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Gauge Theory Bootstrap: computing pion dynamics from QCD

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Seminar at Perimeter Institute, 27/05/2025

Based on:
[YH and Kruczenski, Phys. Rev. Lett. 133, 191601, Phys. Rev. D. 110, 096001]
[YH and Kruczenski, arXiv: 2403.10772]
[YH and Kruczenski, arXiv: 2505.19332]

Low energy physics of asymptotically free gauge theory

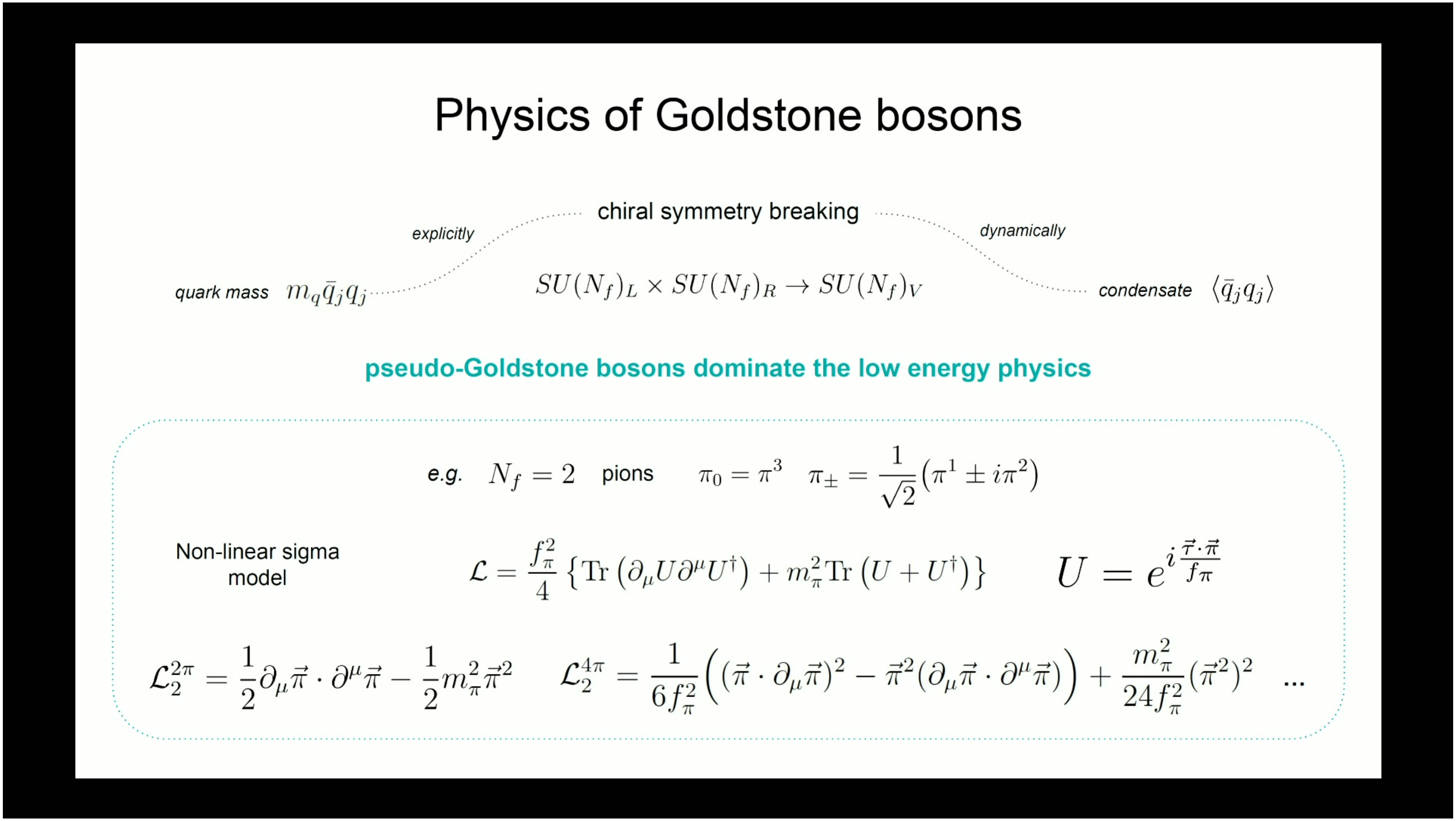
asymptotically free gauge theory $SU(N_c)$ with N_f massive quarks $m_q \ll \Lambda_{\mathrm{QCD}}$

confinement & chiral symmetry breaking

$$\mathcal{L} = i \sum_{j}^{N_f} \bar{q}_j \not{D} q_j - \sum_{j}^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a + \text{gauge fixing} + \text{ghost}$$

gauge theory parameters: $N_c \quad N_f \quad m_q \quad \Lambda_{\mathrm{QCD}}$

What is the low energy physics?



Physics of Goldstone bosons

chiral symmetry breaking

explicitly

dynamically

quark mass $m_q \bar{q}_j q_j$

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$$

condensate $\langle \bar{q}_j q_j \rangle$

pseudo-Goldstone bosons dominate the low energy physics

e.g. $N_f = 2$ pions $\pi_0 = \pi^3 \quad \pi_\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2)$

Non-linear sigma model

$$\mathcal{L} = \frac{f_\pi^2}{4}\left\{\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + m_\pi^2 \mathrm{Tr}\left(U + U^\dagger\right)\right\} \qquad U = e^{i\frac{\vec{\tau}\cdot\vec{\pi}}{f_\pi}}$$

$$\mathcal{L}_2^{2\pi} = \frac{1}{2}\partial_\mu \vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{1}{2}m_\pi^2\vec{\pi}^2 \qquad \mathcal{L}_2^{4\pi} = \frac{1}{6f_\pi^2}\Big((\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\Big) + \frac{m_\pi^2}{24 f_\pi^2}(\vec{\pi}^2)^2 \quad \ldots$$

The strong coupling problem → Gauge Theory Bootstrap

energy

chiSB

weakly coupled pions (EFT)

breaks down

?

strongly coupled regime

asymptotic freedom

weakly coupled quarks/gluons (pQCD)

compute the strongly coupled hadron dynamics:
amplitudes, form factors, correlation functions, spectrum/couplings

Gauge Theory Bootstrap

rules:

assume — *chiral symmetry breaking & confinement*

input — $\underbrace{N_c \quad N_f \quad m_q \quad \alpha_s}_{\textit{defining gauge theory}}$ $\quad m_\pi \quad f_\pi$

m_π set the unit

f_π size of pion

theoretical/numerical computation:

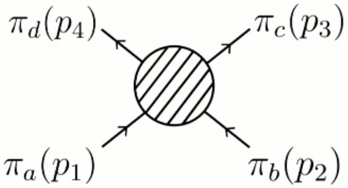
- not using experimental scattering data as input
- no assumption on spectrum

Physical observables

$$|\psi_1\rangle = |p_1, p_2\rangle_{in}\,, \qquad |\psi_2\rangle = |p_1, p_2\rangle_{out}\,, \qquad |\psi_3\rangle = \int dx e^{-i(p_1+p_2)\cdot x}\mathcal{O}(x)|0\rangle$$

[Karateev, Kuhn, Penedones, 2019]

2-to-2 amplitude: $$_{out}\langle p_3, c; p_4, d|p_1, a; p_2, b\rangle_{in} = A(s,t,u)\delta_{ab}\delta_{cd} + A(t,s,u)\delta_{ac}\delta_{bd} + A(u,t,s)\delta_{ad}\delta_{bc}$$



$$\begin{aligned} T^{I=0}(s,t,u) &= 3A(s,t,u) + A(t,s,u) + A(u,t,s) \\ T^{I=1}(s,t,u) &= A(t,s,u) - A(u,t,s) \\ T^{I=2}(s,t,u) &= A(t,s,u) + A(u,t,s) \end{aligned}$$

$$s = (p_1+p_2)^2 \qquad t = (p_1-p_3)^2 \qquad u = (p_1-p_4)^2$$

$$s + t + u = 4m_\pi^2$$

$$f_\ell^I(s) = \frac{1}{4}\int_{-1}^{1} d\cos\theta P_\ell(\cos\theta) T^I(s,t) \qquad S_\ell^I(s) = 1 + i\pi\sqrt{\frac{s-4m_\pi^2}{s}}f_\ell^I(s) = \overset{\text{inelasticity}}{\eta_\ell^I(s)}\, e^{2i\overset{\text{phase shift}}{\delta_\ell^I(s)}} \qquad s > 4m_\pi^2$$

2-particle form factor: $$_{\text{out}}\langle p_1, p_2|\mathcal{O}(0)|0\rangle = F(s)$$

$$s = (p_1+p_2)^2$$

2-point function

and

spectral density:

$$\Pi(s) = i\int \frac{d^4x}{(2\pi)^4} e^{iPx}\langle 0|\hat{T}\left\{\mathcal{O}^\dagger(x)\mathcal{O}(0)\right\}|0\rangle$$

$$\int \frac{d^4x}{(2\pi)^4} e^{iPx}\langle 0|\mathcal{O}^\dagger(x)\mathcal{O}(0)|0\rangle = \rho(s)$$

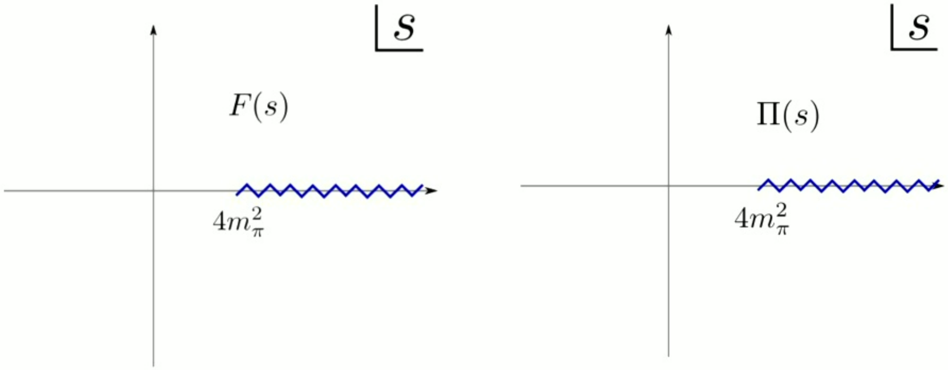
Generic constraints: analyticity, crossing, unitarity (ACU)

Analyticity $A(s,t,u)$ cuts $s,t,u > 4m_\pi^2$

$F(s)$ $\Pi(s)$ cuts $s > 4m_\pi^2$

Crossing

$$A(s,t,u) = A(s,u,t)$$



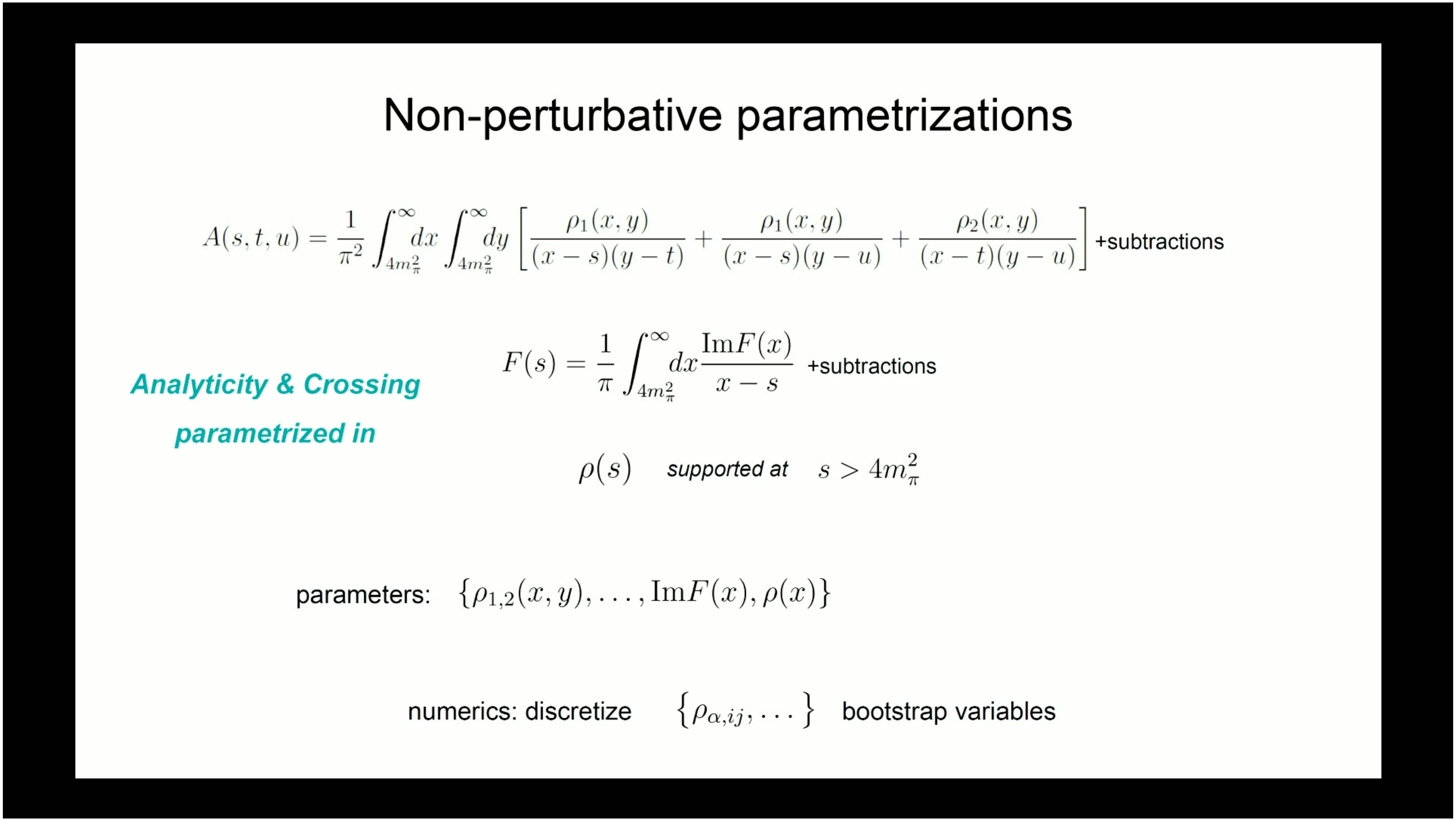
$$\rho(s) = 2\mathrm{Im}\Pi(s+i\epsilon) \quad s > 4m_\pi^2$$

Unitarity

$$\langle\psi_a|\psi_b\rangle = \begin{pmatrix} 1 & S & \mathcal{F} \\ S^* & 1 & \mathcal{F}^* \\ \mathcal{F}^* & \mathcal{F} & \rho \end{pmatrix} \succeq 0$$

positive semidefinite matrix

[Paulos, Penedones, Toledo, van Rees, Vieira 2016 & 2017]
[Karateev, Kuhn, Penedones, 2019]



Non-perturbative parametrizations

$$A(s,t,u) = \frac{1}{\pi^2}\int_{4m_\pi^2}^{\infty} dx \int_{4m_\pi^2}^{\infty} dy \left[\frac{\rho_1(x,y)}{(x-s)(y-t)} + \frac{\rho_1(x,y)}{(x-s)(y-u)} + \frac{\rho_2(x,y)}{(x-t)(y-u)}\right]$$ +subtractions

Analyticity & Crossing parametrized in

$$F(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dx \frac{\mathrm{Im}F(x)}{x-s}$$ +subtractions

$\rho(s)$ *supported at* $s > 4m_\pi^2$

parameters: $\{\rho_{1,2}(x,y), \ldots, \mathrm{Im}F(x), \rho(x)\}$

numerics: discretize $\left\{\rho_{\alpha,ij}, \ldots\right\}$ bootstrap variables

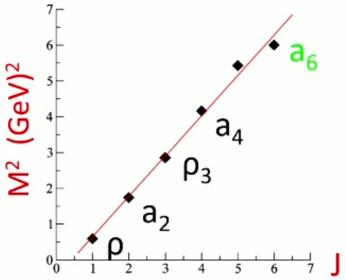
The bootstrap idea: old

bootstrap method (pre-QCD): solve the theory of strong interaction from these constraints [Chew… , 1960s]

amplitudes

Symmetry+Analyticity+Crossing+Unitarity

solve the bootstrap variables $\{\rho_{1,2}(x,y),\ldots\}$

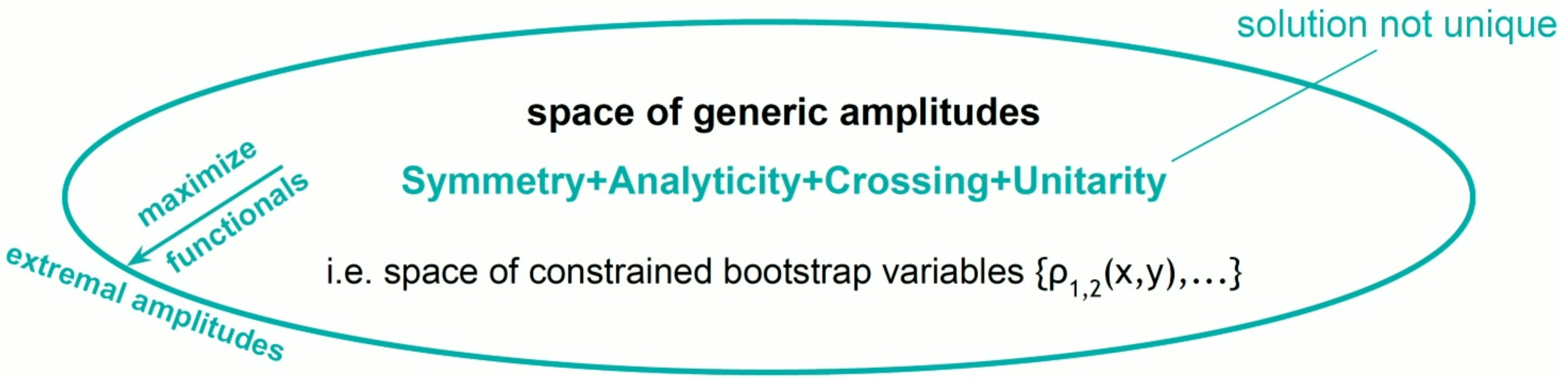


$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

birth of string theory

The bootstrap idea: old and new

bootstrap method (pre-QCD): solve the theory of **strong interaction** *from these constraints* [Chew… , 1960s]



space of generic amplitudes

Symmetry+Analyticity+Crossing+Unitarity

i.e. space of constrained bootstrap variables $\{\rho_{1,2}(x,y),\ldots\}$

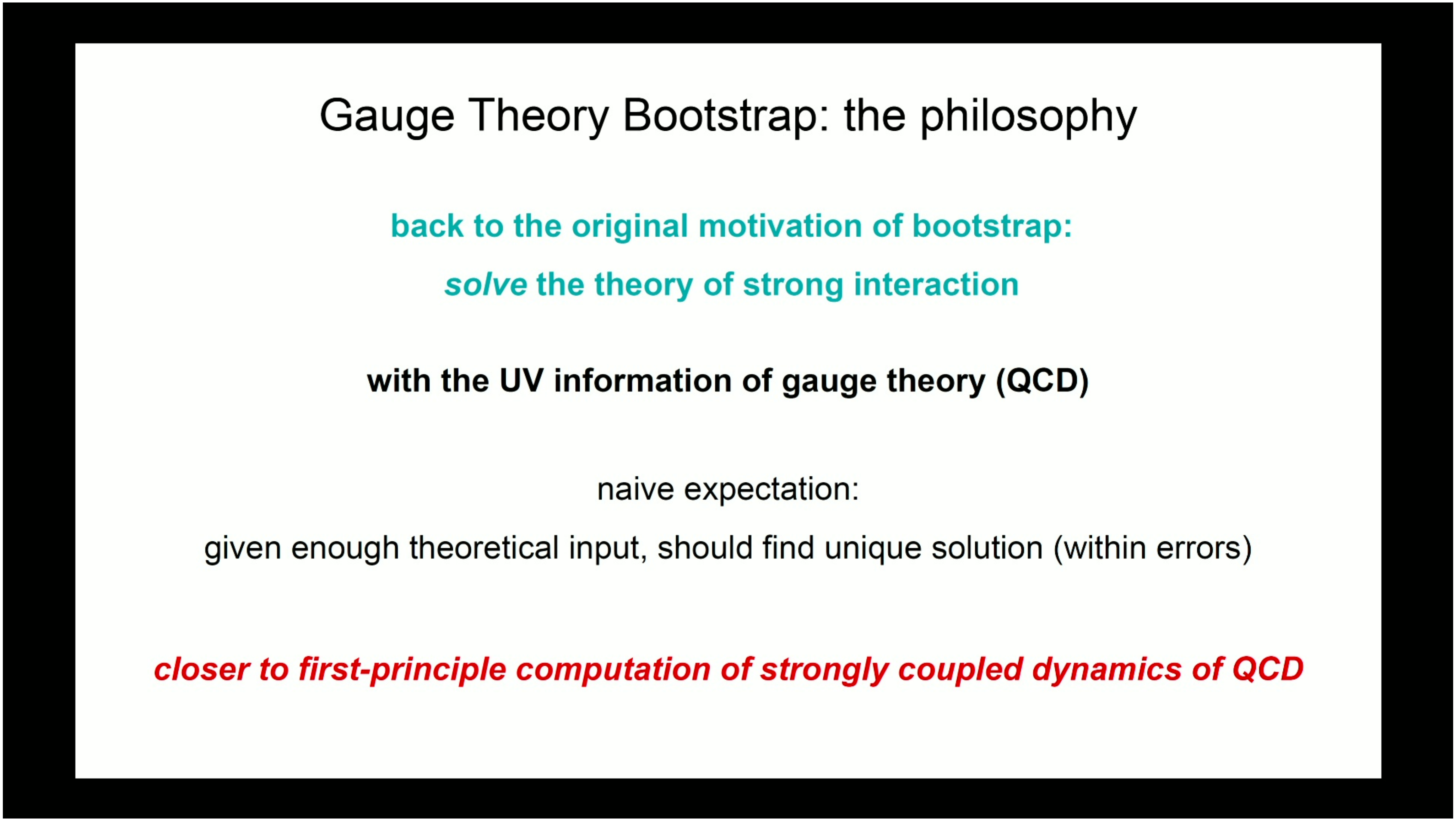
Modern S-matrix bootstrap: maximize functional, examine extremal amplitude, bounding physical quantities

[Paulos, Penedones, Toledo, van Rees, Vieira 2016 & 2017]

pion physics application: bound low energy parameters

[Guerrieri, Penedones, Vieira, 2018 & 2020], [Albert, Rastelli, Henriksson, Vichi, 2022 & 2023], …

Very recent phenomenology application (fit experimental scattering data) [Guerrieri, Haring, Su, 2025]



Gauge Theory Bootstrap: the philosophy

back to the original motivation of bootstrap:

***solve* the theory of strong interaction**

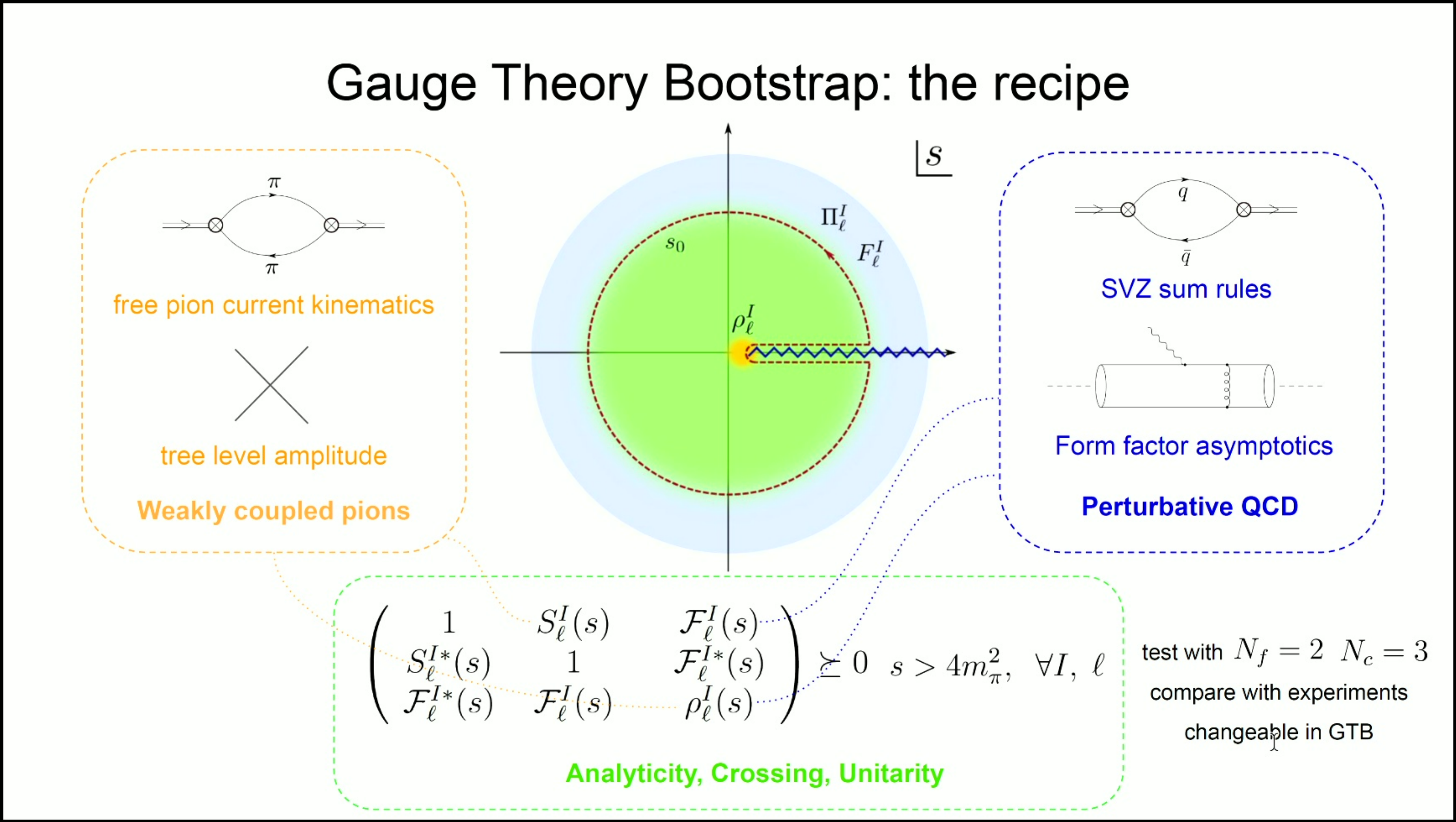
with the UV information of gauge theory (QCD)

naive expectation:

given enough theoretical input, should find unique solution (within errors)

closer to first-principle computation of strongly coupled dynamics of QCD

Gauge Theory Bootstrap: the recipe



free pion current kinematics

tree level amplitude

Weakly coupled pions

SVZ sum rules

Form factor asymptotics

Perturbative QCD

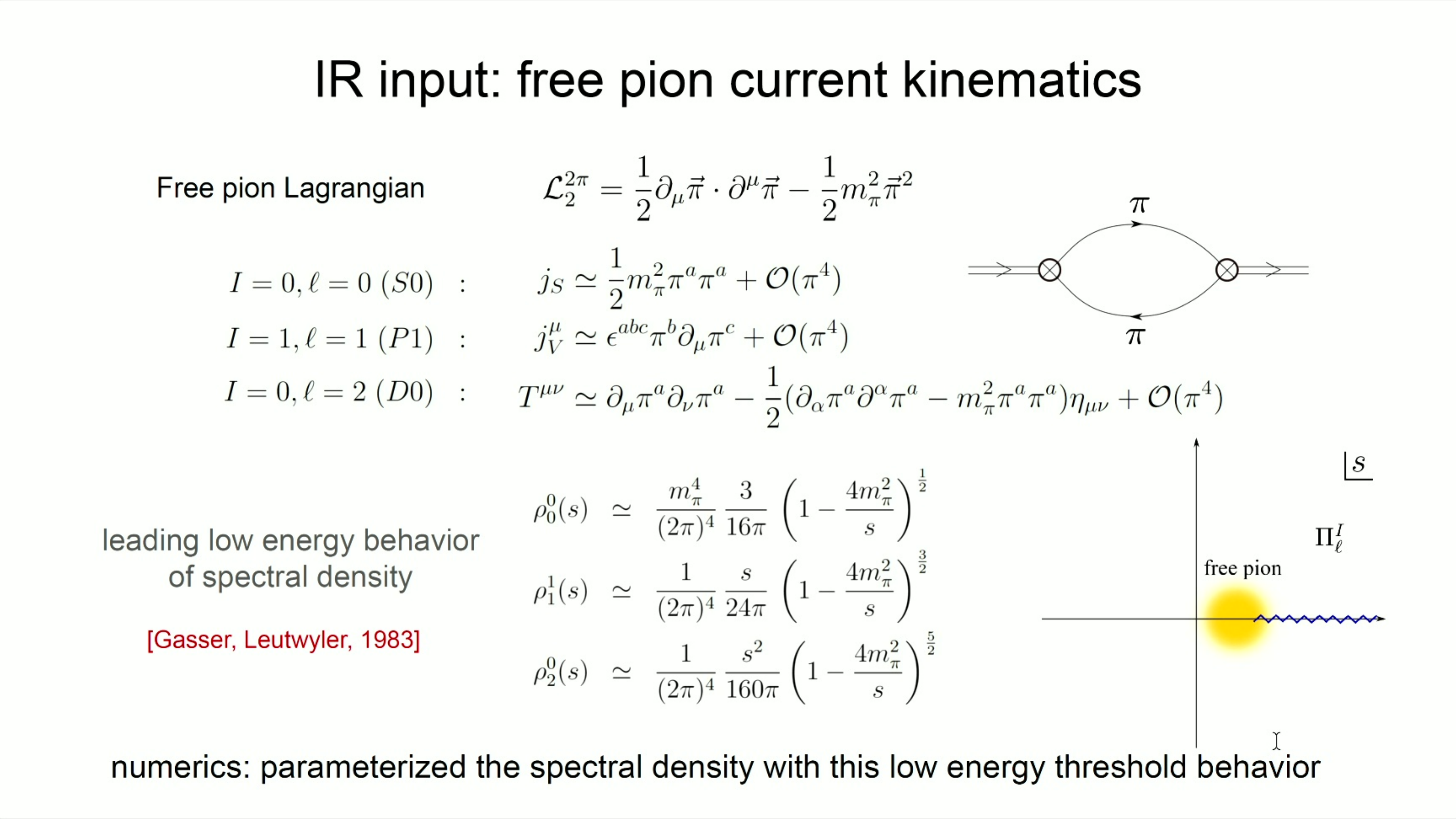
$$\begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I(s) \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*}(s) \\ \mathcal{F}_\ell^{I*}(s) & \mathcal{F}_\ell^I(s) & \rho_\ell^I(s) \end{pmatrix} \succeq 0 \quad s > 4m_\pi^2, \ \ \forall I, \ \ell$$

Analyticity, Crossing, Unitarity

test with $N_f = 2 \ \ N_c = 3$

compare with experiments

changeable in GTB



IR input: free pion current kinematics

Free pion Lagrangian $$\mathcal{L}_2^{2\pi} = \frac{1}{2}\partial_\mu \vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{1}{2}m_\pi^2 \vec{\pi}^2$$

$$I=0,\ell=0\ (S0) \quad : \quad j_S \simeq \frac{1}{2}m_\pi^2 \pi^a\pi^a + \mathcal{O}(\pi^4)$$

$$I=1,\ell=1\ (P1) \quad : \quad j_V^\mu \simeq \epsilon^{abc}\pi^b\partial_\mu\pi^c + \mathcal{O}(\pi^4)$$

$$I=0,\ell=2\ (D0) \quad : \quad T^{\mu\nu} \simeq \partial_\mu\pi^a\partial_\nu\pi^a - \frac{1}{2}(\partial_\alpha\pi^a\partial^\alpha\pi^a - m_\pi^2\pi^a\pi^a)\eta_{\mu\nu} + \mathcal{O}(\pi^4)$$

leading low energy behavior of spectral density

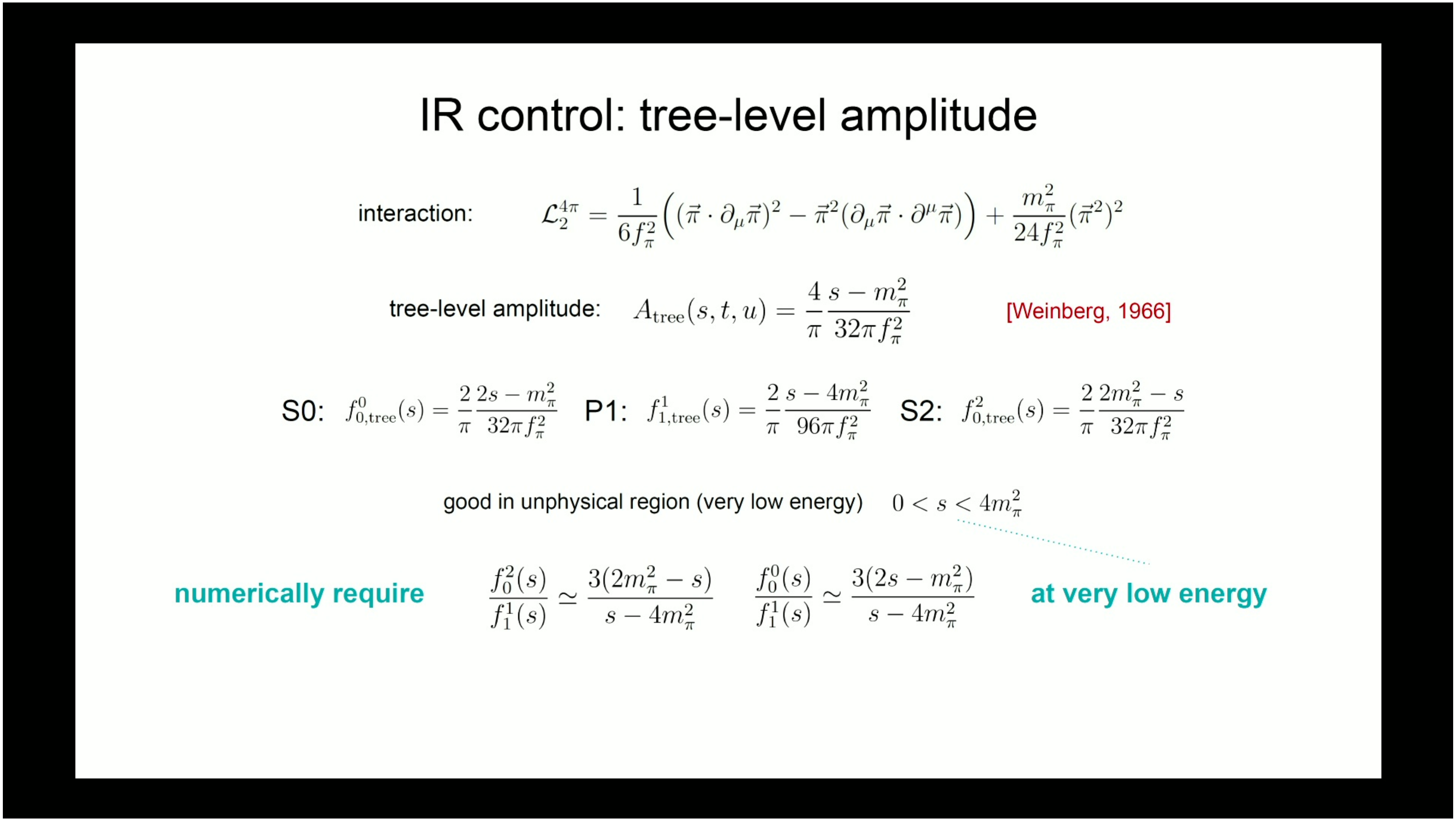
[Gasser, Leutwyler, 1983]

$$\rho_0^0(s) \simeq \frac{m_\pi^4}{(2\pi)^4}\frac{3}{16\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{1}{2}}$$

$$\rho_1^1(s) \simeq \frac{1}{(2\pi)^4}\frac{s}{24\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{3}{2}}$$

$$\rho_2^0(s) \simeq \frac{1}{(2\pi)^4}\frac{s^2}{160\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{5}{2}}$$

numerics: parameterized the spectral density with this low energy threshold behavior



IR control: tree-level amplitude

interaction: $$\mathcal{L}_2^{4\pi} = \frac{1}{6f_\pi^2}\Big((\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\Big) + \frac{m_\pi^2}{24f_\pi^2}(\vec{\pi}^2)^2$$

tree-level amplitude: $$A_{\text{tree}}(s,t,u) = \frac{4}{\pi}\frac{s-m_\pi^2}{32\pi f_\pi^2}$$ [Weinberg, 1966]

S0: $f^0_{0,\text{tree}}(s) = \frac{2}{\pi}\frac{2s-m_\pi^2}{32\pi f_\pi^2}$ P1: $f^1_{1,\text{tree}}(s) = \frac{2}{\pi}\frac{s-4m_\pi^2}{96\pi f_\pi^2}$ S2: $f^2_{0,\text{tree}}(s) = \frac{2}{\pi}\frac{2m_\pi^2-s}{32\pi f_\pi^2}$

good in unphysical region (very low energy) $0 < s < 4m_\pi^2$

numerically require $$\frac{f_0^2(s)}{f_1^1(s)} \simeq \frac{3(2m_\pi^2-s)}{s-4m_\pi^2} \qquad \frac{f_0^0(s)}{f_1^1(s)} \simeq \frac{3(2s-m_\pi^2)}{s-4m_\pi^2}$$ **at very low energy**

pQCD input: asymptotically free current correlator

QCD Lagrangian $$\mathcal{L} = i\sum_j^{N_f} \bar{q}_j \not{D} q_j - \sum_j^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu}$$

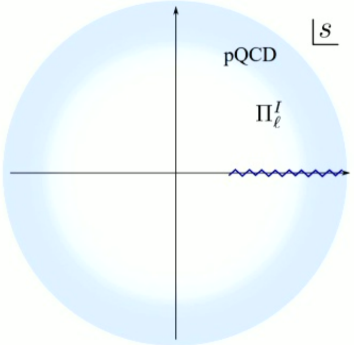
$$I = 0, \ell = 0\ (S0) \quad : \quad j_S = m_q(\bar{u}u + \bar{d}d)$$

$$I = 1, \ell = 1\ (P1) \quad : \quad j_V^\mu = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)$$

$$I = 0, \ell = 2\ (D0) \quad : \quad T_{\mu\nu} = T^q_{\mu\nu} + T^g_{\mu\nu}$$

$$\cdots$$

$$\Pi_\ell^I(s) = i\int \frac{d^4x}{(2\pi)^4} e^{iPx} \langle 0|\hat{T}\left\{ j_\ell^{I\dagger}(x) j_\ell^I(0)\right\}|0\rangle$$



large spacelike momenta: asymptotically free region controlled by pQCD

SVZ expansion

[Shifman, Vainshtein, Zakharov, 1979]

OPE: $$T\{j(x)j(0)\} = C_{\mathbb{1}}(x)\;\mathbb{1} + \sum_{\mathcal{O}} C_{\mathcal{O}}(x)\;\mathcal{O}(0)$$

$$\langle 0|T\{j(x)j(0)\}|0\rangle = C_{\mathbb{1}}(x) + C_{\bar{q}q}(x)\;\langle 0|m_q\bar{q}q|0\rangle + C_{G^2}(x)\;\langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^{a\,\mu\nu}|0\rangle + \ldots$$

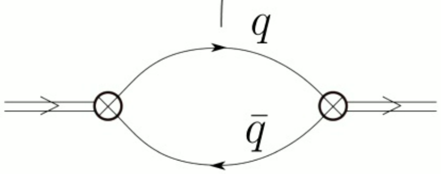
SB vacuum

quark condensate

gluon condensate

pQCD computation

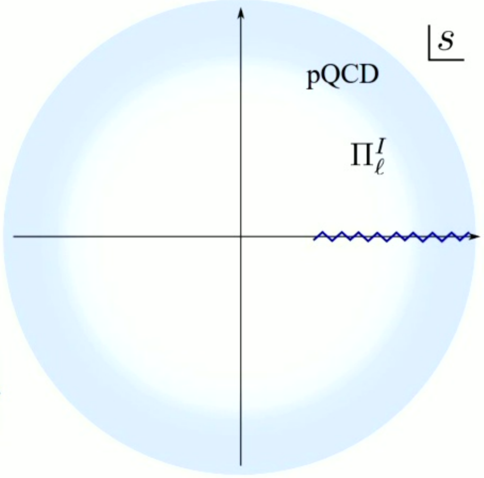
e.g.:



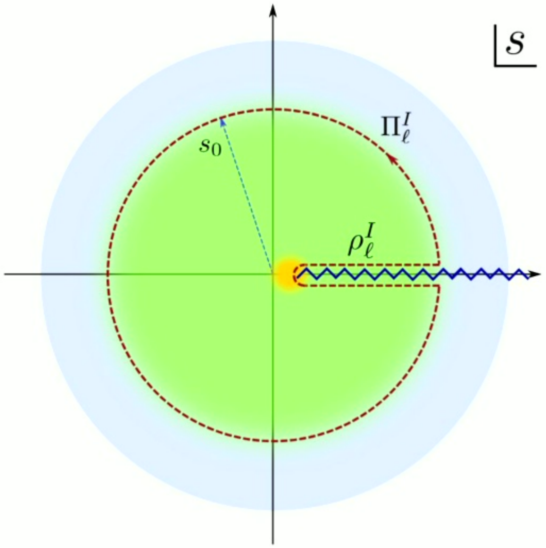
+ …

large s expansion of vacuum polarization: e.g. vector current $N_c = 3$

$$\Pi_1^1(s) = \frac{1}{2}\frac{1}{(2\pi)^4}\left\{-\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)s\ln(-\frac{s}{\mu^2}) + \frac{1}{12s}\langle\frac{\alpha_s}{\pi}G^2\rangle + \frac{1}{s}\langle m_q\bar{q}q\rangle + \ldots\right\}$$



Finite energy sum rule



connect bootstrap with pQCD at s$_0$

contour integral $s^n \Pi(s)$ vanishes

SVZ

$$\int_{4m_\pi^2}^{\infty} \rho(x) x^n dx = -s_0^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi} \Pi(s_0 e^{i\varphi}) d\varphi$$

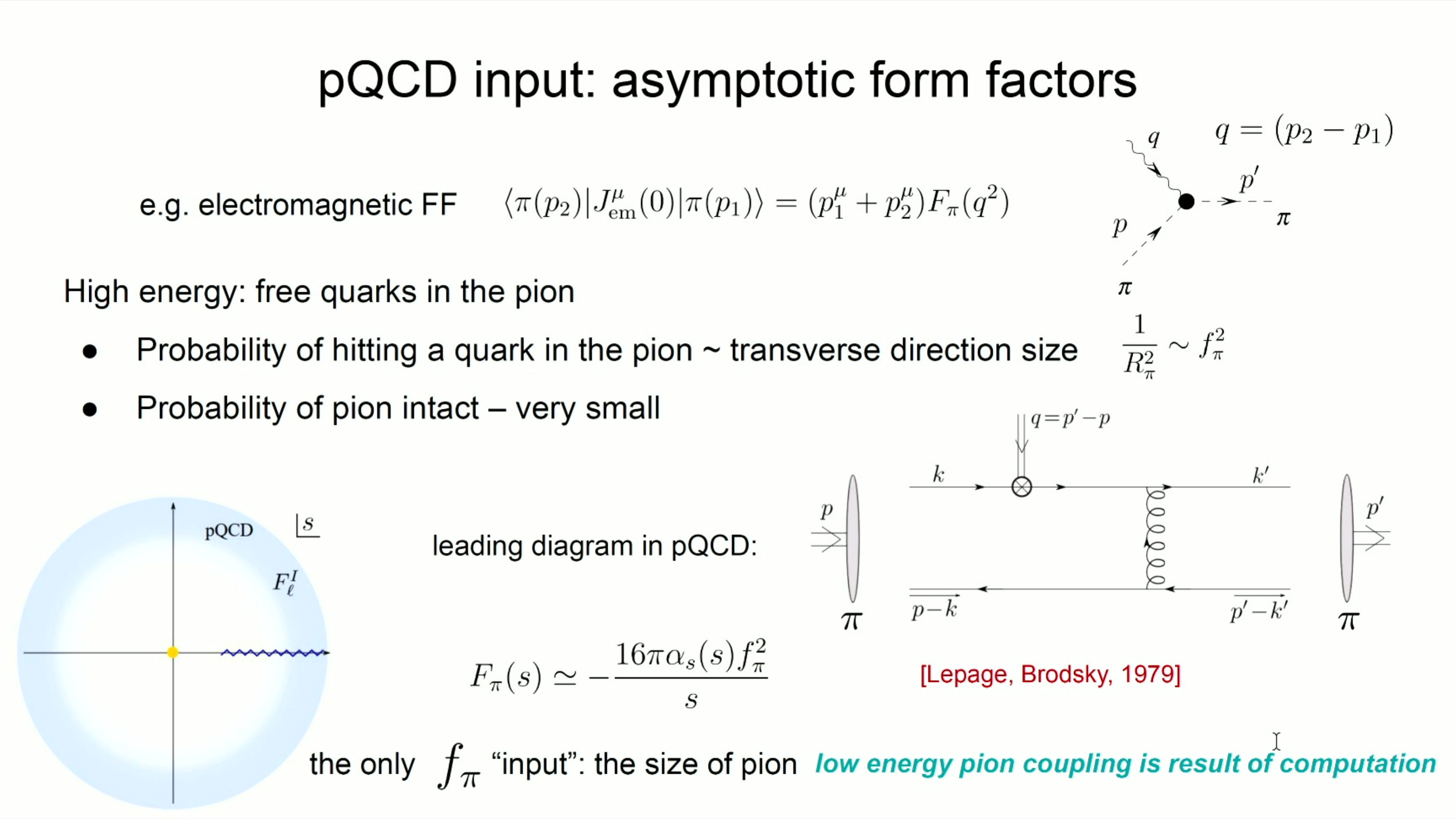
bootstrap variables ← **gauge theory information**

linear constraints

$$P1 \ : \ \frac{1}{s_0^{n+2}} \int_{4m_\pi^2}^{s_0} \rho_1^1(x) x^n dx = \frac{1}{2(2\pi)^4} \left\{ \frac{1}{2\pi(n+2)} \left(1 + \frac{\alpha_s}{\pi}\right) \underbrace{- \cancel{\frac{\delta_n \pi}{6 s_0^2} \langle \frac{\alpha_s}{\pi} G^2 \rangle} - \cancel{\frac{\delta_n 2\pi}{s_0^2} \langle m_q \bar{q} q \rangle}} + \dots \right\}, \ n \geq -1$$

to extract from bootstrap in the future

condensates suppressed at large s_0, not input



pQCD input: asymptotic form factors

$q = (p_2 - p_1)$

e.g. electromagnetic FF $\langle \pi(p_2) | J^\mu_{\rm em}(0) | \pi(p_1) \rangle = (p_1^\mu + p_2^\mu) F_\pi(q^2)$

High energy: free quarks in the pion

- Probability of hitting a quark in the pion ~ transverse direction size $\frac{1}{R_\pi^2} \sim f_\pi^2$
- Probability of pion intact – very small

leading diagram in pQCD:

$$F_\pi(s) \simeq -\frac{16\pi\alpha_s(s) f_\pi^2}{s}$$

[Lepage, Brodsky, 1979]

the only f_π “input”: the size of pion ***low energy pion coupling is result of computation***

Quick summary of the GTB constraints

e.g. $I = 1, \ell = 1 \ (P1)$ $\epsilon^{abc}\pi^b\partial_\mu\pi^c \simeq \ j_V^\mu \ = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)$ *identify current*

IR

UV

free pion kinematics

$$\rho_1^1(s) \ \simeq \ \frac{1}{(2\pi)^4}\frac{s}{24\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{3}{2}}$$

spectral density

pQCD sum rules

$$\frac{1}{s_0^{n+2}}\int_{4m_\pi^2}^{s_0}\rho_1^1(x)x^n dx \simeq \frac{1}{(2\pi)^4}\frac{1}{4\pi(n+2)}\left(1+\frac{\alpha_s}{\pi}\right)$$

normalization

$$F_1^1(0) \ = \ 1$$

form factor

asymptotics

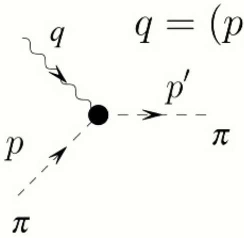
$$F_\pi(s) \simeq -\frac{16\pi\alpha_s(s)f_\pi^2}{s}$$

ACU $$\begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*} \\ \mathcal{F}_\ell^{I*} & \mathcal{F}_\ell^I & \rho_\ell^I(s) \end{pmatrix} \succeq 0 \quad s > 4m_\pi^2, \ \forall I, \ \ell$$

how to find solution?

pQCD input: asymptotic form factors

$$q = (p_2 - p_1)$$



e.g. electromagnetic FF $\langle \pi(p_2)|J^\mu_{\rm em}(0)|\pi(p_1)\rangle = (p_1^\mu + p_2^\mu)F_\pi(q^2)$

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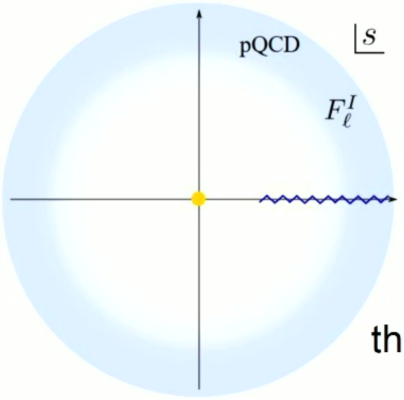
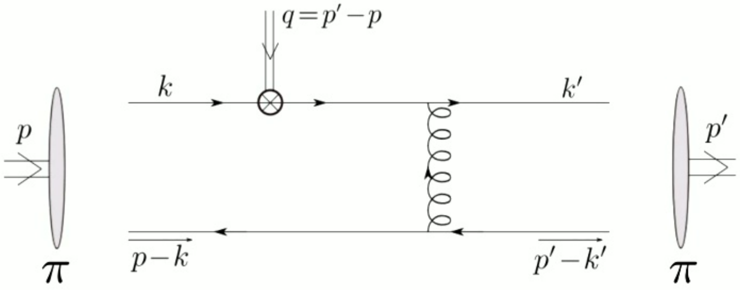


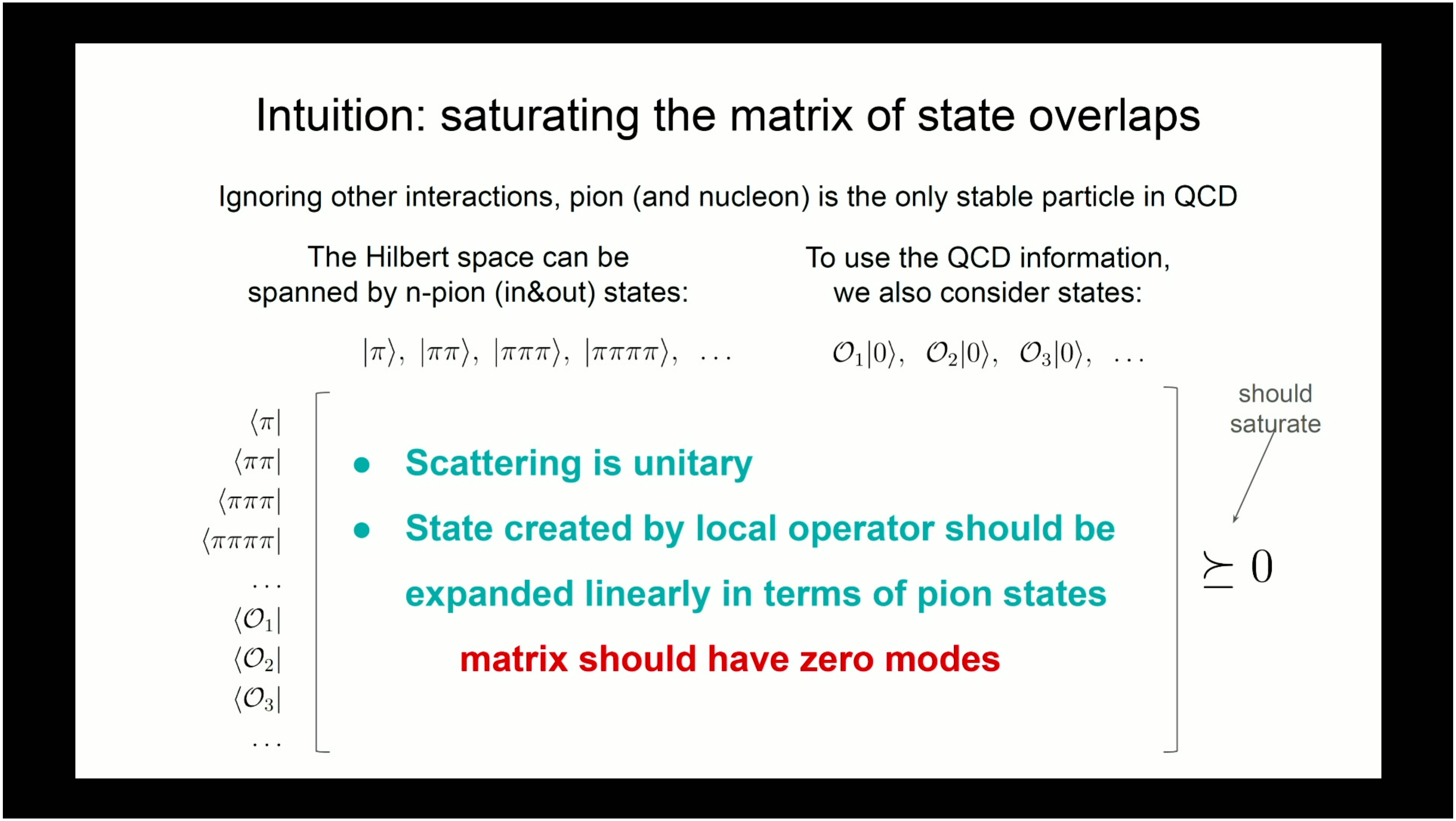
leading diagram in pQCD:



$$F_\pi(s) \simeq -\frac{16\pi\alpha_s(s) f_\pi^2}{s}$$

[Lepage, Brodsky, 1979]

the only f_π "input": the size of pion ***low energy pion coupling is result of computation***



Intuition: saturating the matrix of state overlaps

Ignoring other interactions, pion (and nucleon) is the only stable particle in QCD

The Hilbert space can be spanned by n-pion (in&out) states:

$$|\pi\rangle,\ |\pi\pi\rangle,\ |\pi\pi\pi\rangle,\ |\pi\pi\pi\pi\rangle,\ \ldots$$

To use the QCD information, we also consider states:

$$\mathcal{O}_1|0\rangle,\ \mathcal{O}_2|0\rangle,\ \mathcal{O}_3|0\rangle,\ \ldots$$

$$\begin{array}{c} \langle\pi| \\ \langle\pi\pi| \\ \langle\pi\pi\pi| \\ \langle\pi\pi\pi\pi| \\ \ldots \\ \langle\mathcal{O}_1| \\ \langle\mathcal{O}_2| \\ \langle\mathcal{O}_3| \\ \ldots \end{array}$$

- **Scattering is unitary**
- **State created by local operator should be expanded linearly in terms of pion states**

matrix should have zero modes

$$\succeq 0$$

should saturate

Saturating the positive semidefinite matrix

Currently, only consider two-pion (in&out) states and one operator per channel

$$\begin{pmatrix} 1 & S & \mathcal{F} \\ S^* & 1 & \mathcal{F}^* \\ \mathcal{F}^* & \mathcal{F} & \rho \end{pmatrix} \succeq 0 \qquad \forall I,\ \ell,\ s$$

saturation should be good approximation up to some energy

iff all its principal minors are non-negative [Karateev, Kuhn, Penedones, 2019]

$$\rho + S^*\mathcal{F}^2 + S(\mathcal{F}^*)^2 - 2|\mathcal{F}|^2 - \rho|S|^2 \geq 0$$

$$\rho \geq 0 \qquad |\mathcal{F}|^2 \leq \rho \qquad |S|^2 \leq 1$$

saturation: $\rho = |\mathcal{F}|^2$

phase of the FF is the same as phase shift

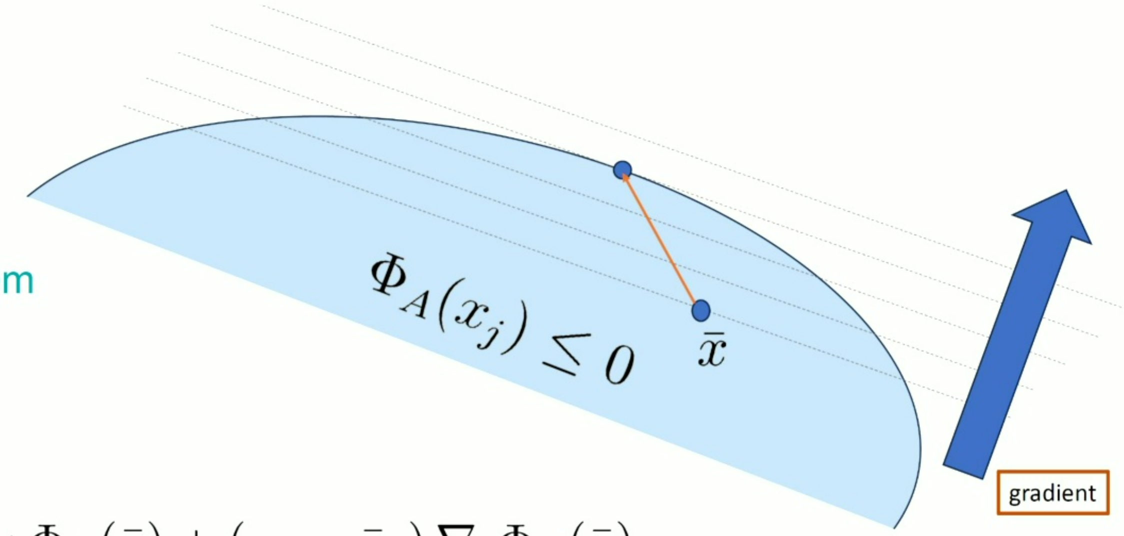
$$e^{2i\delta(s)} = S(s) = \frac{F(s)}{F^*(s)} = e^{2i\alpha(s)}$$

Watson theorem

zero mode of the matrix

GTB solution: finding zero mode iteratively

based on the Watson theorem
an iterative procedure
to find this zero mode



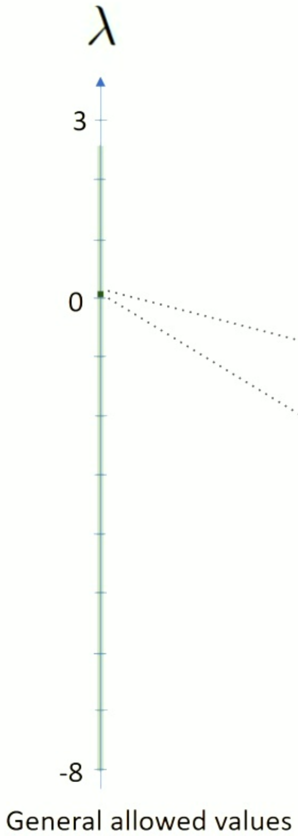
$$\Phi(x) \simeq \Phi_A(\bar{x}) + \underbrace{(x_j - \bar{x}_j)\,\nabla_j \Phi_A(\bar{x})}_{\text{Maximize until convergence}}$$

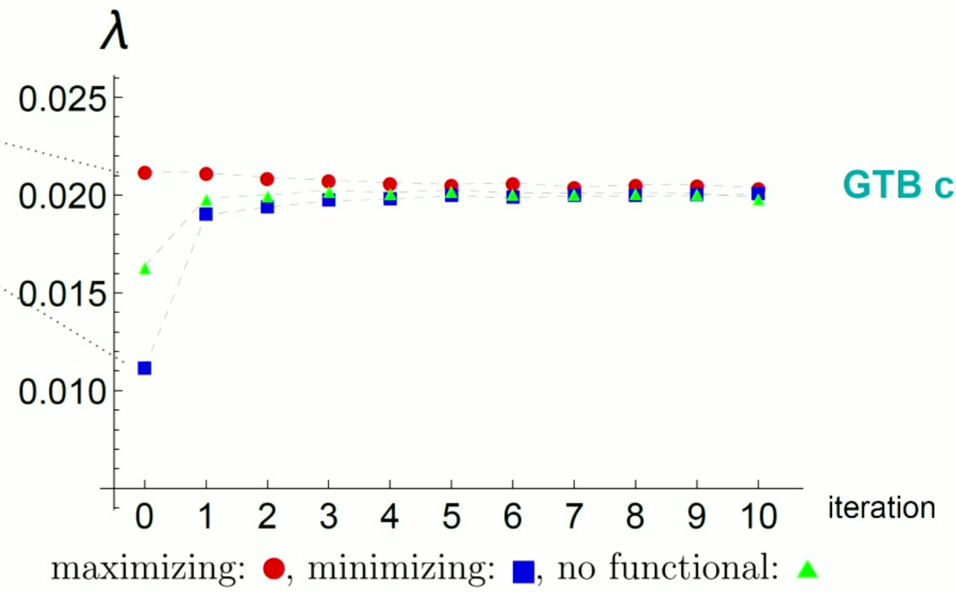
- start with arbitrary initial functional to maximize **e.g. finding a feasible point by min 0**
- construct iterative functional based on initial feasible solution
- quickly converge to a unique solution

Test: Computing pion quartic coupling

$$\lambda = \frac{1}{32\pi}\,\mathcal{M}(\pi^0\pi^0 \to \pi^0\pi^0)\big|_{s=t=u=\frac{4}{3}m_\pi^2} = \frac{3\pi}{4}A\left(\frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2\right)$$

generic bounds from S-matrix bootstrap (ACU): $-8.02 \leq \lambda \leq 2.661$

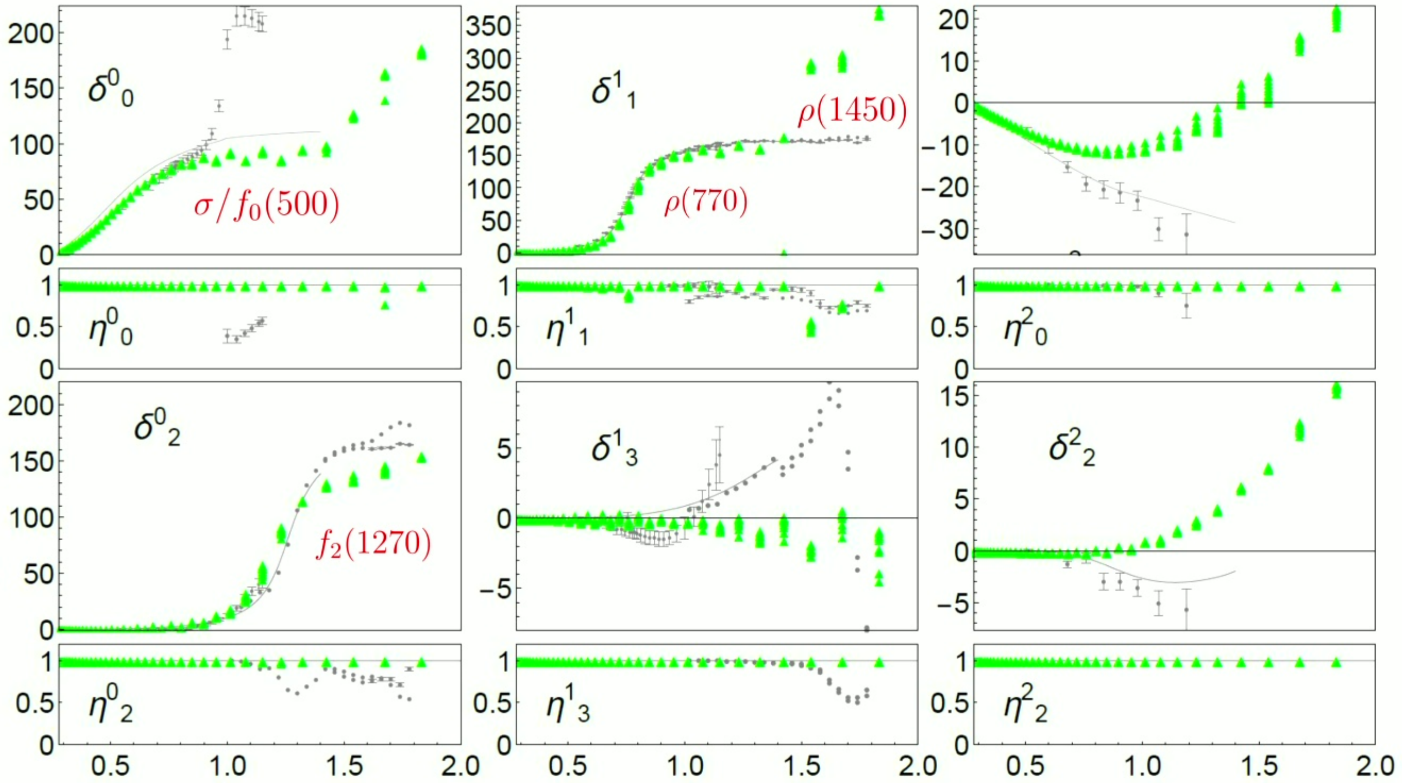




GTB computation

Partial waves from 1-10 iterations

starting from
a generic
feasible point



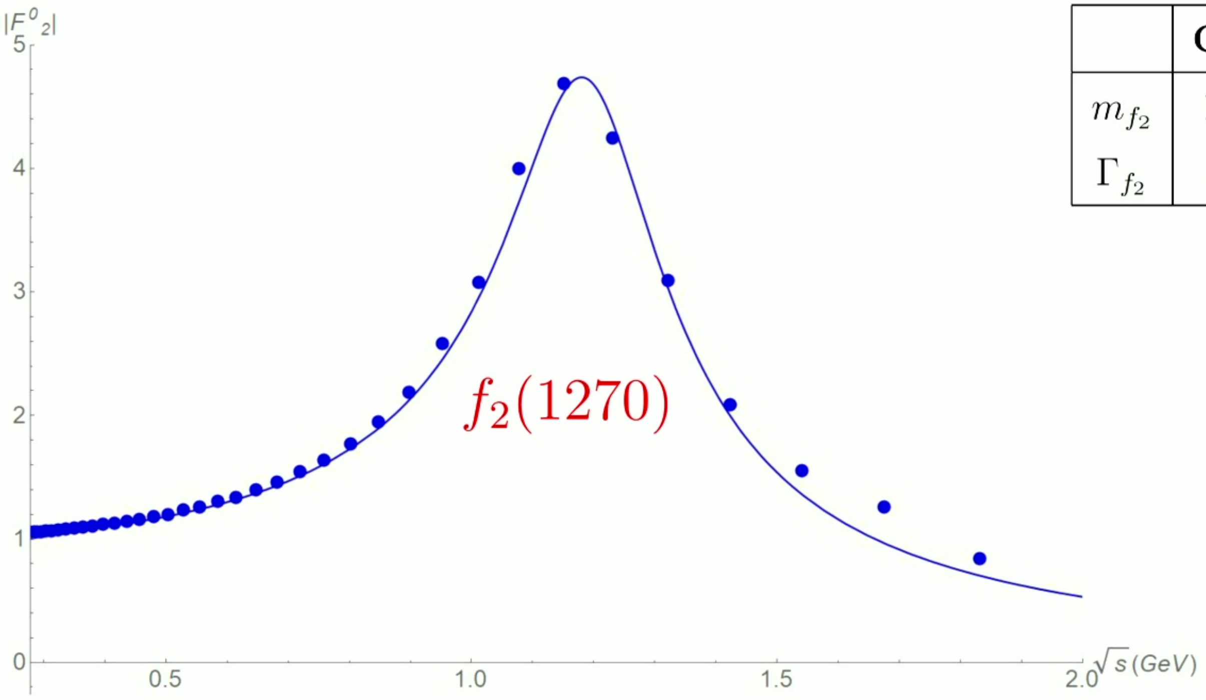
experimental data
(gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]
pheno fit
(gray line)
[Pelaez, Yndurain, 2005]

1-loop amplitude & form factors: computing LEC

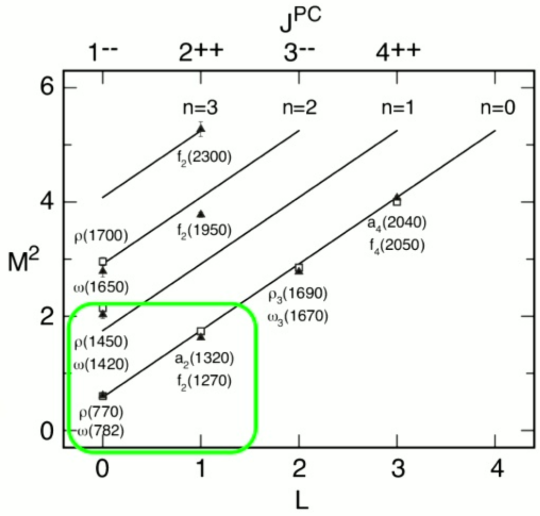
	GTB	**GL**	**Bij**	**CGL**
$\bar{\ell}_1$	1.6	-2.3 ± 3.7	-1.7 ± 1.0	-0.4 ± 0.6
$\bar{\ell}_2$	5.5	6.0 ± 1.3	6.1 ± 0.5	4.3 ± 0.1
$\bar{\ell}_3$	7.8	2.9 ± 2.4		
$\bar{\ell}_4$	4.7	4.3 ± 0.9	4.4 ± 0.3	4.4 ± 0.2
$\bar{\ell}_6$	14.3	18.7 ± 1.1	16.0 ± 0.5 ± 0.7	
		Exp.	**W**	
λ	0.02		0.023	
f_π (MeV)	101	92		

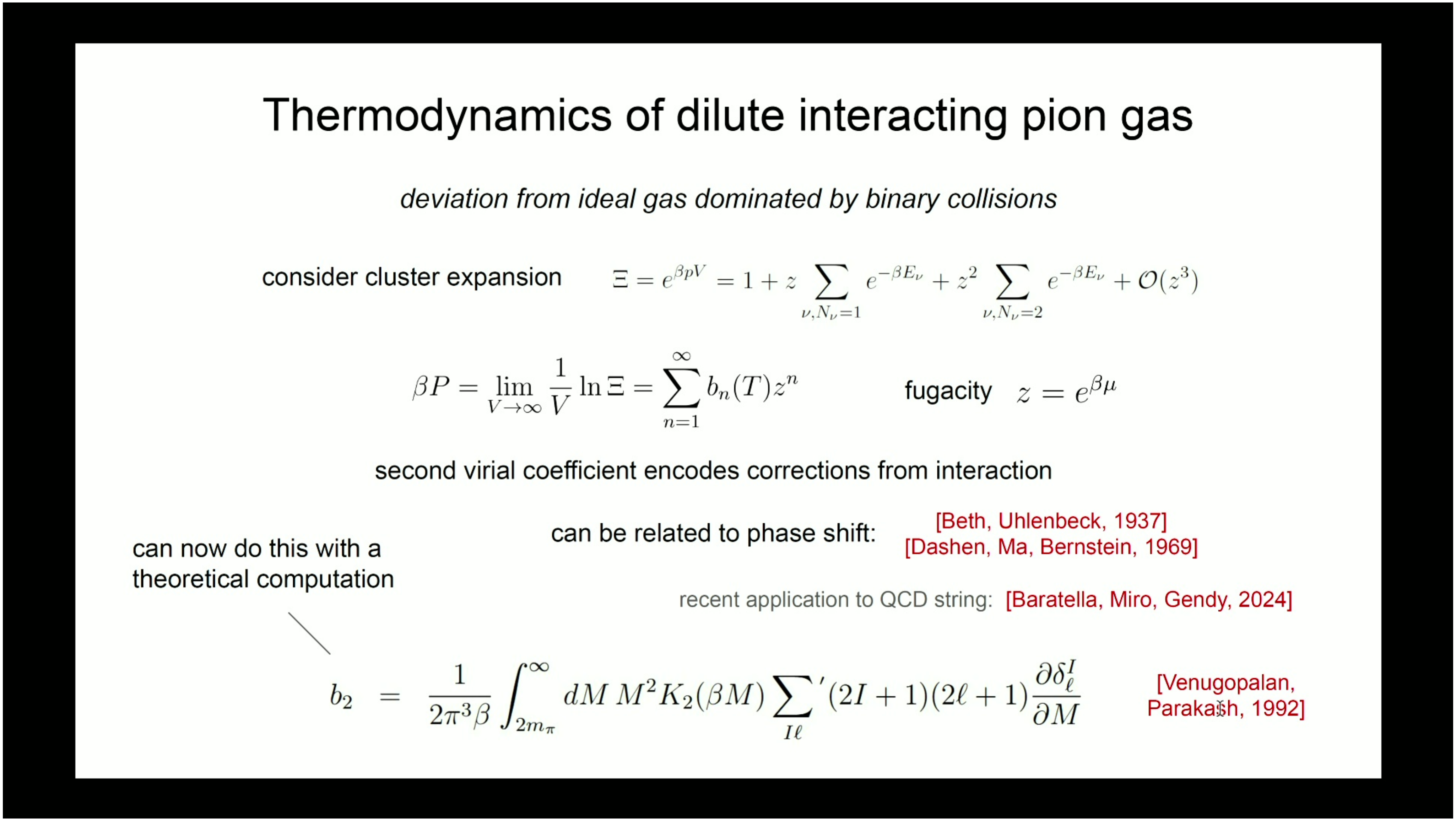
pion coupling

Gravitational form factor



	GTB	PDG
m_{f_2}	1180	1275.4 ± 0.6 MeV
Γ_{f_2}	249	186.6 ± 2.3 MeV





Thermodynamics of dilute interacting pion gas

deviation from ideal gas dominated by binary collisions

consider cluster expansion $$\Xi = e^{\beta p V} = 1 + z \sum_{\nu, N_\nu = 1} e^{-\beta E_\nu} + z^2 \sum_{\nu, N_\nu = 2} e^{-\beta E_\nu} + \mathcal{O}(z^3)$$

$$\beta P = \lim_{V \to \infty} \frac{1}{V} \ln \Xi = \sum_{n=1}^{\infty} b_n(T) z^n$$

fugacity $z = e^{\beta \mu}$

second virial coefficient encodes corrections from interaction

can be related to phase shift: [Beth, Uhlenbeck, 1937] [Dashen, Ma, Bernstein, 1969]

can now do this with a theoretical computation

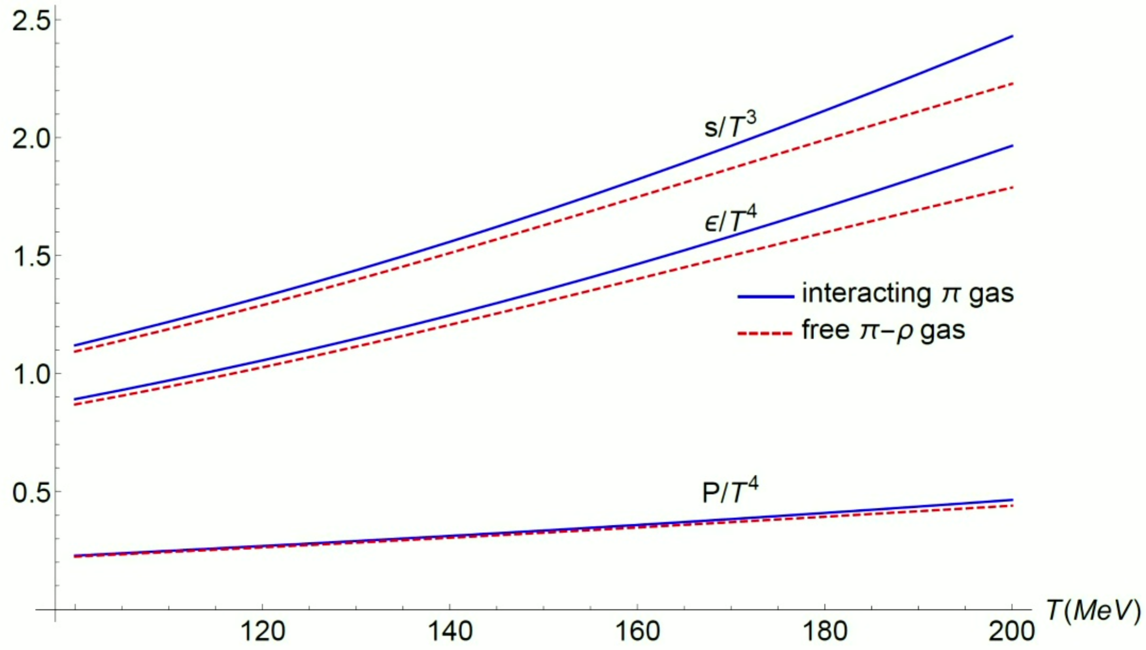
recent application to QCD string: [Baratella, Miro, Gendy, 2024]

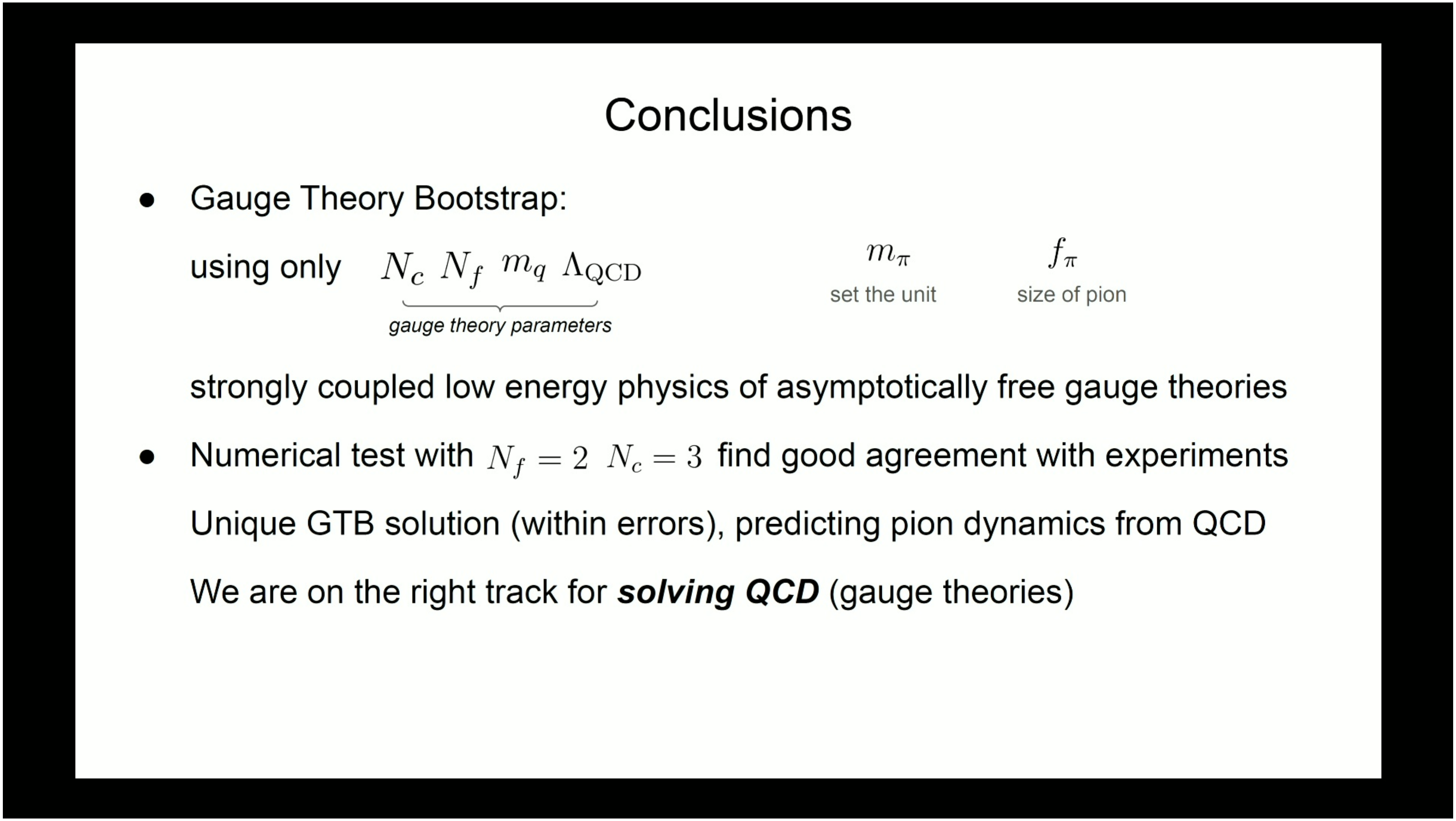
$$b_2 = \frac{1}{2\pi^3 \beta} \int_{2m_\pi}^{\infty} dM \, M^2 K_2(\beta M) \sum_{I\ell}{}' (2I+1)(2\ell+1) \frac{\partial \delta_\ell^I}{\partial M}$$

[Venugopalan, Parakash, 1992]

Thermodynamic quantities at temperature near pion mass

$$P_{\rm int} = T b_2 \qquad \epsilon_{\rm int} = -\frac{\partial b_2}{\partial \beta} \qquad s_{\rm int} = \left[b_2 \left(1 - 2\mu\beta\right) - \beta \frac{\partial b_2}{\partial \beta} \right]$$





Conclusions

- Gauge Theory Bootstrap:

  using only $\underbrace{N_c \;\; N_f \;\; m_q \;\; \Lambda_{\rm QCD}}_{\textit{gauge theory parameters}}$ 　 m_π set the unit 　 f_π size of pion

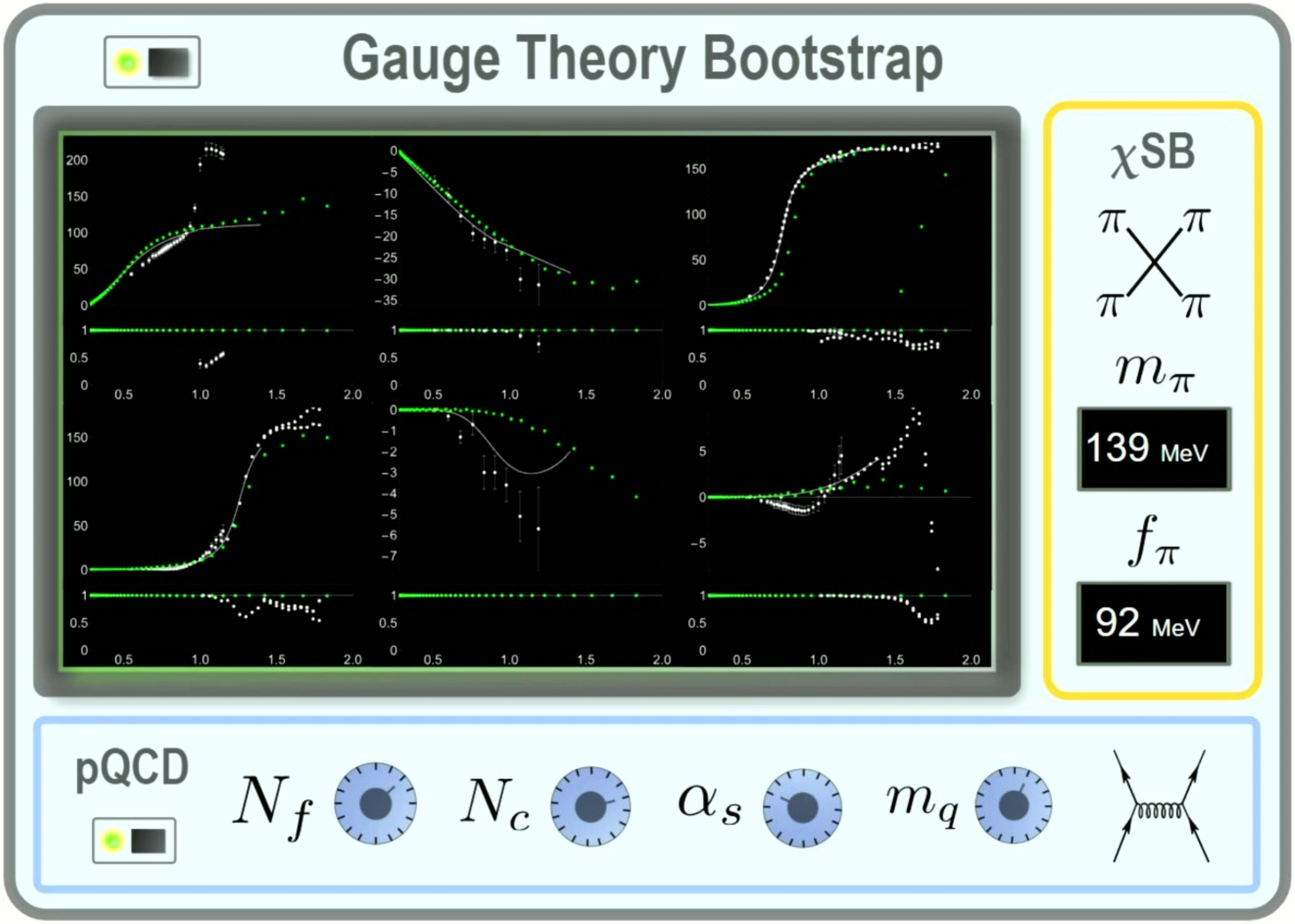
  strongly coupled low energy physics of asymptotically free gauge theories

- Numerical test with $N_f = 2 \;\; N_c = 3$ find good agreement with experiments

  Unique GTB solution (within errors), predicting pion dynamics from QCD

  We are on the right track for ***solving QCD*** (gauge theories)

Gauge Theory Bootstrap



Thank you!