Title: Perverse coherent sheaves and cluster categorifications

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Abstract:

K-theoretical Coulomb branches are expected to have cluster structure. Cautis and Williams categorified this expectation. In particular, they conjecture (and prove in type A) that the category of perverse coherent sheaves on the affine Grassmannian is a cluster monoidal categorification. We discuss recent progress on this conjecture. In particular, we construct cluster short exact sequences of certain perverse coherent sheaves. We do that by constructing a bridge, relating this (geometric) category to the (algebraic) category of finite dimensional modules over the quantum affine group. This is done by relating both categories to the notion of Feigin--Loktev fusion product.

Perverse coherent sheaves and cluster categorification.

General picture.

G-reductive grp. N-representation $/\mathbb{C}$.

- BFN Coulomb branch

- $\mathcal{R}_{G,N}$ - space of triples

$\mathcal{M}_{G,N} \simeq \operatorname{Spec} H^G_\cdot(\mathcal{R}_{G,N})$ (3d, $\mathcal{N}=4$)

$\mathcal{M}^\times_{G,N} \simeq \operatorname{Spec} K^G(\mathcal{R}_{G,N})$ (4d, $\mathcal{N}=2$)

General picture.

G-reductive grp. N-representation $/\mathbb{C}$.

· BFN Coulomb branch

· $\mathcal{R}_{G,N}$ - space of triples

$\mathcal{M}_{G,N} \simeq \mathrm{Spec}\, H^{G}_{\cdot}(\mathcal{R}_{G,N})$ (3d, $\mathcal{N}=4$)

$\mathcal{M}^{\times}_{G,N} \simeq \mathrm{Spec}\, K^{G}(\mathcal{R}_{G,N})$ (4d, $\mathcal{N}=2$)

defined $K_{G,N} \subset \mathcal{D}^{b}(\mathrm{Coh}^{G(\mathcal{O})}(\mathcal{R}_{G,N}))$ (half-BPS line defects)

Koszul-perverse

$K_{G,N}$ - monoidal cluster categorification.

Today: $\mathcal{N}=0$ (pure gauge theory)

$R_{G,0} = \mathrm{Gr}_G$

$K_{G,0} = \mathcal{P}^{G(\mathcal{O})}_{coh}(\mathrm{Gr})$ perverse coherent sheaves on Gr.

<u>Conj</u>. (GW, thm for $G = GL_n$)

$G = G^{ad}$ - simple.

$\mathcal{P}^{G(\mathcal{O})}_{coh}(\mathrm{Gr})$ is a cluster monoidal categorification A_Q

$$Q = \begin{pmatrix} C^T - C & -C^T \\ C & 0 \end{pmatrix}$$

(C - Cartan matrix)

$K_{G,\mathcal{O}} = {}^{coh}$perverse coherent sheaves ...

Conj. (GW, th-m for $G = GL_n$)

$G = G^{ad}$ - simple.

$\mathcal{P}^{G(\mathcal{O})}_{coh}(Gr)$ is a cluster monoidal categorification $\mathcal{A}_Q$

$$Q = \begin{pmatrix} C^T - C & -C^T \\ C & 0 \end{pmatrix}$$

(C - Cartan matrix)

$Gr = G(\mathcal{K})/G(\mathcal{O})$, $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$.

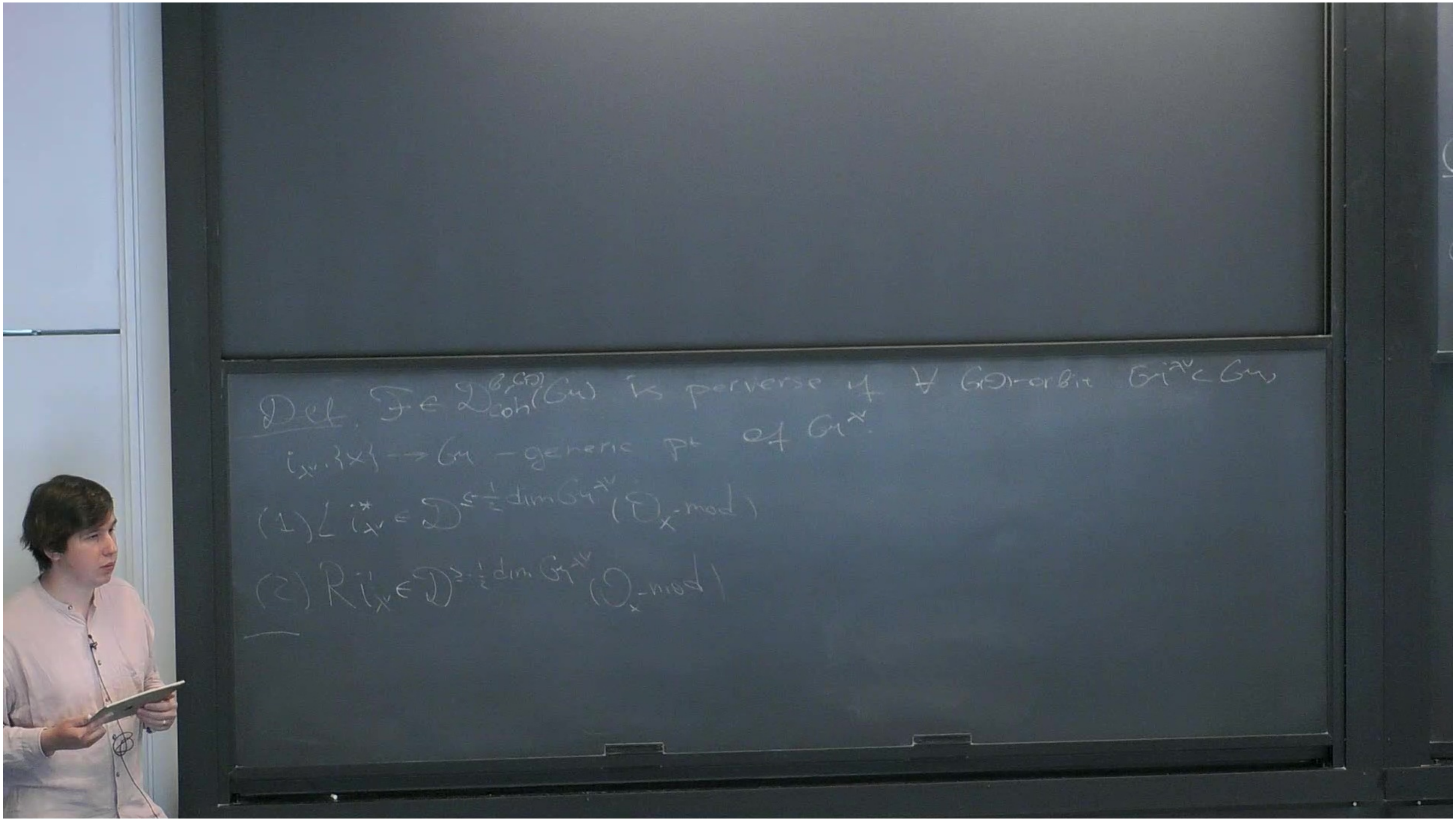
$G(\mathcal{O}) \curvearrowright Gr$, orbits $\longleftrightarrow P^\vee_+$ - dominant coweights

$Gr^{\lambda^\vee} \longleftrightarrow \lambda^\vee$

$\mathcal{P}^{G(\mathcal{O})}_{coh}(Gr) \subset D^b(Coh^{G(\mathcal{O})}(Gr))$

abelian

coherent Satake cat.


Def. $\mathcal{F} \in D^{b, G(\mathcal{O})}_{coh}(Gr)$ is perverse if $\forall$ $G(\mathcal{O})$-orbit $Gr^{\lambda^\vee} \subset Gr$
$i_{x^\vee}: \{x\} \to Gr$ - generic pt of $Gr^{\lambda^\vee}$.
(1) $L i^*_{x^\vee} \in D^{\le -\frac{1}{2} \dim Gr^{\lambda^\vee}}(\mathcal{O}_x\text{-mod})$
(2) $R i^!_{x^\vee} \in D^{\ge -\frac{1}{2} \dim Gr^{\lambda^\vee}}(\mathcal{O}_x\text{-mod})$

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$i_x : \{x\} \to Gr$ - generic pt of $Gr^{\lambda^\vee}$:

(1) $L i_x^* \in \mathcal{D}^{\le -\frac{1}{2}\dim Gr^{\lambda^\vee}}(\mathcal{O}_x\text{-mod})$

(2) $R i_x^! \in \mathcal{D}^{\ge -\frac{1}{2}\dim Gr^{\lambda^\vee}}(\mathcal{O}_x\text{-mod})$

Ex. $\mathcal{P}_{\lambda,\ell} = \mathcal{L}^{\ell}|_{\overline{Gr^{\lambda^\vee}}}[\frac{1}{2}\dim Gr^{\lambda^\vee}] \in \mathcal{P}^{G(\mathcal{O})}_{coh}(Gr)$

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$i_{x^\vee} : \{x\} \to Gr$ — generic pt. of $Gr^{\lambda^\vee}$.

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(2) $R i^!_{x^\vee} \in \mathcal{D}^{\geq -\frac{1}{2}\dim Gr^{\lambda^\vee}}(\mathcal{O}_x\text{-mod})$

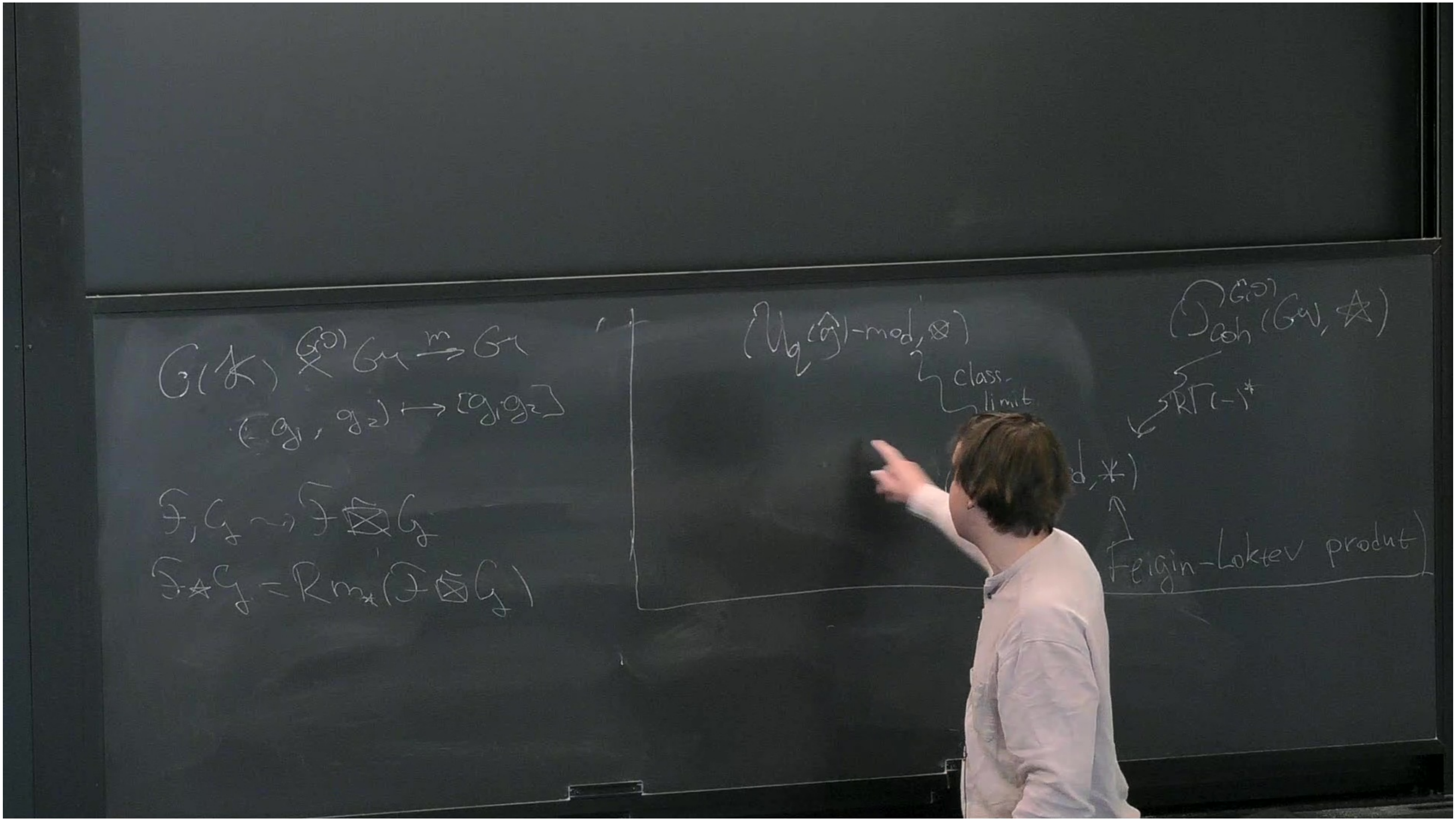
Ex. $\mathcal{J}_{\lambda, \ell} = \mathcal{L}^{\ell}\big|_{\overline{Gr^{\lambda^\vee}}}[-\frac{1}{2}\dim Gr^{\lambda^\vee}] \in \mathcal{P}^{G(\mathcal{O})}_{coh}(Gr)$

$\left(\mathcal{P}^{G(\mathcal{O})}_{coh}(Gr), \star\right)$-monoidal cat)

$\mathcal{J}_{\omega_i^\vee, \ell} =: \mathcal{P}_{i, \ell}$

$$0 \to \mathop{\bigstar}_{j \sim i} \mathcal{J}_{j, \ell} \to \mathcal{J}_{i, \ell+1} \star \mathcal{J}_{i, \ell-1} \to \mathcal{J}_{i, \ell} \star \mathcal{J}_{i, \ell} \to 0$$

$j \sim i$, if i, j are connected in I.


$G(\mathcal{K}) \times^{G(\mathcal{O})} Gr \xrightarrow{m} Gr$
$(g_1, g_2) \longmapsto [g_1 g_2]$
$\mathcal{F}, \mathcal{G} \rightsquigarrow \mathcal{F} \tilde{\boxtimes} \mathcal{G}$
$\mathcal{F} \star \mathcal{G} = Rm_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G})$
$(U_q(\hat{g})\text{-mod}, \otimes)$
Class. limit
$(D^{G(O)}_{coh}(Gr), \star)$
$R\Gamma(-)^*$
d, $*$)
Feigin-Loktev produt

$\mathcal{F}, \mathcal{G} \rightsquigarrow \mathcal{F} \boxtimes \mathcal{G}$

$\mathcal{F} \star \mathcal{G} = R m_* (\mathcal{F} \boxtimes \mathcal{G})$

$(\mathfrak{g}[t]\text{-mod}, *)$

$\uparrow$

Feigin-Loktev produt

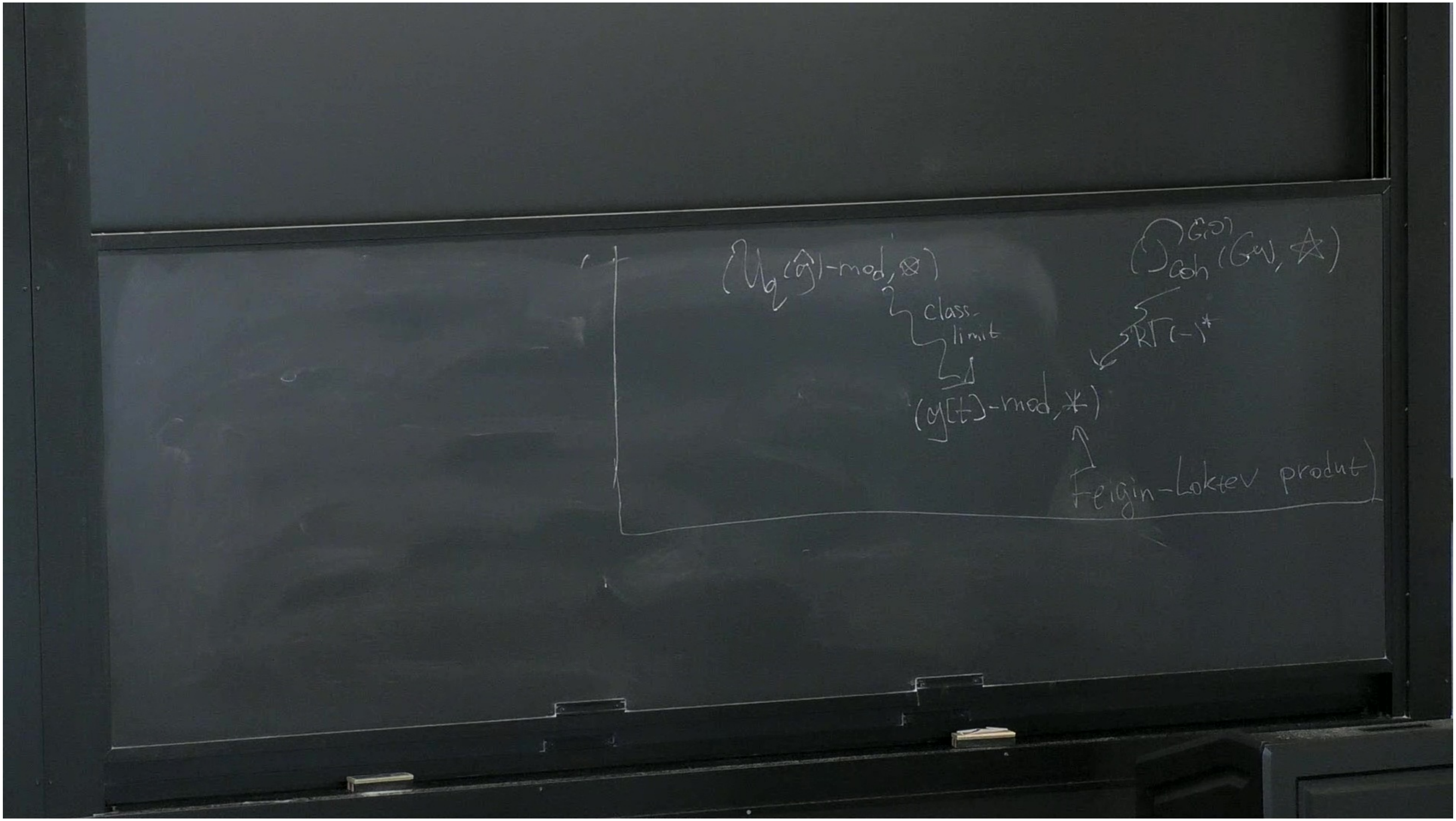
FL-fusion

$\mathfrak{g}$ - simple Lie alg $\mathfrak{g}[t]$, M_1, M_2 = graded, cyclic ($M_i \simeq \mathcal{U}(\mathfrak{g}[t]).(m_i)$), f.d. modules.

$c_1, c_2 \in \mathbb{C}$, define $M_i(c_i) \underset{v.s.}{\simeq} M_i$, xt^s acts as $x(t+c_i)^s$.

Fact: $c_1 \neq c_2$ $\quad M_1(c_1) \otimes M_2(c_2) = \mathcal{U}(\mathfrak{g}[t]).(m_1 \otimes m_2)$, define $F_{\leq n} := \mathcal{U}(\mathfrak{g}[t])_{\leq n}.(m_1 \otimes m_2)$

Def $M_1 * M_2 \simeq \mathrm{gr}_F (M_1(c_1) \otimes M_2(c_2))$


(U_q(ĝ)-mod, ⊗)
class. limit
(g[t]-mod, ⊛)
Feigin-Loktev produt
Coh^{G(O)}(Gr, ⊛)
RΓ(−)*

One step towards tow the Conj

Th (D, 2023) (ADE) $\exists$ S.E.S in $\mathcal{P}^{G(\mathcal{O})}_{coh}(Gr)$ $\forall i, l$

$$0 \to \bigstar_{j \sim i} \mathcal{P}_{j,l} \to \mathcal{P}_{i,l+1} \bigstar \mathcal{P}_{i,l-1} \to \mathcal{P}_{i,l} \bigstar \mathcal{P}_{i,l} \to 0$$

Th $l_1 \geq l_2$: $R\Gamma(\mathcal{P}_{\lambda^\vee, l_1} \bigstar \mathcal{P}_{\mu^\vee, l_2})^* \simeq \Gamma(\mathcal{P}_{\lambda^\vee, l_1})^* * \Gamma(\mathcal{P}_{\mu^\vee, l_2})^*$

$\square$ Idea BD convolution $\square$

$$\overline{Gr}^{\lambda^\vee}_{/\mathbb{A}} \tilde{\times} \overline{Gr}^{\mu^\vee}_{/\mathbb{A}}, \quad \mathcal{L}^{l_1, l_2}$$

$$\downarrow \pi$$

$$(\varepsilon_1, \varepsilon_2) \mapsto \mathbb{A}^2$$

$\mathcal{O}(\mathcal{K}) \times \mathcal{O}(\ldots$

$(g_1, g_2) \mapsto [g_1 g_2]$

$\mathcal{F}, \mathcal{G} \rightsquigarrow \mathcal{F} \boxtimes \mathcal{G}$

$\mathcal{F} \bigstar \mathcal{G} = Rm_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G})$

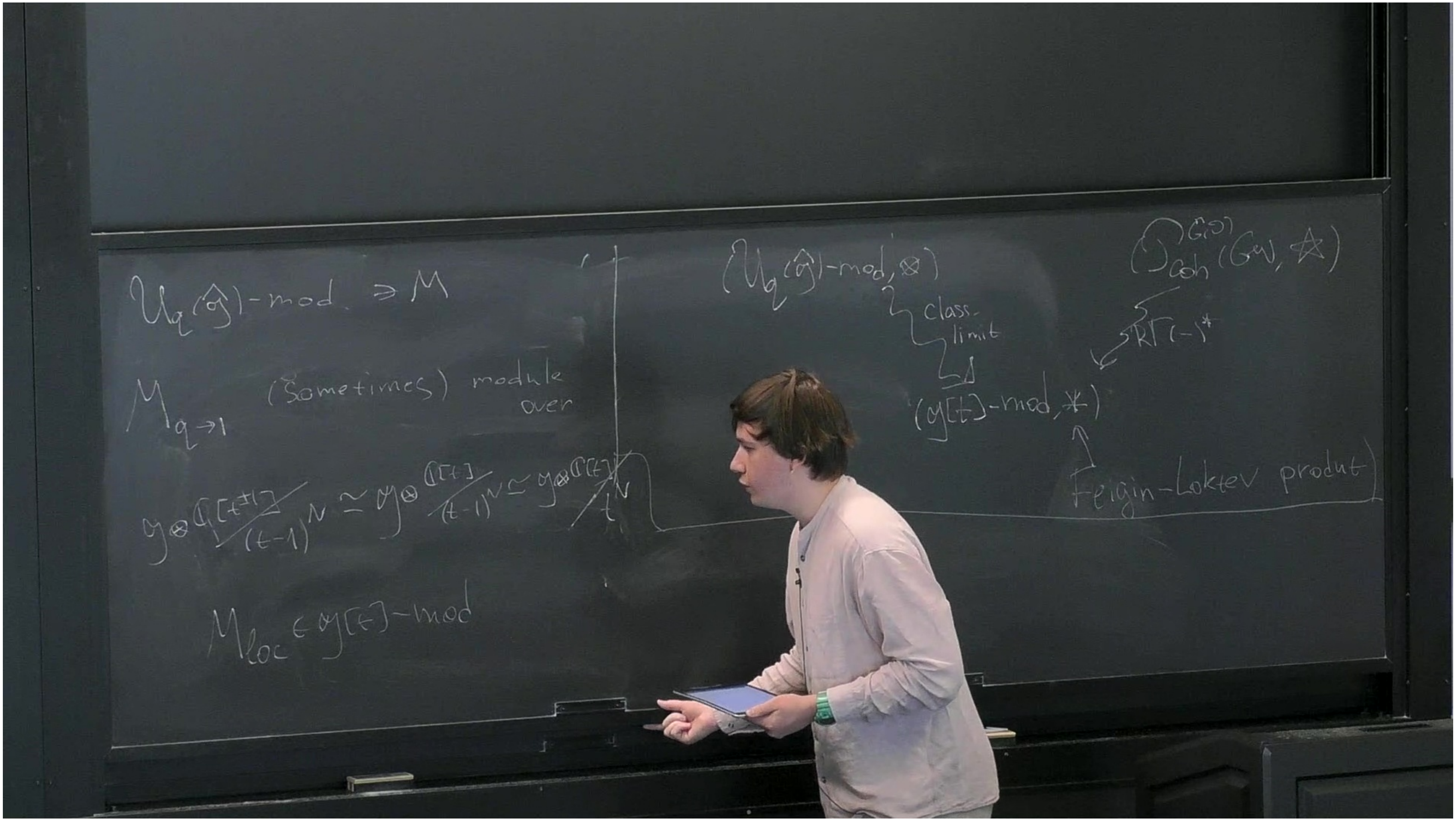
FL-fusion

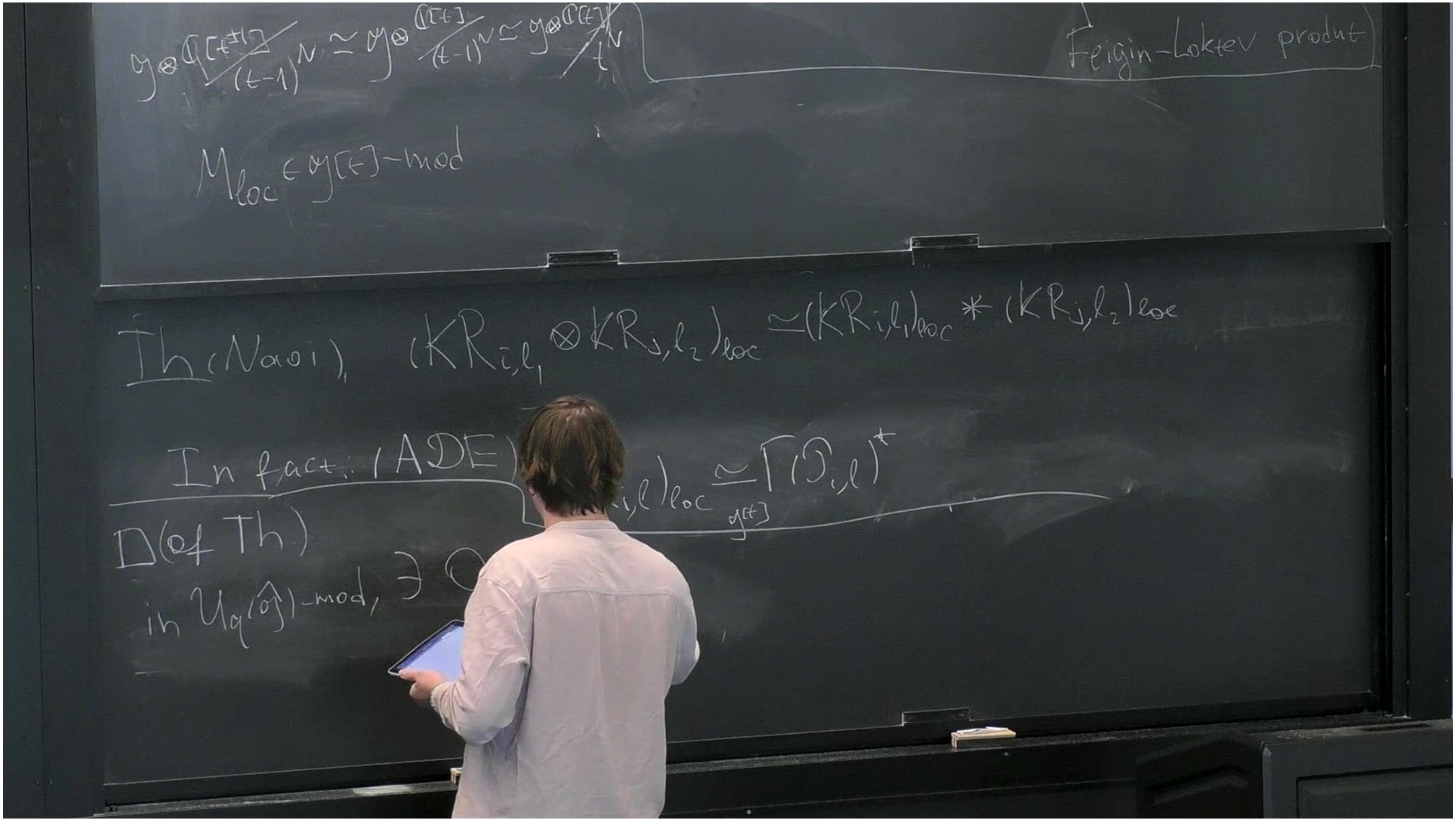
$\mathfrak{g}$ - simple Lie alg $\mathfrak{g}$...

$c_1, c_2 \in \mathbb{C}$, define $M_1(c_1) \underset{v.s}{\simeq}$ ...

Fact: $c_1 \neq c_2$ $M_1(c_1) \otimes M_2($...

Def $M_1 * M_2 \simeq gr_F(M_1($...


$U_q(\hat{\mathfrak{g}})$-mod $\ni M$
$M_{q\to 1}$ (Sometimes) module over
$\mathfrak{g}\otimes \mathbb{C}[t^{\pm 1}]/(t-1)^N \simeq \mathfrak{g}\otimes \mathbb{C}[t]/(t-1)^N \simeq \mathfrak{g}\otimes \mathbb{C}[t]/t^N$
$M_{loc} \in \mathfrak{g}[t]$-mod
$(U_q(\hat{\mathfrak{g}})$-mod$, \otimes)$
Class. limit
$(\mathfrak{g}[t]$-mod$, *)$
Feigin-Loktev produt
$(\mathcal{S}_{Coh}^{G(\mathcal{O})}(Gr, \star))$
$R\Gamma(-)^*$


g ⊗ C[t^{±1}]/(t−1)^N ≃ g ⊗ C[t]/(t−1)^N ≃ g ⊗ C[t]/t^N
Feigin-Loktev produt
M_loc ∈ g[t]-mod
Th (Naoi), (KR_{i,l_1} ⊗ KR_{j,l_2})_loc ≃ (KR_{i,l_1})_loc * (KR_{j,l_2})_loc
In fact: (ADE)
l)_loc ≃_{g[t]} Γ(O_{i,l})^*
D(of Th)
in U_q(ĝ)-mod, ∃

$$\mathfrak{g}\otimes \frac{\mathbb{C}[t^{\pm 1}]}{(t-1)^N} \simeq \mathfrak{g}\otimes \frac{\mathbb{C}[t]}{(t-1)^N} \simeq \mathfrak{g}\otimes \frac{\mathbb{C}[t]}{t^N}$$

Feigin-Loktev produt

$M_{loc} \in \mathfrak{g}[t]$-mod

In fact: (ADE) $(KR_{i,\ell})_{loc} \underset{\mathfrak{g}[t]}{\simeq} \Gamma(\mathcal{O}_{i,\ell})^*$

D(of Th)

in $U_q(\hat{\mathfrak{g}})$-mod, $\exists$ Q-systems

$$0 \to \bigotimes_{j\sim i} KR_{j,\ell} \to KR_{i,\ell}\otimes KR_{i,\ell} \to KR_{i,\ell-1}\otimes KR_{i,\ell+1} \to 0$$

gr. limit

SES