Title: Carroll limit of gravity and spacelike (Belinski-Khalatnikov-Lifshitz) singularities

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Abstract:

Einstein's theory admits interesting limits with different causal structures obtained by letting the speed of light go to infinity (Galilean or "non-relativistic" limit) or to zero (Carrollian, sometimes called "ultrarelativistic", limit). The Carroll limit turns out to be relevant near spacelike (cosmological) singularities, and particularly so when p-form gauge fields are coupled to gravity as in the context of extended supergravities. The resulting differential equations possess a remarkable interpretation in terms of infinite-dimensional Kac-Moody algebras. The talk will review various aspects of Carroll invariant theories, of the Belinski-Khalatnikov-Lifshitz analysis of spacelike singularities and of the related symmetries.

Carroll limit of gravity and spacelike (Belinski-Khalatnikov-Lifshitz) singularities

Marc Henneaux

Perimeter, 1 May 2025

Introduction

There has been some interest recently in the Carrollian limit ($c \to 0$) of general relativity.

This interest is mostly motivated by the close connection that exists between Carroll geometry and null hypersurfaces, the BMS group and the conformal Carroll group etc.

There is (at least) another situation where the Carrollian limit is relevant,

which is the Belinskii, Khalatnikov and Lifschitz ("BKL") analysis of spacelike (cosmological) singularities.

("classic" topic from the 70's)

The BKL analysis (and the relevance of the Carrollian limit) hold also in higher dimensions and for coupling to matter fields.

Introduction

The purpose of the talk is to :

- briefly review the BKL analysis extended to higher dimensions and with the couplings to matter fields relevant to supergravity ;
- discuss its connection with the Carroll limit ;
- show that in spite of the ultralocal character of the relevant equations, non trivial (chaotic) dynamics emerge in the limit,

  which can be described in terms of "cosmological billiards"

  that can be related to hyperbolic Coxeter groups and hyperbolic Kac-Moody algebras (E_{10} for 11-dimensional supergravity)

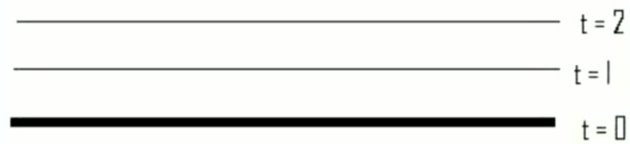
  (structures which will be themselves explained) ;

  (talk is a mixture of old and more recent material)

Spacelike (cosmological) singularities

Is a spacelike ("cosmological") singularity generic? The way followed by BKL to answer this question was constructive.

Assume that there is a spacelike singularity, and take it at $t = 0$ for convenience.



Investigate then the Einstein equations as one approaches the singularity, $t \to 0$.

BKL were able to find the general solution in the limit,

and to show that this solution contained indeed sufficiently many arbitrary physical functions to match generic initial data.

This is remarkable given that the Einstein equations are non linear and remain so in the BKL limit.

Dynamics near a spacelike singularity

BKL considered originally pure gravity in four spacetime dimensions,

but their analysis is very general and has been extended to general coupled gravity-p-forms-scalar fields systems in higher dimensions, as relevant to supergravity models.

We shall sketch the general ideas here, referring for more information to the monograph (with references to the original literature)

V. Belinski and M. Henneaux, "The Cosmological Singularity," Cambridge University Press, 2017.

Dynamics near a spacelike singularity

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We will start right away with the coupled Einstein-3-form system in eleven spacetime dimensions, which is relevant for M-theory and conceptually very clear.

The canonical variables are then $g_{ij}(t,\mathbf{x})$, $\pi^{ij}(t,\mathbf{x})$, $A_{ijk}(t,\mathbf{x})$, $\pi^{ijk}(t,\mathbf{x})$.

The dynamics is controlled by the Hamiltonian constraint $\mathscr{H} \approx 0$, which can be decomposed as a sum of four terms :

$$\mathscr{H} = \mathscr{K} + \mathscr{R} + \mathscr{E}^e + \mathscr{E}^m \qquad \text{where :}$$

- $\mathscr{K}$ = kinetic term for the gravitational field, $\mathscr{K} \sim \pi^{ij}\pi_{ij} - \pi^2$
- $\mathscr{R} \sim$ spatial curvature term R
- $\mathscr{E}^e$ = kinetic term for the 3-form = electric energy density, $\mathscr{E}^e \sim \pi^{ijk}\pi_{ijk}$
- $\mathscr{E}^m$ = magnetic energy density, $\mathscr{E}^m \sim F^{ijkm}F_{ijkm}$

Time derivatives are in $\mathscr{K}$ and $\mathscr{E}^e$, spatial gradients are in $\mathscr{R}$ and $\mathscr{E}^m$.

BKL and Ultralocality

One can show that as $t \to 0$, time derivatives dominate spatial gradients, so that the dynamics is effectively controlled by $\mathscr{K} + \mathscr{E}^e$.

The dynamics decouples therefore at each spatial point (equations become in the limit ODE's with respect to time, a finite number at each point) - "ultralocality".

The dynamics is thus effectively ultralocal.

This is just as in Carrollian field theories!

(Note : Things are more subtle in the case of pure gravity where curvature terms cannot be neglected all the time, even in the limit, but the above statements are strictly true (as $t \to 0$) whenever p-forms are coupled to gravity.)

"Minimal" Carroll geometry

Geometric ingredients :

(i) Degenerate metric $g_{\alpha\beta}$ of rank $D-1$ which is positive semi-definite, i.e. $\det g_{\alpha\beta} = 0$, $g_{\alpha\beta} v^\alpha v^\beta \geq 0$, with $g_{\alpha\beta} v^\alpha v^\beta = 0$ if and only if the vector v^α is along the null direction ("null vector")

(ii) Non-vanishing density Ω of weight one (for integration)

Null vectors can then be "normalized" through

$$\mathscr{G}^{\alpha\beta} = \Omega^2 n^\alpha n^\beta$$

where $\mathscr{G}^{\alpha\beta}$ are the minors of $g_{\alpha\beta}$ ($\mathscr{G}^{\alpha\beta} g_{\gamma\beta} = 0$ since $\det g_{\alpha\beta} = 0$).

In general the metric and the volume element depend on x^μ, $g_{\alpha\beta}(x)$, $\Omega(x)$. If they are constant, the Carroll geometry is flat.

No extra structure (parallel transport, unique foliation by transverse hyperplanes etc) is added to this "minimal definition", but it can be useful to introduce such extra structure as auxiliary tools.

Carroll geometry

Because the metric is degenerate, it has no inverse, i.e., there is no tensor $g^{\alpha\beta}$ such that $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$.

One can nevertheless raise indices by introducing the extra structure of a one-form θ_α such that $\theta_\alpha n^\alpha = 1$.

One then defines the twice contravariant symmetric tensor $G^{\alpha\beta}(g_{\rho\sigma}, n^\lambda, \theta_\mu)$ such that

$$G^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma - n^\alpha\theta_\gamma.$$

If one imposes in addition the condition $G^{\alpha\beta}\theta_\alpha\theta_\gamma = 0$, the tensor $G^{\alpha\beta}$ is completely determined.

Since the one-form θ_α comes on top of the basic Carroll structure defined by the degenerate metric $g_{\alpha\beta}$ and density Ω, we shall insist that "Carrollian physics" should *not* depend on θ_α.

For instance, the scalar product $G^{\alpha\beta}v_\alpha w_\alpha$ does not depend on the choice of θ_α if the covectors v_α and w_α are both transverse ($n^\alpha v_\alpha = 0 = n^\alpha w_\alpha$).

Flat Carroll geometry

A flat Carroll structure is a vector space equipped with a Carroll inner product

and a choice of normalization of the null vectors.

One can take

$$(g_{\alpha\beta}) = \begin{pmatrix} 0 & 0 \\ 0 & I_{d\times d} \end{pmatrix}, \qquad (n^\alpha) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \Omega = 1$$

One finds $\mathscr{L}_n g_{\alpha\beta} (= -2K_{\alpha\beta}) = 0$,

so that metric-preserving, symmetric connections exist.

(One can take $\Gamma^\alpha_{\beta\gamma} = 0$, a choice adapted to the underlying linear structure.)

Carroll groups

The Carroll group $\mathscr{C}(D)$ is the group of diffeomorphisms that preserve this flat structure.

Because the metric is degenerate, it is infinite-dimensional.

The form of the infinitesimal transformations is

$$\delta x^0 = f(x^k), \quad \delta x^k = \omega^k_{\ m} x^m + a^k, \quad \omega_{km} = -\omega_{mk}.$$

It is straightforward to verify that these transformations preserve $ds^2 = \delta_{ij} dx^i dx^j$ and Ω (or $\frac{\partial}{\partial t}$) and are the only ones.

The transformations $\delta x^0 = f(x^k)$ are like BMS supertranslations.

Carroll groups

Carroll limit of gravity and spacelike (Belinski-Khalatnikov-Lifshitz) singularities

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The Carroll group $C(D)$ is the group of (inhomogeneous) linear transformations that preserve this flat structure.
Infinitesimally,

$$\delta x^0 = a^0 + b_k x^k, \quad \delta x^k = \omega^k{}_m x^m + a^k, \quad \omega_{km} = -\omega_{mk},$$

While the flat tensors $g_{\alpha\beta}$ and Ω are numerically invariant under Carroll transformations (by definition of the Carroll group), this is not so for the extra structure θ_α.
One finds

$$\delta\theta_0 = 0, \qquad \delta\theta_m = -b_m + \theta_k \omega^k{}_m.$$

One can use Carroll transformations to set $\theta_k = 0$, so that θ_α reads ($\theta_0 = \theta_\alpha n^\alpha = 1$),

$$(\theta_\alpha) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix},$$

but this special form is not preserved in all Carroll frames.

Carroll groups

When θ_α takes that special form, the contravariant tensor $G^{\alpha\beta}$ reduces to

$$(G^{\alpha\beta}) = \begin{pmatrix} 0 & 0 \\ 0 & I_{d\times d} \end{pmatrix}$$

but in a general frame where $((\theta_\alpha) = (1, \theta_a))$, it reads

$$(G^{\alpha\beta}) = \begin{pmatrix} \delta^{cd}\theta_c\theta_d & -\delta^{bc}\theta_c \\ -\delta^{ac}\theta_c & \delta^{ab} \end{pmatrix}$$

It is thus not numerically invariant under Carroll boosts, for which one can actually verify that

$$\delta G^{\alpha\beta} \equiv \frac{\partial G^{\alpha\beta}}{\partial \theta_\mu}\delta\theta_\mu \sim n^\alpha \lambda^\beta + n^\beta \lambda^\alpha.$$

Flat space actions

Flat space actions can be constructed using tensor calculus.

The coupling to (Carrollian) gravity is then straightforward.

The method consists in building invariant Lagrangians using the dynamical fields and the Carroll invariant tensors (degenerate metric and null vector).

Take the example of a scalar field ϕ.

A quadratic Lagrangian in the first derivatives of ϕ should involve $\partial_\mu\phi\partial_\nu\phi$.

These indices can be contracted using the invariant tensor $n^\mu n^\nu$, yielding the Lagrangian density $\frac{1}{2}(n^\mu\partial_\mu\phi)^2\Omega$,

equal to $\frac{1}{2}(\partial_0\phi)^2$ in flat coordinates (electric limit of KG).

Clearly, $[\mathscr{E}(x),\mathscr{E}(y)] = 0$ with $\mathscr{E}(x) = \frac{1}{2}\pi^2$.

Flat space actions

Carroll limit of gravity and spacelike (Belinski-Khalatnikov-Lifshitz) singularities

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An alternative approach would be to impose through a Lagrange multiplier that $\partial_\alpha \phi$ is transverse, so that $G^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$ is well defined. So we try

$$S^M[\phi,\pi_\phi,\theta_\alpha] = \int d^Dx\big(\pi_\phi n^\alpha \partial_\alpha\phi - \frac{1}{2}G^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\big),$$

where we have made explicit that $G^{\alpha\beta}$ involves θ_α (which might matter off-shell).

The action possesses a gauge invariance that enables one to gauge the extra field θ_α away since a shift in θ_α can be compensated for by a shift in π_ϕ.

The action reduces then to the first order action

$$S^M[\phi,\pi_\phi] = \int d^Dx\big(\pi_\phi\partial_0\phi - \frac{1}{2}\delta^{ab}\partial_a\phi\partial_b\phi\big)$$

(magnetic limit of KG).

Flat space actions

An alternative approach would be to impose through a Lagrange multiplier that $\partial_\alpha \phi$ is transverse, so that $G^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$ is well defined. So we try

$$S^M[\phi, \pi_\phi, \theta_\alpha] = \int d^D x \big(\pi_\phi n^\alpha \partial_\alpha \phi - \frac{1}{2} G^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\big),$$

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(magnetic limit of KG).

Clearly again, $[\mathscr{E}(x), \mathscr{E}(y)] = 0$ with $\mathscr{E}(x) = \frac{1}{2}\delta^{ab}\partial_a\phi\partial_b\phi$.

Flat space actions

Carroll limit of gravity and spacelike (Belinski-Khalatnikov-Lifshitz) singularities

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This can be generalized to p-forms, which have also an electric or magnetic limit.

The electric limit keeps the electric energy density in the Hamiltonian;

the magnetic limit keeps the magnetic energy density.

So, for instance, the Lagrangian for the electric limit for the 3-form is $\sim (F_{\mu\nu\rho} n^\rho)^2 \Omega$.

More complicated (tachyonic) interactions can be defined ("swiftons")

but these interacting theories will not be needed here.

By construction, the Carroll-invariant theories constructed by tensor methods are invariant under all the transformations that preserve the flat Carroll structure, i.e., the infinite-dimensional group $\mathscr{C}$.

Gravitational actions

Carroll limit of gravity and spacelike (Belinski-Khalatnikov-Lifshitz) singularities

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One can also consider electric and magnetic limits of Einstein theory.

The easiest way is to start with the Hamiltonian formulation,

with $\mathscr{H} = \mathscr{K} + \mathscr{R}$.

In the first case one keeps the term $\mathscr{K}$ quadratic in the time derivatives in the Hamiltonian constraint;

in the second case, one keeps the curvature term $\mathscr{R}$.

The electric case has covariant Lagrangian $\Omega(K_{\mu\nu}K^{\mu\nu} - K^2)$.

It can also be viewed as the strong coupling limit of gravity $G \to \infty$.

BKL and Carroll

The Lagrangian describing the asymptotic behaviour of the bosonic sector of 11 D SUGRA is the sum of the electric limit of Einstein gravity and the electric limit of the 3-form theory.

In covariant form :

$$\mathscr{L}^{BKL} = \mathscr{L}^{e,Carroll}$$

with

$$\mathscr{L}^{e,Carroll} \sim \Omega\left[(K_{\mu\nu}K^{\mu\nu} - K^2) + (F_{\mu\nu\rho}n^\rho)^2\right]$$

(the Chern-Simons term can be dropped in the limit.)

If the theory is effectively Carrollian and the equations of motion reduce to ODE with respect to time... does this mean that the dynamics is trivial?

Dynamics near a spacelike singularity

No! At each spatial point, the dynamics is equivalent to the motion of a billiard ball in a portion of hyperbolic space,

and exhibits strong chaotic properties.

To show this, one diagonalizes the metric (by an "Iwasawa" = triangular change of frame)

$$ds^2 = -dt^2 + \sum_i a_i^2(t,\mathbf{x})\mathbf{l}_i{}^2$$

where a_i are the "scale factors" and $\mathbf{l}_i = l_{ij}(t,\mathbf{x})\,dx^j$ define the frame,

and one writes the 3-form potential and its conjugate in that frame.

Cosmological billiards

One then proves that as $t \to 0$,

the non-diagonal components of the metric (l_{ij}) and electromagnetic variables (A_{ijk}) freeze so that the interesting dynamics is carried by the scale factors a_i;

The electric energy density of the 3-form and the kinetic energy density of the off-diagonal terms can be neglected most of the time.

When both are negligible, the scale factors have the "Kasner behaviour" $a_i \sim t^{p_i}$.

As $t \to 0$, some term in the electric energy density or in the off-diagonal terms inevitably grows, however, and induces a "collision" from one Kasner regime to a new one. The collision is localized in time. After the collision has taken place, the term responsible for the collision decays, but another term will then grow, leading to another collision.

Cosmological billiards

This leads to a billiard picture, where the Kasner regime corresponds to free flight motion, and where the effect of the electric energy density and of the off-diagonal terms is to introduce "walls" against which there are collisions, changing one Kasner regime to another in a never-ending way.

By appropriately parametrizing the scale factors, one can actually map this asymptotic billiard description into a region of 9-dimensional hyperbolic space $\mathbb{H}_9$.

(9-dimensional because the 10 scale factors are not independent due to the Hamiltonian constraint).

The Kasner motion is a geodesic on $\mathbb{H}_9$.

The walls are hyperplanes in $\mathbb{H}_9$.

The dynamics is chaotic if the billiard volume is finite, which is the case for 11-d SUGRA.

Cosmological billiards

The billiard table for the four-dimensional Einstein-Maxwell theory (one billiard system at each point in space) in the Poincaré disk model. Straight lines are symmetry walls. Arc of circles are electric (also gravitational) walls. Thick lines indicate the dominant walls. The billiard table, which lies on the positive side of each wall, is indicated in grey. It forms a triangle with angles $0, \frac{\pi}{3}, \frac{\pi}{2}$.

Cosmological billiards

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Introduction

BKL Analysis

Carrollian field theories

BKL and Carroll

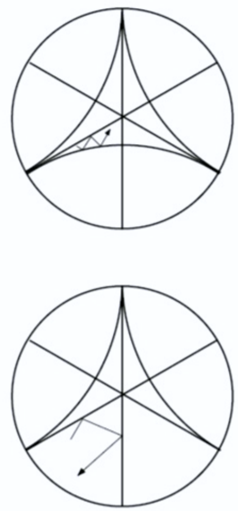
Cosmological billiards

Coxeter polyhedra

Kac-Moody algebras

Conclusions

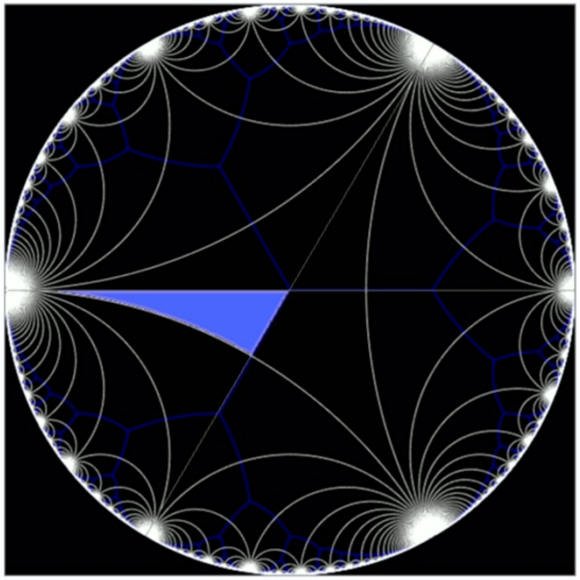
If some walls are removed, the never-ending behaviour might stop.



Free flight = Kasner behaviour ($a_i \sim t^{p_i}$, $\sum p_i = \sum p_i^2 = 1$) ; Collision against a wall (geometric reflection) = change from one Kasner regime to another

Coxeter Polyhedra

The billiard table possesses remarkable properties, encoded in the angles between the billiard walls :



Angles between billiard walls : $0 = \frac{\pi}{\infty}, \frac{\pi}{3}, \frac{\pi}{2}$. In this case, the triangle is the fundamental region of the group $PGL(2, \mathbb{Z})$ (A_1^{++}).

Coxeter Polyhedra

The billiard picture with walls that are hyperplanes forming a convex polyhedron is valid for general gravitational theories.

There are in general four types of walls defining the billiard region : symmetry walls, gravitational walls, p-form electric walls, p-form magnetic walls but gravitational walls are generically sub-dominant - precise rules for determining the walls from the Lagrangian have been given.

For pure gravity in any D, or supergravity models, the billiard table possesses the following additional properties :

Any two intersecting hyperplanes H_i, H_j defining the table form an acute dihedral angle equal to

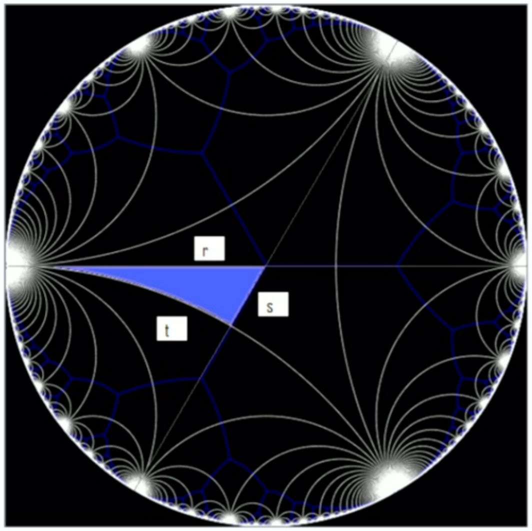
$$\frac{\pi}{m_{ij}} \qquad \text{(“Coxeter polyhedron")}$$

where m_{ij} is an integer, so that the product $s_i s_j$ of the reflections s_i and s_j is a rotation by the angle $\frac{2\pi}{m_{ij}}$

It is a simplex.

Coxeter Polyhedra

Coxeter groups



The reflections r, s and t, subject to the Coxeter relations

$$s^2 = 1, \ r^2 = 1, \ t^2 = 1, \ (rs)^3 = (sr)^3 = 1, \ (ts)^2 = (st)^2 = 1,$$

generate the symmetry group (the definition of a Coxeter group). [There are no other relations (standard Coxeter presentation) and the region in blue is a fundamental domain.]

Connection with Kac-Moody algebras

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The fact that the billiard domain is a finite-volume Coxeter simplex in hyperbolic space provides a direct connection with hyperbolic Kac-Moody algebras.

Kac-Moody algebras are algebras with a structure that generalizes the well-known triangular decomposition of finite-dimensional simple Lie algebras, e.g.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ g & h & 0 \end{pmatrix} + \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix} + \begin{pmatrix} 0 & b & c \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix}$$

for $sl(3)$.

Connection with Kac-Moody algebras

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Standard concepts of finite-dimensional Lie algebras based on this triangular decomposition generalize to Kac-Moody algebras.

(Cartan subalgebras, roots, simple roots, Cartan matrices, Chevalley relations, Serre relations etc)

In particular, one can define Weyl reflections and the Weyl group generated by the Weyl reflections, which is a Coxeter group.

It turns out that the billiard Coxeter groups of supergravity theories can be identified with the Weyl groups of hyperbolic (infinite-dimensional) Kac-Moody algebras.

Connection with Kac-Moody algebras

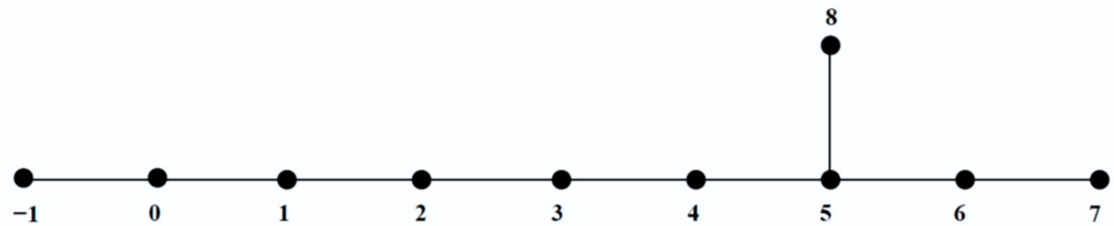
Dictionary

Gravity side		Kac-Moody side
Scale factors	↔	Cartan degrees of freedom
Billiard motion	↔	Lightlike motion in Cartan subalgebra
Walls	↔	Hyperplanes orthogonal to simple roots
Reflection against a wall	↔	Fundamental Weyl reflection

Connection with Kac-Moody algebras

For $11D$ SUGRA, the Kac-Moody algebra that emerges is

$E_{10} \equiv E_8^{++}$ (11-dimensional supergravity)



Connection with Kac-Moody algebras

Does this signal a hidden, infinite-dimensional symmetry that would exist independently of the cosmological context (the BKL analysis acting as a revelator) ?

In line with previous work on hidden symmetries and dimensional reduction (E_7, E_8, E_9, ...)

Efforts to answer positively the question based on non-linear realizations (P. West, T. Damour -MH -H. Nicolai) are promising ...

... but still largely incomplete.

Conclusions and Prospects

- The study of the dynamics of the gravitational field near a spacelike singularity (BKL) can be described in Carrollian terms

  because the spatial derivatives become negligible with respect to the time derivatives.
- This is one very strong motivation for studying Carrollian models (in addition to flat space holography).
- In spite of this drastic reduction, the dynamics remains extremely rich

  and reveals remarkable connections with hyperbolic Coxeter groups and Kac-Moody algebras.
- Open question : is this the tip of an iceberg indicating a huge symmetry ?
- Other open question : can we go beyond Carroll in a perturbative expansion ?

THANK YOU !