Title: Topological Feynman integrals and the odd graph complex

Speakers: Paul-Hermann Balduf

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Abstract:

Recent work by Davide Gaiotto and collaborators introduced a new type of parametric Feynman integrals to compute BRST anomalies in topological and holomorphic quantum field theories. The integrand of these integrals is a certain differential form in Schwinger parameters. In a new article together with Simone Hu, we showed that this "topological" differential form coincides with a "Pfaffian" differential form that had been used by Brown, Panzer, and Hu, to compute cohomology of the odd graph complex and of the linear group. In my talk, I will review some aspects of the graph complex and the role played by the Pfaffian form there, sketch the proof of equivalence, and comment on various observations on either side of the equivalence and their natural counterparts on the other side.

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Topological Feynman integrals and the odd graph complex

Paul-Hermann Balduf
Mathematical Institute, University of Oxford

Based on arXiv 2503.09558 (Documenta Mathematica ...) together with Simone Hu,
and arXiv 2408.03192 (JHEP ...) together with Davide Gaiotto.
Slides and links are available from paulbalduf.com/research.

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Introduction

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Theorem (PHB and Hu 2025). The topological form is the Pfaffian form,

$$\alpha_G = \phi_G \qquad \text{(up to constants).}$$

1. What is the topological form α_G? What does it compute in topological QFT?

   $\sim$ interlude about graph matrices $\sim$

2. What is the Pfaffian form ϕ_G? How is it used in the odd graph complex?
3. What can we learn from them being equal?

(I will not present details of the proof for $\alpha_G = \phi_G$)

TQFT Propagator $P_n(\vec{x})$

- Consider n-dimensional topological QFT, position space $\vec{x} = (x^{(1)}, \ldots, x^{(n)})^\intercal$ with field differential operator = de Rham operator

$$\mathrm{d} = \mathrm{d}x^{(1)}\partial_{x^{(1)}} + \mathrm{d}x^{(2)}\partial_{x^{(2)}} + \ldots + \mathrm{d}x^{(n)}\partial_{x^{(n)}}.$$

- Propagator is Green function of d, defined by $\mathrm{d}P_n(\vec{x}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\,\delta^n(\vec{x})\;\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. It is

$$P_n(\vec{x}) = \frac{\Omega_n}{|\vec{x}|^n} = \frac{\sum_{j=1}^n (-1)^j x^{(j)}\,\mathrm{d}x^{(1)} \wedge \widehat{\mathrm{d}x^{(j)}} \wedge \mathrm{d}x^{(n)}}{\sqrt{\vec{x}\cdot\vec{x}}^{\;n}}.$$

- Ω_n is the projective n-dimensional volume form (= infinitesimal surface element in spherical coordinates). In particular

$$P_1 = \frac{x}{|x|} = \mathrm{sgn}(x), \qquad P_2 = \frac{x^{(2)}\,\mathrm{d}x^{(1)} - x^{(1)}\,\mathrm{d}x^{(2)}}{{x^{(1)}}^2 + {x^{(2)}}^2} = \frac{r^2\sin^2\phi\,\mathrm{d}\phi + r^2\cos^2\phi\,\mathrm{d}\phi}{r^2} = \mathrm{d}\phi.$$

Parametric representation of the TQFT propagator $P_n(\vec{x})$

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- Recall integral representation of Euler gamma function,

$$\frac{1}{|\vec{x}|^n} = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{1}{a^{\frac{n}{2}+1}} e^{-\frac{\vec{x}^2}{a}} \, \mathrm{d}a.$$

(notice: often "Schwinger trick" done with $t = \frac{1}{a}$. Here, UV limit is $a \to 0$)

- For each component $x^{(j)}$ introduce $s^{(j)} := \frac{x^{(j)}}{\sqrt{a}}$. Then, $\mathrm{d}s^{(j)} = \frac{\mathrm{d}x^{(j)}}{a^{\frac{1}{2}}} - \frac{x^{(j)}}{2a^{\frac{3}{2}}} \mathrm{d}a$ [Gaiotto, Kulp, and Wu 2024; Budzik et al. 2023]. Wedge product:

$$\mathrm{d}s^{(1)} \wedge \ldots \wedge \mathrm{d}s^{(n)} = \frac{\mathrm{d}x^{(1)} \wedge \ldots \wedge \mathrm{d}x^{(n)}}{a^{\frac{n}{2}}} + \frac{\mathrm{d}a \wedge \Omega_n}{2a^{\frac{n}{2}+1}}.$$

- If one integrates a, first term vanishes, and

$$\int_0^\infty e^{-\vec{s}^2} \mathrm{d}s^{(1)} \wedge \ldots \wedge \mathrm{d}s^{(n)} = \frac{\Gamma\left(\frac{n}{2}\right)}{2} \frac{\Omega_n}{(\vec{x}^2)^{\frac{n}{2}}} = \frac{\Gamma\left(\frac{n}{2}\right)}{2} P_n(\vec{x}).$$

- Notice that the integrand factorizes: $e^{s^{(1)2}} \mathrm{d}s^{(1)} \wedge e^{s^{(2)2}} \mathrm{d}s^{(2)} \wedge e^{s^{(3)2}} \mathrm{d}s^{(3)} \wedge \ldots$.

Brackets

- Use BRST formalism: BRST differential Q such that gauge-invariant "physical" observables A are 0^{th} cohomology group. That is,

$$QA = 0 \qquad \text{and} \qquad \nexists B : A = QB.$$

- A classically gauge invariant observable might violate gauge invariance at quantum level ("anomaly"). Work in perturbation theory, let $\mathcal{O}_j$ be local operators. Define *bracket* [Gaiotto, Kulp, and Wu 2024]

$$\{\mathcal{O}_1, \ldots, \mathcal{O}_k\} := Q\left(\int_{\mathbb{R}^{n(k-1)}} \mathcal{O}_1 \cdots \mathcal{O}_k\right).$$

- The integral is a sum over Feynman integrals with k vertices in the n-dimensional TQFT,

$$\{\mathcal{O}_1, \mathcal{O}_2, \ldots\} = \sum_{\text{Graphs } G} \frac{1}{|\text{Aut}(G)|} I_G \prod_{v \in V_G} \prod_i \varphi_{i,v}.$$

symmetry factor → $\frac{1}{|\text{Aut}(G)|}$; Feynman integral → I_G; External leg structure → $\prod_i \varphi_{i,v}$

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The topological form

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- Recall that parametric integrand factorizes along dimension $\Rightarrow$ consider 1-dimensional integrand α_G. Schwinger parameter a_e for each edge. Start/end coordinates $x_e^{\pm} \in \mathbb{R}$. Then $I_G = \int \alpha_G \wedge \alpha_G \wedge \ldots$ with the *topological form* (differential form of degree ℓ)

$$\alpha_G := \frac{1}{\pi^{\frac{|E_G|}{2}}} \int \cdots \int_{\mathbb{R}^{|V_G|-1}} \bigwedge_{e \in E_G} e^{-s_e^2} \, \mathrm{d}s_e, \qquad \text{where} \quad s_e := \frac{x_e^+ - x_e^-}{\sqrt{a_e}}.$$

- Key results of [Balduf and Gaiotto 2024]:

$$\alpha_G = \frac{1}{\pi^{\frac{\ell}{2}} 4^\ell \left(\frac{\ell}{2}\right)! \cdot \psi_G^{\frac{\ell+1}{2}}} \sum_{T \text{ spanning tree}} \det\left(\mathbb{I}[T]\right) \left(\sum_{\sigma \in \mathfrak{S}_T} \psi_G^{\sigma(f_1),\sigma(f_2)} \cdots \psi_G^{\sigma(f_{\ell-1}),\sigma(f_\ell)} \right) \bigwedge_{f \notin T} \mathrm{d}a_f,$$

$\alpha_G \wedge \alpha_G = 0$ for all graph (Kontsevich Formality theorem).

Here $\mathbb{I}$ is the edge-vertex incidence matrix, ψ_G is the Symanzik polynomial, ψ^{e_1,e_2} are edge-induced Dodgson polynomials (all of these can be produced easily with a computer).

Graph matrices 1: Incidence matrix and Laplacian

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- Always assume that the graph G is connected. Edge set E, vertex set V.
- $|E| \times (|V|-1)$ *incidence matrix* $\mathbb{I}$ has entry $\mathbb{I}_{e,v} = +1$ if edge e ends at vertex v, and -1 if e starts at v, and 0 else. Column of one vertex $v_\star$ left out.
- $|E| \times |E|$ *edge variable matrix* $\mathbb{D} = \operatorname{diag}\left(a_1, \dots, a_{|E|}\right)$ contains Schwinger parameters.
- $(|V|-1) \times (|V|-1)$ *vertex Laplacian*

$$\mathbb{L} := \mathbb{I}^\intercal \mathbb{D}^{-1} \mathbb{I}.$$

- *First Symanzik polynomial*

$$\psi_G := \det \mathbb{L} \cdot \det \mathbb{D} = \det \mathbb{L} \cdot \prod_{e \in E} a_e = \sum_{T \text{ spanning}} \prod_{e \notin T} a_e$$

is homogeneous of degree ℓ in the variables a_e.

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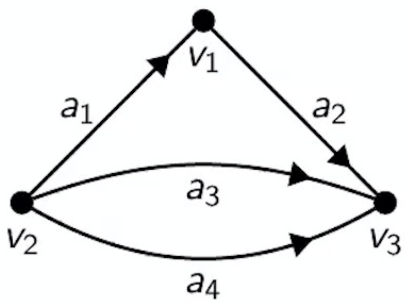
Example: The dunce's cap

"Dunce's cap" G is a graph on 3 vertices and 4 edges, with $\ell = 2$ loops. Labels and directions are chosen as:



We further choose $v_3 =: v_\star$ as the vertex to remove from $\vec{x}$.
Remaining: $|V| = 2, |E| = 4$.
$\mathbb{I}$ is 4×2 and $\mathbb{D}$ is 4×4.

With these choices:

$$\mathbb{I} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}, \qquad \mathbb{D} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}.$$

This gives the Laplacian $\mathbb{L} = \mathbb{I}^\intercal \mathbb{D} \mathbb{I}$:

$$\mathbb{L} = \begin{pmatrix} \frac{1}{a_1} + \frac{1}{a_2} & -\frac{1}{a_1} \\ -\frac{1}{a_1} & \frac{1}{a_1} + \frac{1}{a_3} + \frac{1}{a_4} \end{pmatrix}.$$

Symanzik polynomial:

$$\psi_G = \det \mathbb{L} \cdot \prod_{e \in E} a_e = a_3 a_4 + a_1(a_3 + a_4) + a_2(a_3 + a_4).$$

(Notice *matrix tree theorem*: The terms of ψ are the complements of spanning trees, $\psi = \sum_T \prod_{e \notin T} a_e$).

Topological differential form for the dunce's cap

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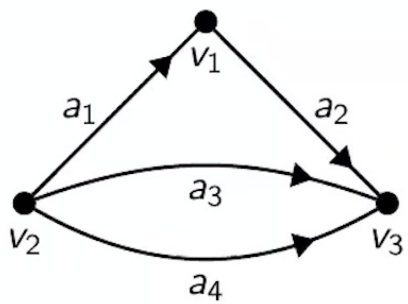
$$\alpha_G = \frac{1}{\pi^{\frac{\ell}{2}} 4^\ell \left(\frac{\ell}{2}\right)! \cdot \psi_G^{\frac{\ell+1}{2}}} \sum_{T \text{ spanning tree}} \det\left(\mathbb{I}[T]\right) \left(\sum_{\sigma \in \mathfrak{S}_T} \psi_G^{\sigma(f_1),\sigma(f_2)} \cdots \psi_G^{\sigma(f_{\ell-1}),\sigma(f_\ell)} \right) \bigwedge_{f \notin T} \mathrm{d}a_f .$$



G has five spanning trees T. For example, consider $T = \{2,4\}$.
Then $E \setminus T = \{f_1, f_2\} = \{1,3\}$ and $\mathbb{I}[T] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and
$\psi^{1,3} = -a_4$ (I didn't introduce how to compute this).
One obtains the contribution

$$\frac{(+1)}{16\pi(a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4)^{3/2}} \cdot (-2a_4)\,\mathrm{d}a_1 \wedge \mathrm{d}a_3 .$$

End result:

$$\alpha_G = \frac{-a_4(\mathrm{d}a_1 \wedge \mathrm{d}a_3 + \mathrm{d}a_2 \wedge \mathrm{d}a_3) + a_3(\mathrm{d}a_1 \wedge \mathrm{d}a_4 + \mathrm{d}a_2 \wedge \mathrm{d}a_4) - (a_1 + a_2)\,\mathrm{d}a_3 \wedge \mathrm{d}a_4}{8\pi(a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4)^{3/2}} .$$

Graph matrices 2: Cycle incidence matrix

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- A *circuit* is a closed path of edges (regardless of edge directions). May visit vertex, but not edge, multiple times.
- Circuits can be added and subtracted, form a vector space over $\mathbb{Z}$ (mod ± 2). *Cycle space*, dimension: $|E| - |V| + 1 = \ell$ is *loop number*.
- A choice of basis for cycle space determines a *cycle incidence matrix* $\mathcal{C}$: Entry $\mathcal{C}_{e,c} = +1$ if edge e is in cycle c in positive direction, -1 if in negative direction.
- Analogously, vertex incidence matrix $\mathbb{I}$ represents a choice of basis in *cut space*.
- The spaces, and hence the matrices $\mathcal{C}$ and $\mathbb{I}$ are orthogonal, $\mathbb{I}^\intercal \mathcal{C} = \mathbb{0}_{(|V|-1)\times \ell}$, $\mathcal{C}^\intercal \mathbb{I} = \mathbb{0}_{\ell \times (|V|-1)}$.

Example: Cycles in the dunce's cap

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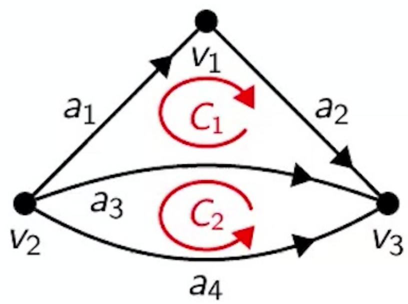

$\ell = 2 \Rightarrow 2$ linearly independent circuits to be chosen as basis of cycle space. This choice is not unique.

With C_1 and C_2 as drawn,
$C_1 = \{+a_1, +a_2, -a_3\}$ and $C_2 = \{-a_3, +a_4\}$.

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \text{recall} \quad \mathbb{I} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}.$$

Columns of $\mathcal{C}$ are basis vectors in cycle space, columns of $\mathbb{I}$ are basis vectors in cut space.
Cut space and cycle space are orthogonal, i.e.

$$\mathcal{C}^\intercal \mathbb{I} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Graph matrices 3: Cycle Laplacian

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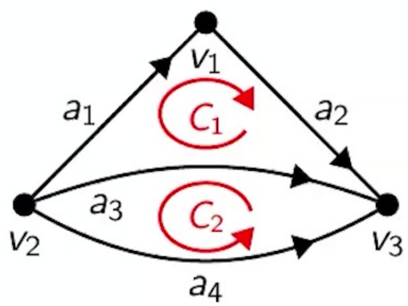
- Recall the vertex Laplacian $\mathbb{L} := \mathbb{I}^\intercal \mathbb{D}^{-1} \mathbb{I}$, is a $(|V|-1) \times (|V|-1)$ sym. matrix.
- Analogously *cycle Laplacian* is the $\ell \times \ell$ symmetric matrix $\mathbb{\Lambda} := \mathcal{C}^\intercal \mathbb{D} \mathcal{C}$.
- Determinant is $\det \mathbb{\Lambda} = \psi_G$ (regardless of the choice of $\mathcal{C}$). Hence, $\mathbb{\Lambda}$ is invertible.



$C_1 = \{+a_1, +a_2, -a_3\}$ and $C_2 = \{-a_3, +a_4\}$.

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{\Lambda} = \begin{pmatrix} a_1 + a_2 + a_3 & a_3 \\ a_3 & a_3 + a_4 \end{pmatrix}.$$

Inverse matrix denominator is Symanzik polynomial $\det \mathbb{\Lambda} = \psi_G$,

$$\mathbb{\Lambda}^{-1} = \frac{1}{\psi_G} \begin{pmatrix} a_3 + a_4 & -a_3 \\ -a_3 & a_1 + a_2 + a_3 \end{pmatrix}.$$

Graph matrices 4: Path matrices

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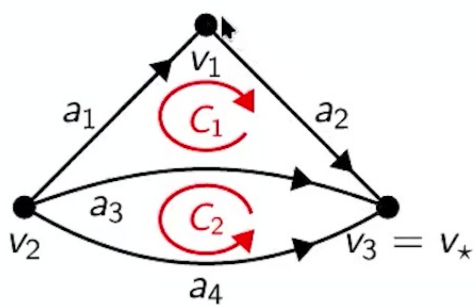
- A *path matrix* $\mathcal{P}$ is a $|E| \times (|V|-1)$–matrix where column j is a directed path of edges from $v_\star$ to v_j.
- $\mathcal{P}$ has the same shape as $\mathbb{I}$, but they are distinct. In fact, $\mathcal{P}^\intercal \mathbb{I} = \mathbb{1}_{(|V|-1)\times(|V|-1)}$.
- One can show that $\det(\mathcal{C} \mid \mathcal{P}) \in \{+1,-1\}$. This determinant encodes a (relative) sign ambiguity that arises from the choice of cycle basis in $\mathcal{C}$ [Conant and Vogtmann 2003].



Let $v_\star = v_3$ and paths $P_1 = \{a_1, -a_3\}$ and $P_2 = \{-a_4\}$.

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{I} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}.$$

The concatenation $(\mathcal{C} \mid \mathcal{P})$ has full rank and $\det(\mathcal{C} \mid \mathcal{P}) = +1$.
One also checks that $\mathcal{P}^\intercal \mathbb{I} = \mathbb{1}_{2\times 2}$.
It is coincidence that all matrices have the same shape.

Pfaffians

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- Let M be a $2n \times 2n$ skew-symmetric matrix. The *Pfaffian* is

$$\operatorname{Pf}(M) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot M_{\sigma(1),\sigma(2)} \cdots M_{\sigma(2n-1),\sigma(2n)}.$$

- If a skew-symmetric M has odd dimensions, set $\operatorname{Pf}(M) = 0$.
  Then $\operatorname{Pf}(M)^2 = \det(M)$ for all skew-symmetric matrices.
- This (like the determinant) assumes that the entries of M commute.
- Examples:

$$\operatorname{Pf}\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b, \qquad \operatorname{Pf}\begin{pmatrix} 0 & b & c & d \\ -b & 0 & g & h \\ -c & -g & 0 & l \\ -d & -h & -l & 0 \end{pmatrix} = bl - ch + dg.$$

The Pfaffian form

- Consider a graph with even loop number ℓ, and differential wrt Schwinger parameters

$$\mathrm{d}\mathbb{\Lambda} = \mathrm{d}\left(\mathcal{C}^{\intercal}\mathbb{D}\mathcal{C}\right) = \mathcal{C}^{\intercal}\,\mathrm{d}\mathbb{D}\mathcal{C}.$$

Then the matrix $\mathrm{d}\mathbb{\Lambda}\cdot\mathbb{\Lambda}^{-1}\cdot\mathrm{d}\mathbb{\Lambda}$ is a $\ell\times\ell$ (=even), skew-symmetric matrix whose entries are 2-forms (hence they commute).

- The *Pfaffian form* is defined as [Brown, Hu, and Panzer 2024]

$$\phi_G := \frac{1}{(-2\pi)^{\frac{\ell}{2}}}\frac{\mathrm{Pf}\left(\mathrm{d}\mathbb{\Lambda}\cdot\mathbb{\Lambda}^{-1}\cdot\mathrm{d}\mathbb{\Lambda}\right)}{\sqrt{\det\mathbb{\Lambda}}}.$$

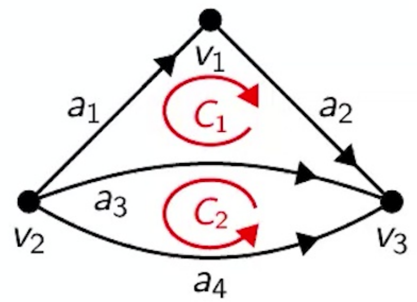
- Change of cycle basis $\mathcal{C}' = A^{\intercal}\mathcal{C}A$ with constant matrix A leads to

$$\mathrm{d}\mathbb{\Lambda}'\mathbb{\Lambda}'^{\intercal}\,\mathrm{d}\mathbb{\Lambda} = A^{\intercal}\,\mathrm{d}\mathbb{\Lambda}A\,\left(A^{\intercal}\mathbb{\Lambda}A\right)^{-1}A^{\intercal}\,\mathrm{d}\mathbb{\Lambda}A = A^{\intercal}\,\mathrm{d}\mathbb{\Lambda}\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\,A$$

known: $\mathrm{Pf}(A^{\intercal}BA) = \det(A)\,\mathrm{Pf}(B)$.

$\Rightarrow \phi_G$ changes sign by $\det(A)$ under change of basis.

Example: Pfaffian form of the dunce's cap



$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{\Lambda}^{-1} = \frac{1}{\psi_G} \begin{pmatrix} a_3 + a_4 & -a_3 \\ -a_3 & a_1 + a_2 + a_3 \end{pmatrix}$$

$$\mathrm{d}\mathbb{\Lambda} = \begin{pmatrix} \mathrm{d}a_1 + \mathrm{d}a_2 + \mathrm{d}a_3 & \mathrm{d}a_3 \\ \mathrm{d}a_3 & \mathrm{d}a_3 + \mathrm{d}a_4 \end{pmatrix}.$$

$\mathrm{d}\mathbb{\Lambda}\, \mathbb{\Lambda}^{-1}\, \mathrm{d}\mathbb{\Lambda}$ is a $\ell \times \ell$ matrix, hence 2×2. Recall $\operatorname{Pf} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b$.

We only need the top right entry of

$$\mathrm{d}\mathbb{\Lambda}\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda} = \frac{1}{\psi_G} \begin{pmatrix} \mathrm{d}a_1 + \mathrm{d}a_2 + \mathrm{d}a_3 & \mathrm{d}a_3 \\ \mathrm{d}a_3 & \mathrm{d}a_3 + \mathrm{d}a_4 \end{pmatrix} \begin{pmatrix} (a_3 + a_4)(\mathrm{d}a_1 + \mathrm{d}a_2) + a_4\,\mathrm{d}a_3 & a_4\,\mathrm{d}a_3 - a_3\,\mathrm{d}a_4 \\ -a_3(\mathrm{d}a_1 + \mathrm{d}a_2) + (a_1 + a_2)\,\mathrm{d}a_3 & (a_1 + a_2)(\mathrm{d}a_3 + \mathrm{d}a_4) + a_3\,\mathrm{d}a_4 \end{pmatrix}$$

This yields

$$\phi_G = \frac{a_4\,\mathrm{d}a_1\,\mathrm{d}a_3 + a_4\,\mathrm{d}a_2\,\mathrm{d}a_3 - a_3\,\mathrm{d}a_1\,\mathrm{d}a_4 - a_3\,\mathrm{d}a_2\,\mathrm{d}a_4 + (a_1 + a_2)\,\mathrm{d}a_3\,\mathrm{d}a_4}{-2\pi\psi_G^{\frac{3}{2}}}.$$

The main result

Compare the two example calculations for the dunce's cap:

$$\phi_G = \frac{a_4 \, \mathrm{d}a_1 \, \mathrm{d}a_3 + a_4 \, \mathrm{d}a_2 \, \mathrm{d}a_3 - a_3 \, \mathrm{d}a_1 \, \mathrm{d}a_4 - a_3 \, \mathrm{d}a_2 \, \mathrm{d}a_4 + (a_1 + a_2) \, \mathrm{d}a_3 \, \mathrm{d}a_4}{-2\pi \psi_G^{\frac{3}{2}}}$$

$$\alpha_G = \frac{-a_4(\, \mathrm{d}a_1 \, \mathrm{d}a_3 + \, \mathrm{d}a_2 \, \mathrm{d}a_3) + a_3(\, \mathrm{d}a_1 \, \mathrm{d}a_4 + \, \mathrm{d}a_2 \, \mathrm{d}a_4) - (a_1 + a_2) \, \mathrm{d}a_3 \, \mathrm{d}a_4}{8\pi(a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4)^{3/2}} = \frac{1}{4}\phi_G.$$

Theorem (PHB and Hu 2025). Let $\mathcal{C}$ be any choice of cycle incidence matrix and $\mathcal{P}$ any choice of path matrix, then $\det(\mathcal{C} \,|\, \mathcal{P}) \in \{+1, -1\}$ and for all graphs

$$\alpha_G = \frac{\det(\mathcal{C} \,|\, \mathcal{P})}{2^\ell} \cdot \phi_G.$$

Proof: Linear algebra, expansion formulas for Pfaffians, match the Dodgson polynomial formula for the topological form α_G.

What is the Pfaffian form good for?

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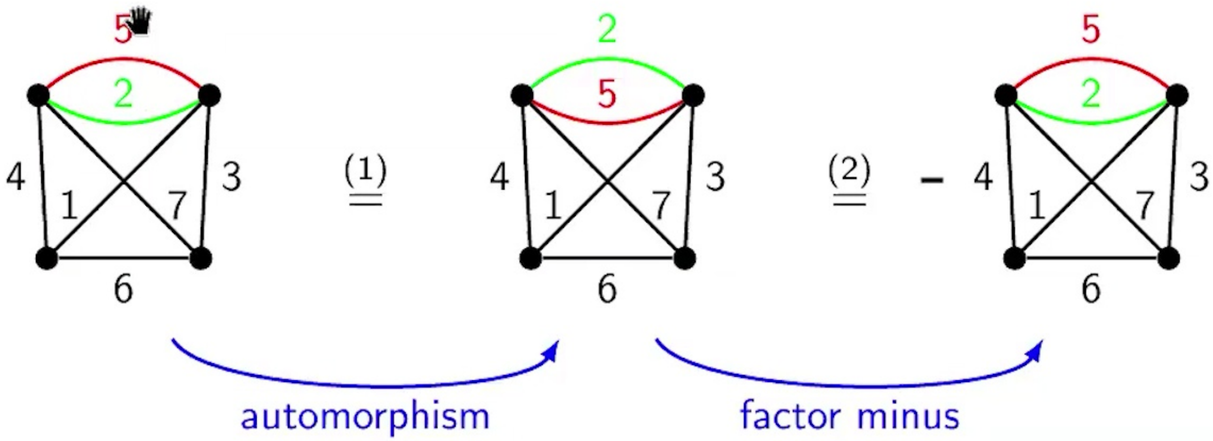
It acts on the odd graph complex...

The even graph complex

- Graph complexes are important combinatorial objects in math and physics.
- The *even* commutative graph complex GC_N is vector space over $\mathbb{Q}$, freely generated by (G, η) where G: connected graph without 1- or 2-valent vertices, η: orientation (=permutation sign of ordering of edges) [Kontsevich 1993].
- Grading $\deg(G) = |E| - N \cdot \ell$ (we choose $N = 2$ since then grading = sdd). All even N give isomorphic complexes.
- Modulo isomorphism $f : G \mapsto \tilde{G}$ by $(G, \eta) \overset{(1)}{=} (\tilde{G}, f(\eta))$ and $-(G, \eta) \overset{(2)}{=} (G, -\eta)$. This implies that all graphs with double edges (or other odd automorphisms) vanish.



Boundary map of graph complexes

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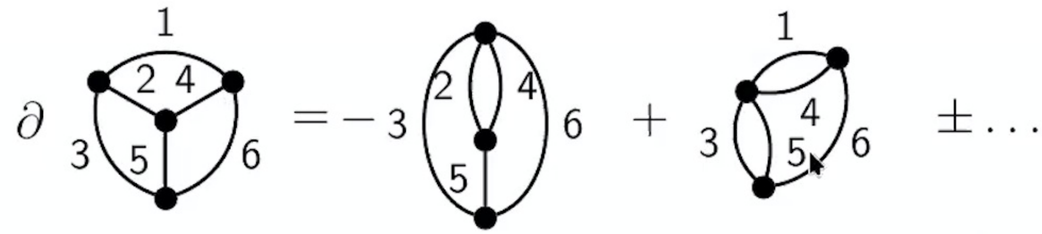
- Let G/γ denote shrinking of subgraph $\gamma \subset G$ to a vertex. Define the boundary operator

$$\partial(G,\eta) = \sum_{j=1}^{n} (-1)^j \left(G/e_j, \eta/e_j\right).$$

Example:



- *Graph homology* is $H_\bullet(\mathrm{GC}_N) = \ker \partial / \operatorname{im} \partial$. It is graded by (homological) degree, H_n where $n = \deg(G) = |E| - N\ell$, and by loop number .
- Example: The above graph W_3 (=wheel on 3 spokes) has $\partial W_3 = 0$ since all resulting graphs contain double edges. $\deg(W_3) = 6 - 2 \times 3 = 0$. Turns out it is not exact, $\nexists F : \partial F = W_3$. Hence $W_3 \in H_0(\mathrm{GC}_2)$.

Homology of the even graph complex

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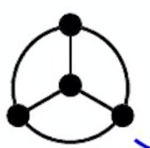
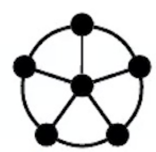
- Example: Wheel with n spokes W_n is closed $\partial W_n = 0$ since contracting any edge yields a double edge (which vanishes).
- However, $W_{2n} = 0$ due to odd automorphism. Can show that $[W_{2n+1}] \in H_0(\mathrm{GC}_2)\ \forall n \geq 1$. They all have degree 0 in GC_2 since $4n + 2 - 2(2n+1) = 0$.
- Homologies are known up to $\ell \approx 10$ [Brun and Willwacher 2024]. One finds only few classes, but for $\ell \to \infty$, their dimension grows super-exponentially [Borinsky and Zagier 2024].

Homologies of GC_2:

H_6	vanishes due to						0	0	
H_4	2-valent vertex					0	0	0	
H_3					0	1	0	1	
H_2				0	0	0	0	0	
H_1			0	0	0	0	0	0	
H_0		0	1	0	1	0	1	1	
ℓ	1	2	3	4	5	6	7	8	...





Brown's canonical differential forms

- Let G be a connected graph with cycle Laplacian $\mathbb{\Lambda} = \mathcal{C}^\intercal \mathbb{D} \mathcal{C}$. Define *canonical form* [Brown 2021]

$$\beta_G^n := \operatorname{tr}\left(\left(\mathbb{\Lambda}^{-1} \, \mathrm{d}\mathbb{\Lambda} \right)^n \right).$$

(distinct objects are called "canonical forms" in the literature. This one is canonical because it is invariant under multiplying $\mathbb{\Lambda}$ by any invertible matrix A with $\mathrm{d}A = 0$.)

- Linear algebra: β_G^n is zero unless $n = 4k+1$ for $k \in \mathbb{N}_0$.
- If $k > 0$, the form is projectively invariant; $\mathrm{d}\beta^{4k+1} = 0$.
- These are the *primitive* canonical forms (i.e. define a coproduct Δ such that $\Delta \beta^{4k+1} = \mathbb{1} \otimes \beta^{4k+1} + \beta^{4k+1} \otimes \mathbb{1}$). They generate an algebra of canonical forms, where products might have different degree. E.g.

$$\beta^5 \wedge \beta^9 \quad \text{has degree } 14 \neq 4k+1.$$

- If ω_G is a canonical form of degree n and $|E| = n+1$, then ω is proportional to the projective volume form $\Omega_{|E|}$,

$$\omega_G = \frac{\text{some polynomial}}{\psi^j} \Omega_{|E|}.$$

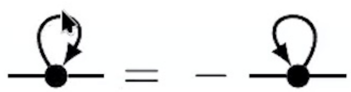
Canonical integrals

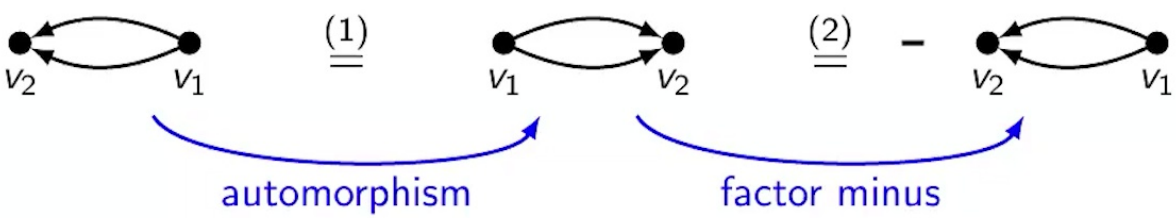
- Canonical forms can be used to find cohomology classes in the graph complex. Let G be some (linear combination of) graphs such that $\partial G = 0$ (this can be checked by explicit computation). Hard part: How to establish whether $\exists F$ such that $\partial F = G$?
- As $\mathrm{d}\beta = 0$, it is $\int_F \mathrm{d}\beta = 0$ for every graph F, where $\int_F = \int_{\sigma_F}$ with $\sigma_F = [a_1 : \ldots : a_{|E|}] \in \mathbb{P}(\mathbb{R}^{|E|})_+$ ("graph simplex").
- Stokes theorem:

$$0 = \int_F \mathrm{d}\beta = \int_{\partial F} \beta = \int_G \beta \qquad (\text{if } \partial F = G).$$

  This integral vanishes for all primitive canonical forms β.
  (There are more terms for a non-primitive $\omega = \beta \wedge \beta \wedge \ldots$, but it still vanishes).
- Conversely, if one finds *any* β such that $\int_G \beta \neq 0$, one knows that $G \neq \partial F$. That is, G is not exact, and since $\partial G = 0$, this G defines a cohomology class in the even graph complex.
- Equivalently, one can view the integrals as elements of the *dual* of the complex ($I_G(\omega) := \int_G \omega$ is a linear map from GC_2 to $\mathbb{R}$).

The odd graph complex

- The *odd graph complex* consists of oriented graphs (G, η), where the orientation η is a labeling of vertices + a choice of edge directions (= the information which is contained in the incidence matrix $\mathbb{I}$, mod 2).
  η equivalent to (cycle basis + edge order) [Conant and Vogtmann 2003].
- Again, vertex valence at least 3, modulo graph isomorphism. All odd N give rise to isomorphic complexes GC_N. Choose GC_3 with grading $\deg(G) = |E| - 3\ell$.
- Tadpoles vanish: 
- Multi edges no longer vanish automatically, but graphs which are *only* multi edges with even number of edges (=odd number of loops) vanish.



Homology of the odd graph complex

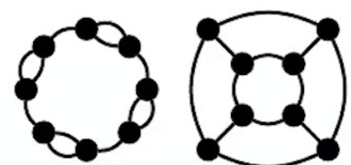
- Same boundary operator as for GC_2:

$$\partial(G,\eta) = \sum_{j=1}^{n} (-1)^j \left(G/e_j, \eta/e_j\right).$$

- Example: All even-loop multi edges are closed.
- H_{-3} is "algebra of 3-graphs" [Duzhin, Kaishev, and Chmutov 1998; Vogel 2011].

Homologies of GC_3:

H_{-8}		known to						0	
H_{-7}		vanish					0	0	
H_{-6}						1	1	2	
H_{-5}					0	0	0	0	
H_{-4}				0	0	0	0	0	
H_{-3}		1	1	1	2	2	3	4	
ℓ	1	2	3	4	5	6	7	8	...



The role of the Pfaffian form

- Recall that canonical forms β_G^{4k+1} operate on the *even* graph complex.
- The odd graph complex requires a form that flips sign in the same way as the graphs do.
- The Pfaffian form ϕ_G has this property [Brown, Hu, and Panzer 2024], it is an "orientation form". Concretely, for a cycle Laplacian $\mathbb{\Lambda} \mapsto A^\intercal \mathbb{\Lambda} A$ we have

$$\beta^{4k+1} \mapsto \beta^{4k+1}, \qquad \text{but} \qquad \phi \mapsto \det(A)\phi.$$

$\Rightarrow \int_G \phi_G \wedge \omega$ is well-defined on the odd graph complex, where ω is any product of β forms.
- Can use $\int_G \phi \wedge \omega$ to compute homology. But note $\Delta\phi = \phi \otimes \phi \Rightarrow$ more terms in Stokes relation.
- Example from [Brown, Hu, and Panzer 2024]: For $\ell = 6$, the form $\beta^5 \wedge \phi$ is of degree 11. There is a linear combination of graphs with $\ell = 6$ and $|E| = 12$ where the integral is non-vanishing, it spans the homology H_{-6} at $\ell = 6$.

Consequences

Paul Balduf (

Now we know what α is and what ϕ is, and that they are the same.
What can we learn from this?

Consequences

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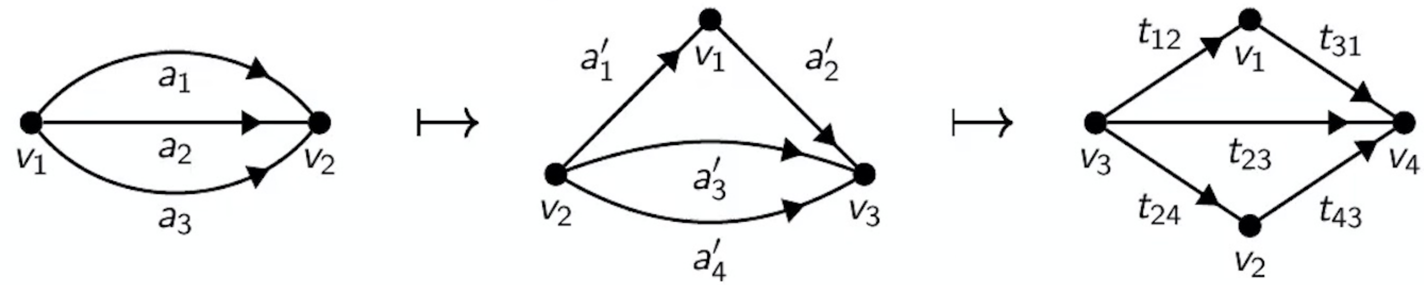
- Obtained a new representation for α_G. Since ϕ_G is (directly) given by matrices, many of its properties follow easily from linear algebra.
- $\mathrm{d}\alpha_G = 0$, and $\int \alpha_G$ is finite, projective, well-defined under change of labelings, etc.
- For example: Contracting (non-tadpole) edge, or inserting 2-valent vertex into edge, is canonical isomorphism s of cycle space. Then $\alpha_{G'} = \pm s^*(\alpha_G)$.



$$\frac{a_3\,\mathrm{d}a_1 \wedge \mathrm{d}a_2}{(a_1a_2 + a_2a_3 + a_1a_3)^{3/2}} \mapsto \frac{a'_4(\,\mathrm{d}a'_1 + \,\mathrm{d}a'_2) \wedge \,\mathrm{d}a'_3}{\left((a'_1 + a'_2)a'_3 + (a'_1 + a'_2)a'_4 + a'_3a'_4\right)^{3/2}}$$

$$\mapsto \frac{(t_{24} + t_{43})(\,\mathrm{d}t_{12} + \,\mathrm{d}t_{31}) \wedge \,\mathrm{d}t_{23}}{\left((t_{12} + t_{31})t_{23} + (t_{12} + t_{31})(t_{24} + t_{43}) + t_{23}(t_{24} + t_{43})\right)^{3/2}}.$$

Formality theorem

- Kontsevich formality theorem [Kontsevich 2003] $\alpha_G \wedge \alpha_G = 0$ (there are no anomalies in TQFTs with $D \geq 2$) proved with some effort in [Balduf and Gaiotto 2024; Wang and Williams 2024].
- Now use that $\mathrm{Pf}(A)^2 = \det(A)$:

$$\phi_G \wedge \phi_G \propto \frac{1}{\det \mathbb{A}} \left(\mathrm{Pf}\left(\mathrm{d}\mathbb{A}\mathbb{A}^{-1}\,\mathrm{d}\mathbb{A}\right)\right)^2 = \det\left(\mathbb{A}^{-1}\right)\det\left(\mathrm{d}\mathbb{A}\ \mathbb{A}^{-1}\,\mathrm{d}\mathbb{A}\right) = \det\left(\mathbb{A}^{-1}\,\mathrm{d}\mathbb{A}\ \mathbb{A}^{-1}\,\mathrm{d}\mathbb{A}\right)$$

$$= \det\left(\left(\mathbb{A}^{-1}\,\mathrm{d}\mathbb{A}\right)^2\right) =: \det(M) = \frac{1}{(\ell/2)!} B_n\left(s_1, s_2, \ldots\right),$$

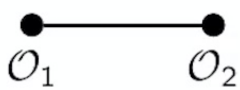
where B_n are Bell polynomials and

$$s_j = -\frac{(j-1)!}{2}\,\mathrm{tr}\left(M^j\right) = -\frac{(j-1)!}{2}\,\mathrm{tr}\left(\left(\mathbb{A}^{-1}\,\mathrm{d}\mathbb{A}\right)^{2j}\right) = -\frac{(j-1)!}{2}\beta_G^{2j} = 0 \quad \forall j.$$

(recall that only $\beta^{4k+1} \neq 0$ due to cyclicity of trace and symmetry of $\mathbb{A}$).
- Hence $\phi_G \wedge \phi_G = 0$, and therefore $\alpha_G \wedge \alpha_G = 0$.

Dipole sums in TQFT

- Recall $$\{\mathcal{O}_1, \mathcal{O}_2, \dots\} = \sum_{\text{Graphs } G} \frac{1}{|\mathrm{Aut}(G)|} I_G \prod_{v \in V_G} \prod_i \varphi_{i,v}.$$
  symmetry factor → $\frac{1}{|\mathrm{Aut}(G)|}$; Feynman integral → I_G; External leg structure → $\prod_i \varphi_{i,v}$
- α_G is of form degree ℓ, so $\int_G \alpha_G \neq 0$ only if $|E| = \ell + 1$, which are multi-edge graphs (=dipoles) D_{2i+1} with $\ell = 2i$. Then $\alpha_{D_{2i+1}} \propto \Omega_{\ell+1}/\psi_G^{i+\frac{1}{2}}$ and $I_G = \int_G \alpha = \frac{1}{2^\ell}$.
- For local operators $\mathcal{O}_1, \mathcal{O}_2$ (polynomials in p and q), propagator connects p with q,



$$= (\partial_p \mathcal{O}_1)(\partial_q \mathcal{O}_2) - (\partial_q \mathcal{O}_1)(\partial_p \mathcal{O}_2) =: \eta^{ij}(\partial_i \mathcal{O}_1)(\partial_j \mathcal{O}_2), \quad \eta := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$= \quad (\partial_p^3 \mathcal{O}_1)(\partial_q^3 \mathcal{O}_2) - (\partial_q^2 \partial_p \mathcal{O}_1)(\partial_p^2 \partial_q \mathcal{O}_2) \pm \dots = \eta^{ij}\eta^{kl}\eta^{mn}(\partial_i \partial_k \partial_m \mathcal{O}_1)(\partial_j \partial_l \partial_n \mathcal{O}_2).$$

- The sum becomes the *Moyal commutator* $\{\mathcal{O}_1, \mathcal{O}_2\} = \mathcal{O}_1 \star \mathcal{O}_2 - \mathcal{O}_2 \star \mathcal{O}_1 = [\mathcal{O}_1, \mathcal{O}_2]_\star$,
$$\{\mathcal{O}_1, \mathcal{O}_2\} = \sum_{n=0}^{\infty} \frac{1}{4^n} \frac{1}{(2n+1)!} \left(\eta^{ij}\right)^{2n+1} \left(\partial^{2n+1} \mathcal{O}_1\right) \left(\partial^{2n+1} \mathcal{O}_2\right).$$
- Subtract anomaly to obtain a quantum corrected differential $Q' = Q - \{\cdot, \mathcal{O}\}$.

Dipole sums in graph complexes

Perimeter-A

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- Dual complex of GC_3 has codifferential δ which acts by splitting a vertex, i.e. inserting an edge (=0-loop dipole D_0) into a vertex. Can be expressed as Lie bracket $\delta G \equiv [G, D_0] = G \circ D_0 - D_0 \circ G$.
- The integral $\int_G \phi_G$ is non-zero iff G is a (linear combination of) dipoles. Hence, the Pfaffian can be viewed as a pairing with dipoles,

$$\int_G \phi_G = \langle G, \mathfrak{m} \rangle \qquad \text{with the dipole sum} \qquad \mathfrak{m} := \sum_{i=1}^{\infty} \frac{D_{2i+1}}{2(2i+1)!}.$$

- Curious fact: The Lie bracket with $\mathfrak{m}$+(edge) is a codifferential, too [Khoroshkin, Willwacher, and Živković 2017], *twisted differential*

$$\delta' = \delta + [\cdot, \mathfrak{m}] .$$

(i.e. insert dipole sum instead of just one edge)
The cohomology of GC_3 wrt *this* codifferential δ' (instead of the usual δ) is 1-dimensional, with the only class is a sum of dipoles itself.

Stokes relations

- For Pfaffian-only $I_G = \int_G \phi_G$ (i.e. not wedged with canonical forms ω), have the Stokes relation [Brown, Hu, and Panzer 2024]

$$0 = I_{\partial G} + \frac{1}{2} [I_G, I_G] .$$

- Choose G=triangle with dipole sides, then ∂G= dipole and one obtains recurrence

$$I_{D_{2i+1}} = (I_{D_3})^i .$$

- On the other hand, brackets $\{\cdot\}$ form L_∞ structure. Amounts to quadratic identities

$$\sum_{S \subset G,\ |V_S|=2} \operatorname{sgn}(G,S) \Delta_{G[S]} \times \Delta_{G/S} = 0$$

for their integration domain Δ (the operatope) [Gaiotto, Kulp, and Wu 2024; Budzik et al. 2023]. These are equivalent to the Stokes relations above.

Conclusion

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- There is a certain, "topological", differential form α_G of degree ℓ in Schwinger parameters which computes BRST anomalies in TQFTs.
- There is another, "Pfaffian", differential form, ϕ_G, of degree ℓ which realizes the combinatorial sign of the odd graph complex GC_3 and therefore makes integrals $\int_G \phi_G \wedge \omega_G$ well-defined, where ω_G is a canonical form (which on its own lives on the even graph complex GC_2).
- The two forms are the same.
- This leads to some simplified proofs for α_G, and to a physical interpretation for ϕ_G.
- The sum of dipole/multi-edge graphs plays a special role on both sides.
- Stokes relations have been known, and are important, on both sides.
- On both sides, one is interested in products between this form and some other forms.

Further investigations are currently in progress (with Simone Hu).


Perimeter-A
Thank you!
95%