = -\frac{1}{2}\mathcal{S} \Rightarrow \mathcal{L} = \mathcal{L}(\mathcal{S}, \dots)$

What is non-linear electrodynamics (NLE)?

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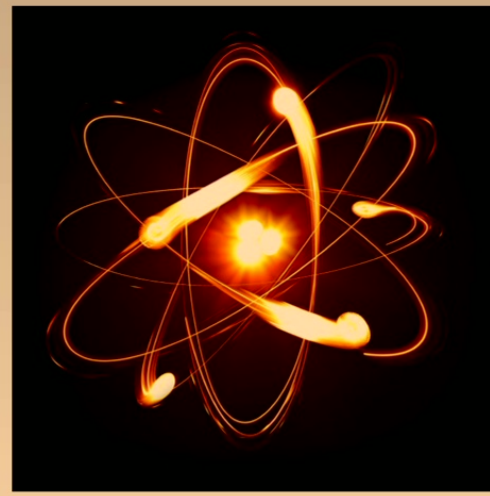
- Idea: **Modify Maxwell equations** in strong field regime to get finite field and self-energy (Mie 1912)



NLE framework:

In general: invariants can be extracted from eigenvalues of F

$$\mathrm{Tr}(F^2)\,,\quad \mathrm{Tr}(F^4)\,,\quad \ldots\quad \mathrm{Tr}(F^{2[d/2]})$$

In 4d: $\mathcal{S}=\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\,,\quad \mathcal{P}=\frac{1}{2}F_{\mu\nu}(*F)^{\mu\nu}$

What is non-linear electrodynamics?

$$\mathcal{L} = \mathcal{L}(\mathcal{S}, \mathcal{P}^2)$$

- (generalized) **Maxwell equations**

$$d * D = 0\,, \quad dF = 0 \qquad D = D(F, *F)$$

$$D_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} = 2\Big(\mathcal{L}_{\mathcal{S}} F_{\mu\nu} + \mathcal{L}_{\mathcal{P}} * F_{\mu\nu}\Big)$$

- **Einstein equations** $G_{\mu\nu} - 8\pi T_{\mu\nu} = 0$

$$T^{\mu\nu} = -\frac{1}{4\pi}\Big(2F^{\mu\sigma}F^{\nu}{}_{\sigma}\mathcal{L}_{\mathcal{S}} + \mathcal{P}\mathcal{L}_{\mathcal{P}}g^{\mu\nu} - \mathcal{L}g^{\mu\nu}\Big)$$

What are the criteria for selecting $\mathcal{L} = \mathcal{L}(\mathcal{S}, \mathcal{P}^2)$?

- Principle of **correspondence (POC)**

$$\lim_{F_{\mu\nu}\to 0} \mathcal{L} = \frac{1}{2}\mathcal{S} + O(\mathcal{S}^2, \mathcal{P}^2)$$

- Absence of **birefringence:**

  = 2 dof of EM field in general propagate in different speeds, according to the **effective optical metrics**.

- **Symmetries:** Electromagnetic duality, Weyl/Conformal invariance, …

- **Other criteria:** Regularity of fields, exact self-gravitating solutions, ….

Born-Infeld theory

M. Born and L. Infeld, Nature 132, 970 (1933); Foundations of new field theory, Proc. Roy. Soc A 144, 425 (1934).

- **Principle of "finiteness"**

$$L = \frac{1}{2}mv^2 \;\rightarrow\; L = mc^2\Big(1 - \sqrt{1 - \frac{v^2}{c^2}}\Big)$$

$$\mathcal{L} = \frac{1}{2}\mathcal{S} = -\frac{1}{2}E^2 \;\rightarrow\; \mathcal{L}_{\rm BI} = b^2\Big(\sqrt{1 - \frac{E^2}{b^2}} - 1\Big) = b^2\Big(\sqrt{1 + \frac{\mathcal{S}}{b^2}} - 1\Big)$$

- **Principle of "covariance"**

$$\mathcal{L}_{\rm BI} = -\frac{b^2}{\sqrt{-g}}\sqrt{-\det\Big(g_{\mu\nu} + \frac{F_{\mu\nu}}{b}\Big)} + b^2 = \sqrt{-g}\Big(\sqrt{1 + \frac{\mathcal{S}}{b^2} - \frac{\mathcal{P}^2}{4b^4}} - 1\Big)$$

Born-Infeld theory

M. Born and L. Infeld, Nature 132, 970 (1933); Foundations of new field theory, Proc. Roy. Soc A 144, 425 (1934).

$$\mathcal{L}_{\rm BI} = -\frac{b^2}{\sqrt{-g}}\sqrt{-\det\left(g_{\mu\nu} + \frac{F_{\mu\nu}}{b}\right)} + b^2$$

- **Phoenix from the ashes:** unique theory without birefringence (60s), string theory (80s), DBI action (80s), early Universe cosmology (2000), …

$$E_{\rm BI} = \frac{Q}{\sqrt{r^4 + Q^2/b^2}}$$

… regularizes the field in the origin

Other NLEs

- **ModMax theory:** maximally symmetric NLE

  Theorem: The most general NLE theory that possesses **SO(2) duality** invariance and **conformal symmetry** is the following **ModMax theory**:

$$\mathcal{L} = -\frac{1}{2}\left(\mathcal{S}\cosh\gamma - \sqrt{\mathcal{S}^2 + \mathcal{P}^2}\sinh\gamma\right)$$

I. Benados, K. Lechner, D. Sorokin, P.K. Townsend, *A nonlinear duality-invariant conformal extension of Maxwell's equations*, Phys. Rev. D102 121703 (2020).

- **RegMax theory:**

  "Close to Maxwell" regarding the existence of analytic **self-gravitating solutions**:

$$E_{\mathrm{RM}} = \frac{Q}{(r+r_0)^2}$$

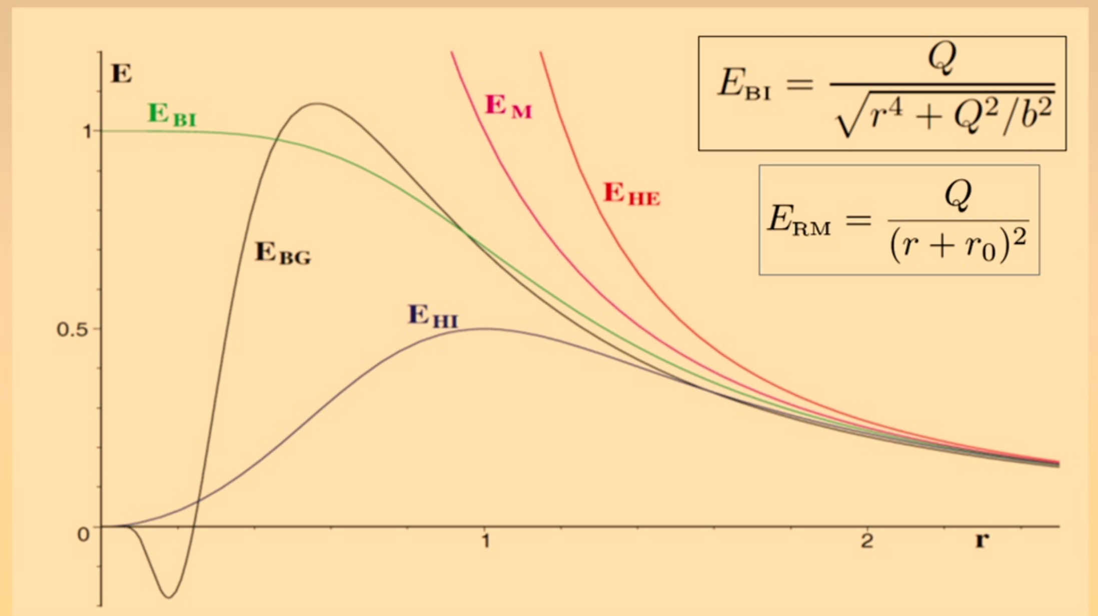
T. Hale, DK, O. Svitek, T. Tahamtan, *Solutions and basic properties of regularized Maxwell theory*, PRD 107 (2023) 12, 124031; Arxiv:2303.16928.

Other NLEs

- **Heisenber-Euler theory:** mimics features of QED

$$\mathcal{L}_{\rm HE} = \frac{1}{2}\mathcal{S} + \alpha\Big(\mathcal{S}^2 + \frac{7}{4}\mathcal{P}^2\Big) + \dots\,, \quad \alpha = \frac{2}{45}\frac{\hbar e^4}{m_e^4} \approx (\text{length})^2\,.$$

- **"Regular" electric solutions:**



Regular black holes in NLE?

$$ds^2 = -Nf\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2$$

$$A = e\phi dt$$

- Characterized by **single metric** function f

$$N = 1 \qquad \left[T_{\mu\nu} l^\mu l^\nu = 0 \right]$$

T. Jacobson, *When is g_tt g_rr=-1?* CQG24 (2007) 5717 [0707.3222].

- E=dϕ/dr determined **algebraically**
- f given by an **integral**
- Is it **singular?**

Regular black holes in NLE?

- **Born-Infeld (1934)** -- E field regular, but curvature singularity remains
- **Hoffmann & Infeld** (1937) – 1st ever example of regular BH spacetime
- **Ayon-Beato & Garcia (1998)** regular electrically charged (no single L throughout the spacetime)

  (No Lagrangian formulation - E not monotonous)

Theorem 4: (Bronnikov 2001). The coupled system of NLE-Einstein equations, satisfying POC ($\pounds \to 0$, $\pounds_{,F} \to 1$ as $F \to 0$), does not admit a static, spherically symmetric solution with a regular center and a nonzero electric charge.

- **Bardeen black holes** sourced by *magnetic monopoles* in NLE (Ayon-Beato & Garcia 2000)

NLE regular BHs: reverse engineering

Z.Y. Fan and X. Wang, *Construction of regular black holes in general relativity*, Phys.Rev. D94, 124027 (2016)

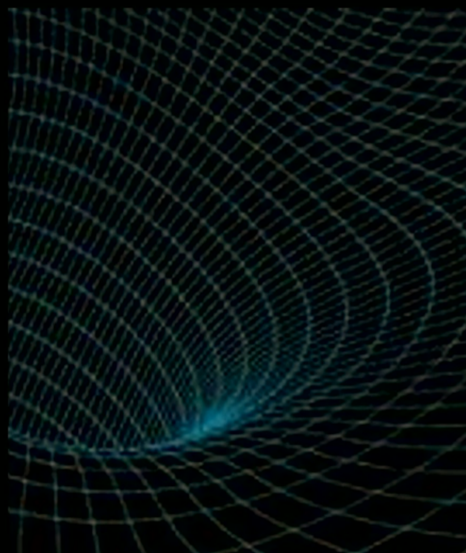
- **Magnetically charged:** Maxwell equations automatically satisfied
- **The only** Einstein equation reads

$$\mathcal{L} = -2\Big(\frac{f'}{r} + \frac{f-1}{r^2}\Big) \qquad \mathcal{F} = \frac{2Q_m^2}{r^4}$$

…for any f, we can find L=L(r) and thence L=L(F).

NLE regular black hole industry
(only spherical solutions!)

- **Problem:** the theory depends on parameters of the solution (no free mass parameter). The same is true for "full solutions" (typically one has to set M=0 to have regular BH).



II) Pure gravity models

Gravitational action

- To write the **Gravitational Action** we want
  - **Scalar Lagrangian** -- diffeomorphism invariance
  - **Second-order (E-L)** equations for the metric

- One possibility is to write the **Einstein-Hilbert action**

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int \sqrt{-g}\mathcal{L}\,, \quad \mathcal{L} = R$$

- Is this just the simplest choice or can we add other scalars?

$$R^2\,,\;\; R_{\mu\nu}R^{\mu\nu}\,,\;\; R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}\,,\;\; \nabla_\mu R \nabla^\mu R\,,\ldots$$

Lovelock's Theorem

D. Lovelock, *The Einstein Tensor and Its Generalizations",*
Journal of Mathematical Physics. **12** *(3): 498–501 (1971).*

In 4D, the Einstein-Hilbert action is the only local action (apart from the cosmological constant and topological terms) that leads to the **second order** differential equations for the metric. **In higher D,** we can have **Gauss-Bonnet** (Lovelock) theories.

Einstein's theory is the unique theory in 4D!

Gauss-Bonnet gravity

$$\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

- In **D<4** it **identically vanishes**!
- In 4D, the **Gauss-Bonnet term** is **topological** (total derivative!?!?).
- In **D=5 and higher** dimensions it yields non-trivial EOMs:

$$\begin{aligned} H_{\alpha\beta} \;&=\; -\frac{1}{2}g_{\alpha\beta}\mathcal{G} + 2RR_{\alpha\beta} - 4R_{\alpha\gamma}R_{\beta}{}^{\gamma} \\ &\quad +4R_{\gamma\alpha\beta\delta}R^{\gamma\delta} + 2R_{\alpha}{}^{\gamma\delta\kappa}R_{\beta\gamma\delta\kappa} = 0\,. \end{aligned}$$

2nd-order PDEs !!!

Lovelock gravity

= Unique higher-curvature (with local action) gravity that yields **2nd-order PDEs** for the metric

$$\mathcal{L} = \frac{1}{16\pi G_N} \sum_{k=0}^{K} \alpha_k \mathcal{L}^{(k)} \qquad K = \lfloor \frac{d-1}{2} \rfloor$$

where $\mathcal{L}^{(k)}$ are the 2k-dimensional Euler densities

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k} R_{a_1 b_1}{}^{c_1 d_1} \ldots R_{a_k b_k}{}^{c_k d_k}$$

- k=0: cosmological term $\Lambda = -\alpha_0/2$
- k=1: Einstein-Hilbert term R **(topological in 2D)**
- k=2: Gauss-Bonnet term $\mathcal{G}$ **(topological in 4D)**
- k=3: 3rd-order Lovelock **(topological in 6D)**

"Natural generalization of Einstein's theory in higher dimensions"

Quasi-topological gravities

- General higher-curvature theory of the form: $\mathcal{L}(g^{ab}, R_{abcd})$
- The field equations read

$$\mathcal{E}_{ab} = P_{a}{}^{cde} R_{bcde} - \frac{1}{2} g_{ab} \mathcal{L} - 2\nabla^c \nabla^d P_{acdb} = 0 \,, \quad P^{abcd} \equiv \frac{\partial \mathcal{L}}{\partial R_{abcd}}$$

- Quasitop obey the following for **spherically symmetric spacetimes**: $\nabla^d P_{acdb} = 0$
(provide a **basis** for effective grav. action)
- Their Lagrangian can be written as: $D \geq 5$

$$I_{\rm QT} = \frac{1}{16\pi G} \int \mathrm{d}^D x \sqrt{|g|} \left[R + \sum_{n=2}^{K} \alpha_n \mathcal{Z}_n \right]$$

Quasitop: spherical solutions

- Considering spherical spacetimes:

$$\mathrm{d}s^2 = -N(r)^2 f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\, \mathrm{d}\Omega^2_{D-2}$$

- The field equations read:

$$\frac{dN}{dr} = 0\,, \qquad \frac{d}{dr}\left[r^{D-1}h(\psi)\right] = 0$$

$$h(\psi) \equiv \psi + \sum_{n=2}^{K} \alpha_n \psi^n, \quad \psi \equiv \frac{1-f(r)}{r^2}$$

- The solution is: $N = 1$ $\quad h(\psi) = \dfrac{m}{r^{D-1}}$

$\Rightarrow$ $f = 1 - r^2\, h^{-1}(m/r^{D-1})$

Ingenious idea

P. Bueno, P. A. Cano, R. A. Hennigar, *Regular Black Holes from Pure Gravity*, Arxiv:2403:04827.

$$f = 1 - r^2 \, h^{-1}(m/r^{D-1})$$

$$h(\psi) \equiv \psi + \sum_{n=2}^{K} \alpha_n \psi^n$$

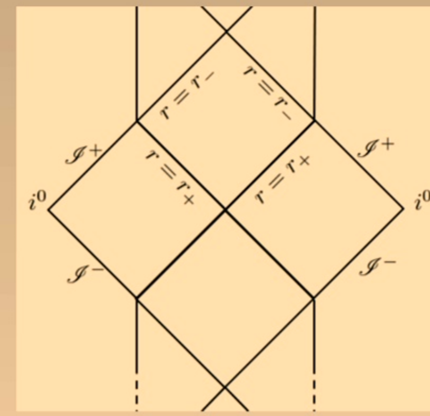
- **Example**: take $\alpha_n = \alpha^{n-1}$

$$\Rightarrow \quad \frac{\psi}{1 - \alpha\psi} = \frac{m}{r^{D-1}} \qquad f = 1 - \frac{mr^2}{r^{D-1} + \alpha m}$$

… Hayward black hole in D dimensions

- Similarly, one can recover other models (as long as the form of f above)

Mass inflation instability of inner horizon



- We know that regular black holes look like **RN**, which suffer from **mass inflation instability**

  Poisson & Israel, *Internal structure of black holes*, Phys. Rev. D41 (1990) 1796.

…saves **strong cosmic censorship** (together with additional semi-classical instability)

- Regular black holes also unstable

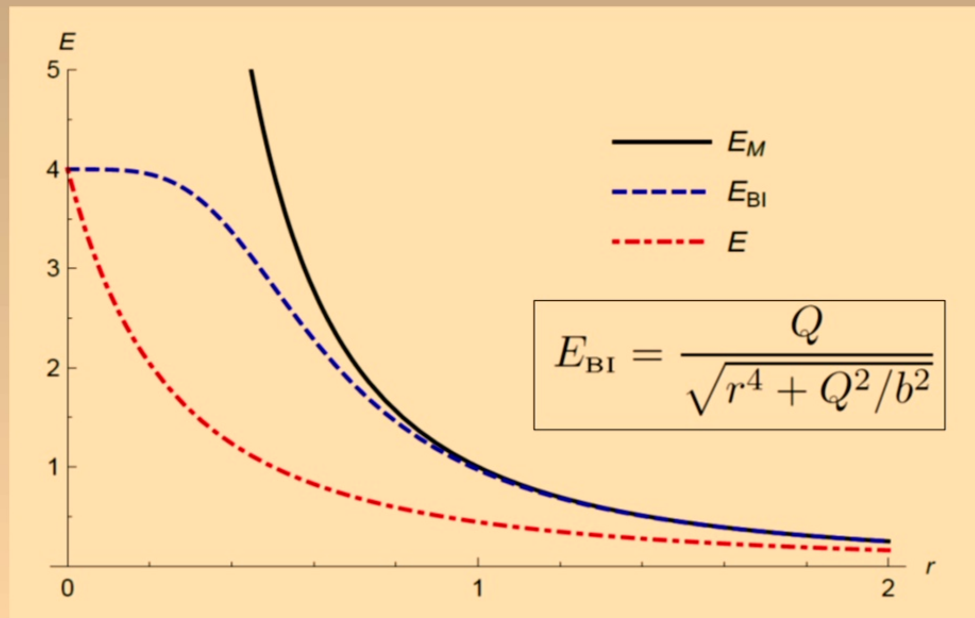
  Carballo Rubio, Di Filippo, Liberati, Pacilio, Visser, *Inner horizon instability and unstable cores of regular black holes*, JHEP 05 (2021) 132.

  Using null shell formalism, showed exponentially growing Misner-Sharp mass at the horizon….

$$m \propto e^{|\kappa_-|v}$$

IV) A remark on singular BH models from non-linear electrodynamics

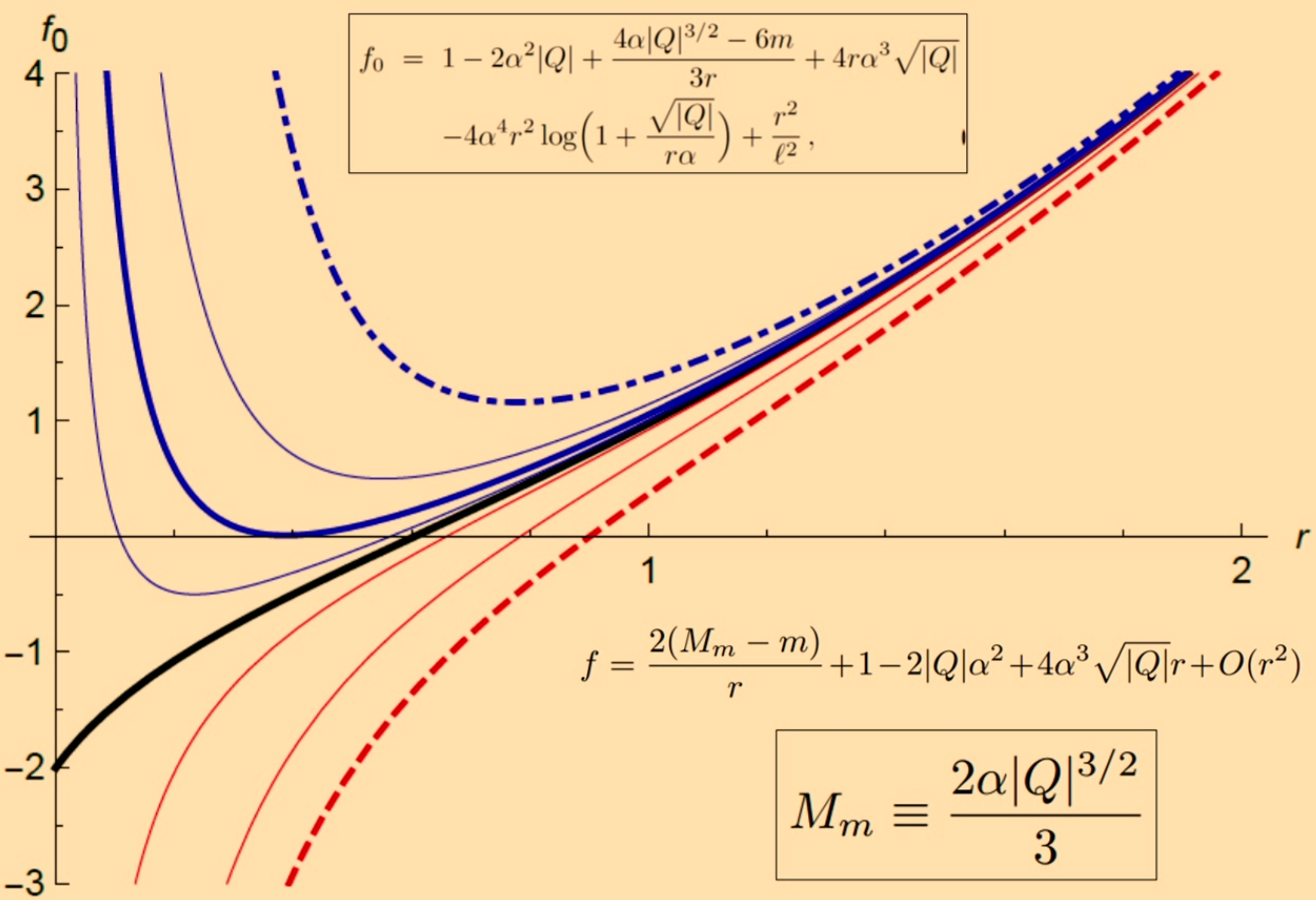
Look closer at two "finite models"



"Simplest regularization", keeping the quadratic profile with a "shifted origin":

$$E = \frac{Q}{(r + r_0)^2}, \quad r_0 = \frac{\sqrt{|Q|}}{\alpha}$$

RegMax spherical solution



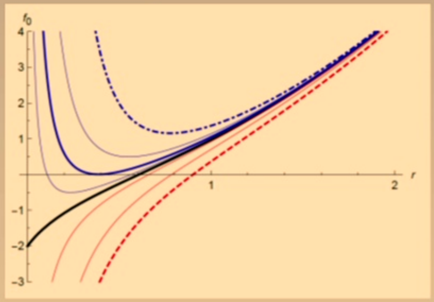
$$f_0 = 1 - 2\alpha^2|Q| + \frac{4\alpha|Q|^{3/2} - 6m}{3r} + 4r\alpha^3\sqrt{|Q|} - 4\alpha^4 r^2 \log\left(1 + \frac{\sqrt{|Q|}}{r\alpha}\right) + \frac{r^2}{\ell^2},$$

$$f = \frac{2(M_m - m)}{r} + 1 - 2|Q|\alpha^2 + 4\alpha^3\sqrt{|Q|}r + O(r^2)$$

$$M_m \equiv \frac{2\alpha|Q|^{3/2}}{3}$$

RegMax spherical solution

- when $m < M_m$ "RN-like branch"
- when $m > M_m$ "S-like branch"



- **Astrophysical BHs:**
  - Have self-energy smaller than mass (weakly charged).
  - Will feature no Cauchy horizon and associated with it instabilities – **effective strong cosmic censorship**? (no problems with **extremal black hole pathologies**)
  - There seems to be a more strict **effective upper bound** on how large the charge can be!

$$M_m \equiv \frac{2\alpha|Q|^{3/2}}{3} \quad \Rightarrow \quad |Q| < \left(\frac{3M}{2\alpha}\right)^{2/3}$$

Summary

1) It is easy to cook up **regular black hole geometries.** These can be **embedded** in classical theories. Allows to study TDs,…

2) **Nonlinear electrodynamics** is a very popular framework for regular BHs. Due to **no-go theorems** – limited to magnetically charged solutions or models violating POC. However, recent reformulation allows for electrically charged solutions as well.

3) Recent new construction of regular black holes **from pure gravity** – using infinite series of **quasitop gravities** (fine-tuned couplings). These can form **dynamically** (Arxiv:2412.02742).

   This construction also allows for (extremely tuned) **inner-extremal black holes** that do not seem to suffer from mass inflation instability.

Summary

4) Note also that in NLE models with finite self energy, we can have “**singular black holes**” without **Cauchy horizons** when “self energy” is smaller than “mass”. This seems relevant for astrophysical black holes!

5) When coupled to “re-summed quasi-topological gravity”, the two types of black holes give raise to regular black holes with **AdS, dS,** cores, respectively.

6) Recent indications of “generic” regular black holes from 6^{th} derivative gravity (Giacchini & Kolar, Arxiv:2406:00997 & ArXiv:2503.17318)