Title: Lecture - Causal Inference, PHYS 777

Speakers: Robert Spekkens

Collection/Series: Causal Inference (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Quantum Foundations

Date: May 02, 2025 - 11:30 AM

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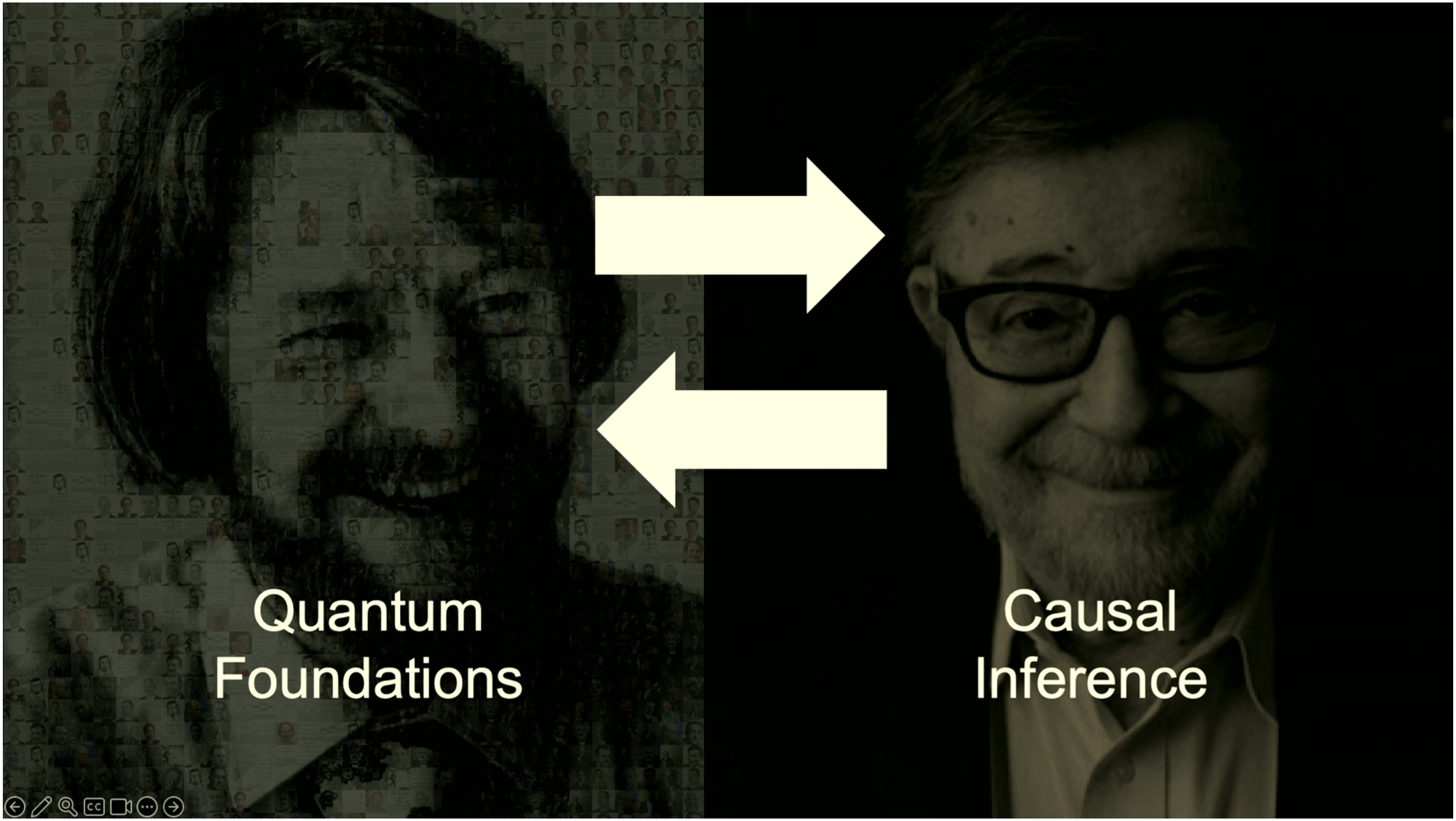
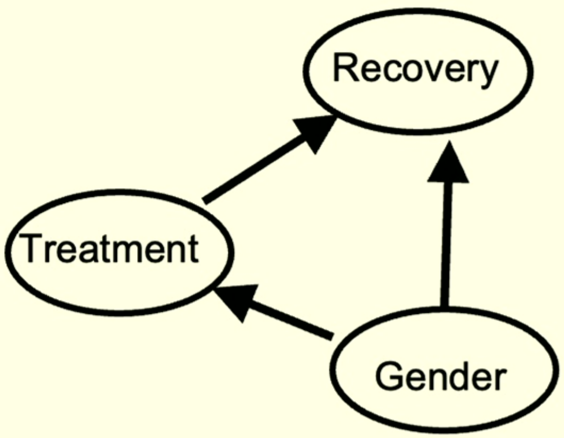

Quantum
Foundations
Causal
Inference

P(recovery | drug) > P(recovery | no drug)

P(recovery | drug, male) < P(recovery | no drug, male) ✓

P(recovery | drug, female) < P(recovery | no drug, female) ✓

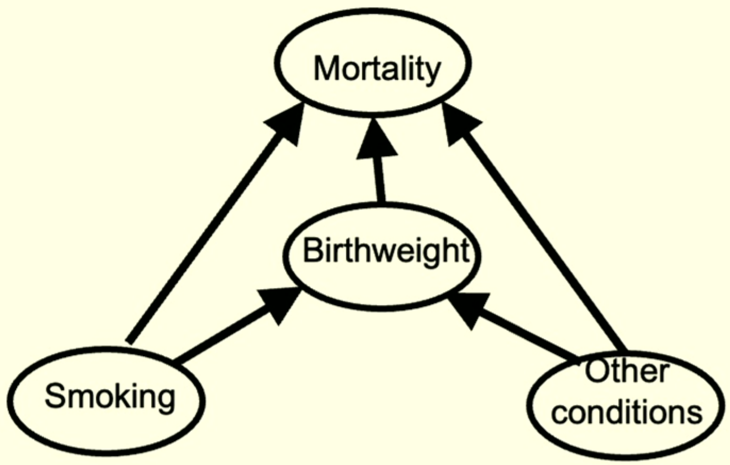


Therefore: stratify the data by the common cause

Birth weight paradox

P(mortality | born to smoker) > P(mortality | born to nonsmoker) ✓

P(mortality | born to smoker, LBW) < P(mortality | born to nonsmoker, LBW)

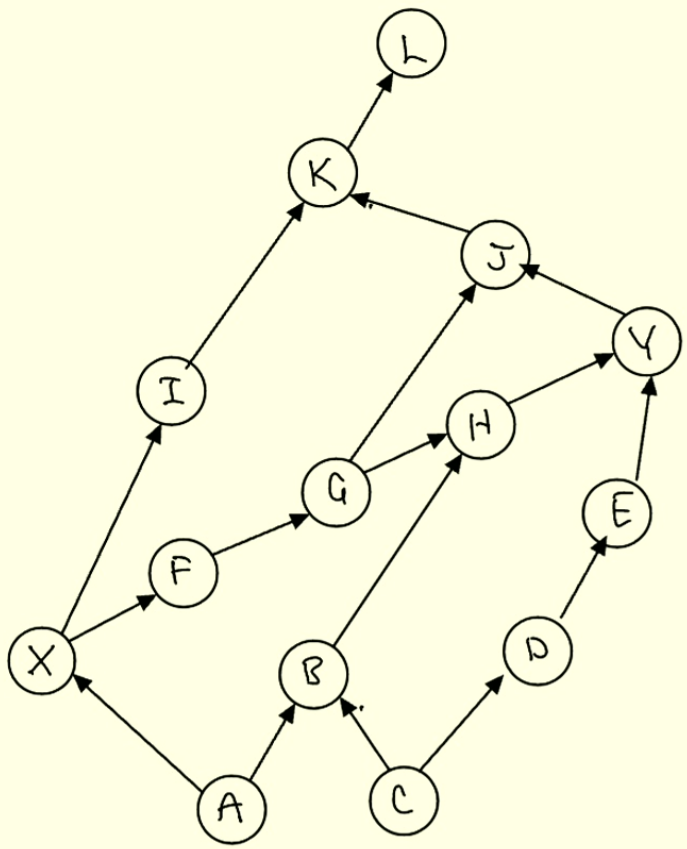


Therefore: *marginalize* over colliders on the "backdoor path"

Definition (path blocking) A path between node X and node Y is blocked by a set of vertices **Z** if at least one of the following conditions holds:

1. The path contains a chain whose mediary node is in **Z**
2. The path contains a fork whose tail node is in **Z**

2. The path contains a collider whose head node is **not** in **Z** and no descendant of which is in **Z**.


L
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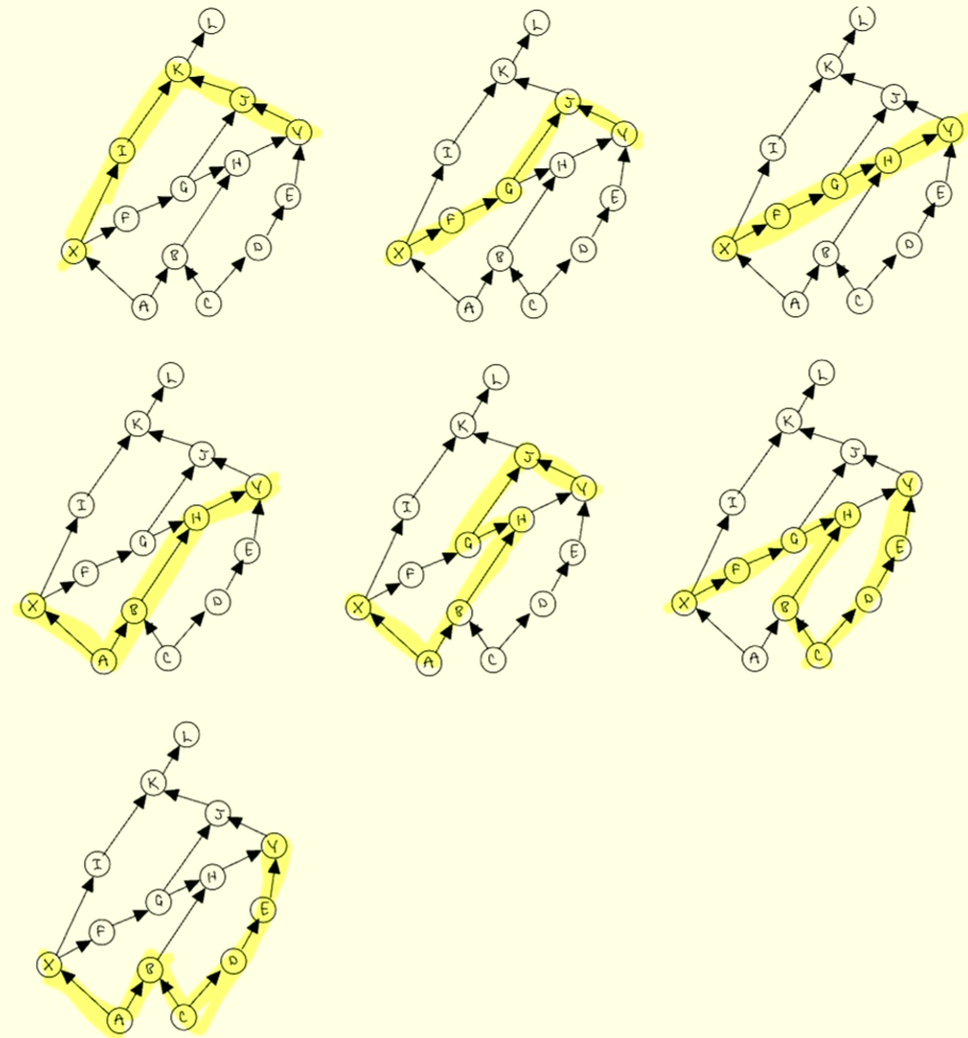
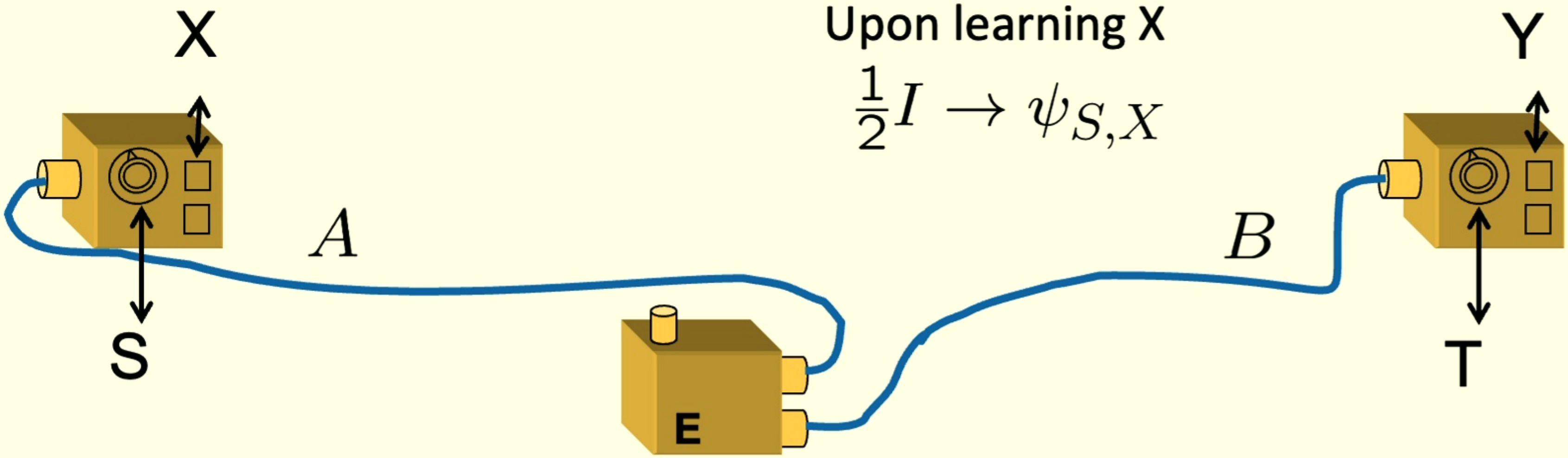
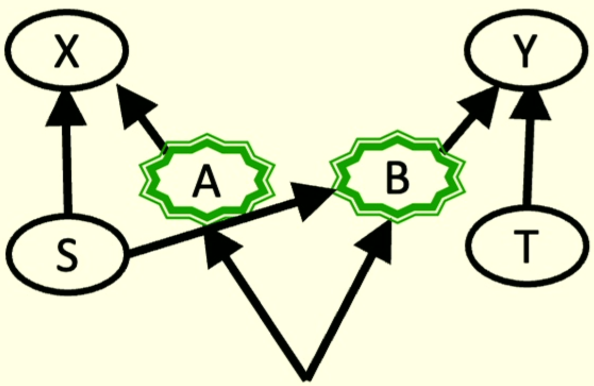


Puzzles in quantum theory and their causal resolution

Upon learning X

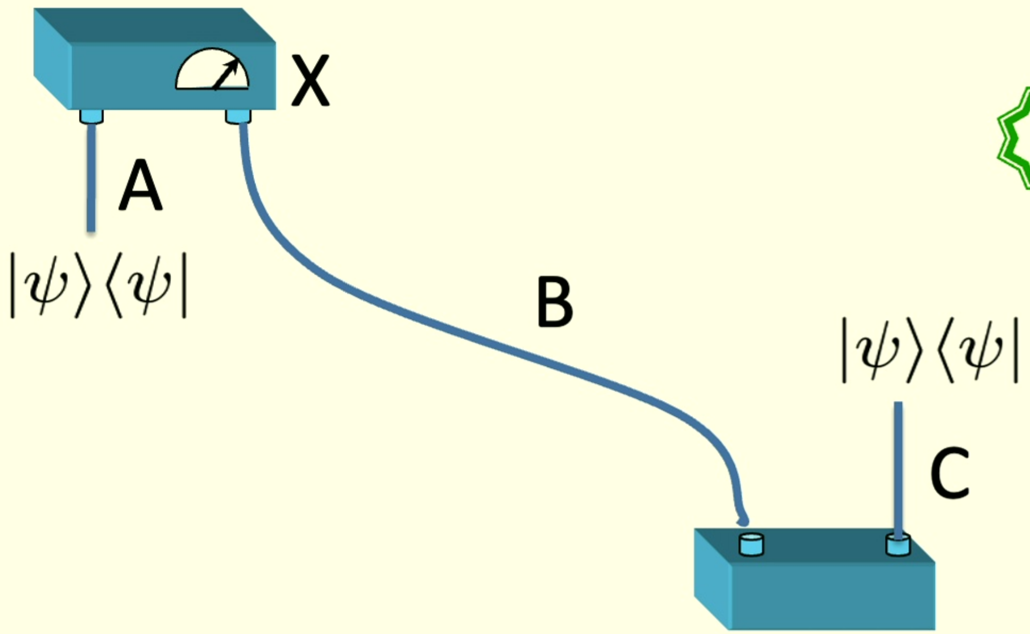
$$\frac{1}{2}I \rightarrow \psi_{S,X}$$



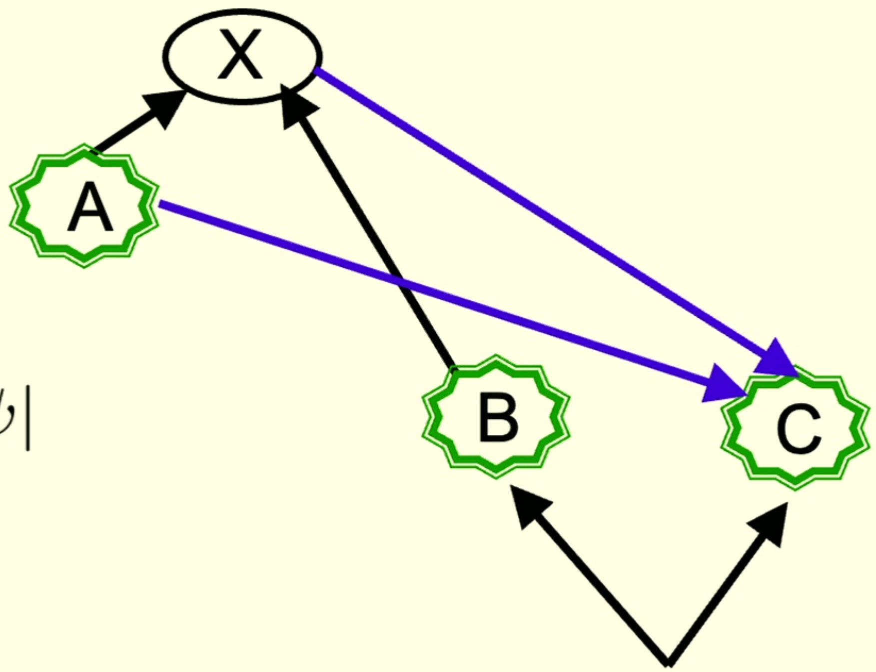


Like “treatment influences recovery”

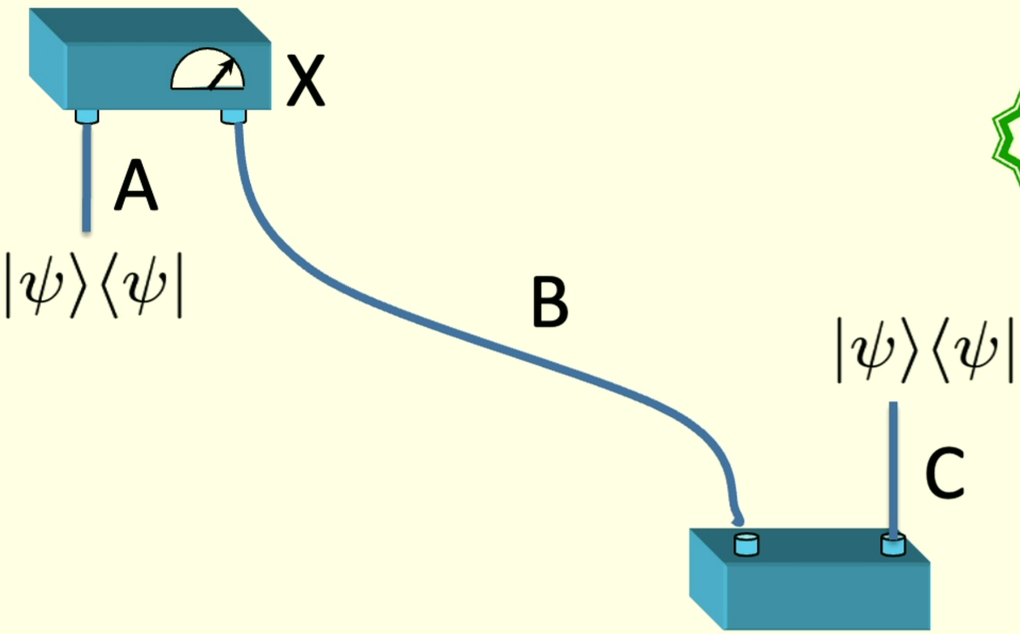
Post-select on outcome



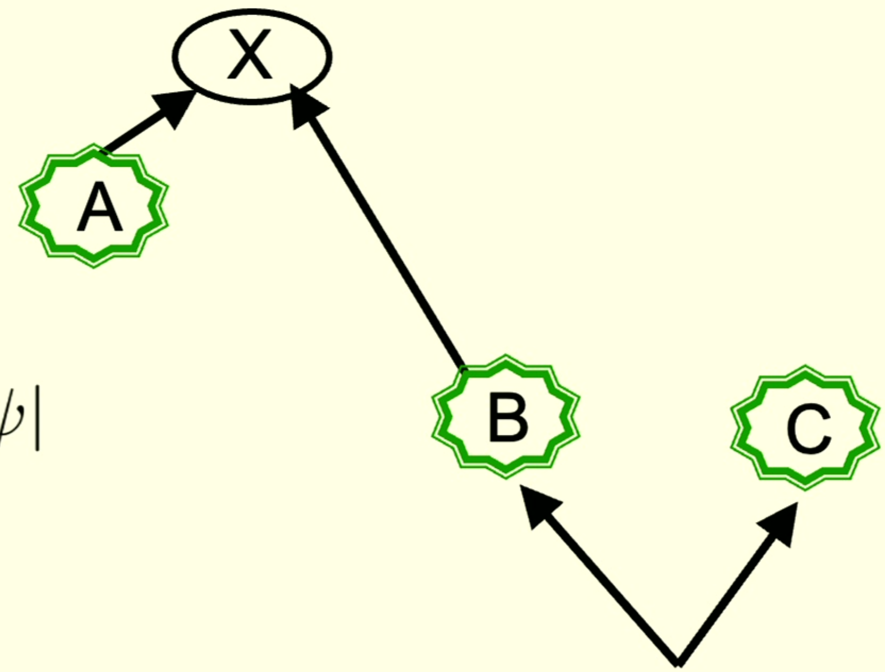
ψ is ontic



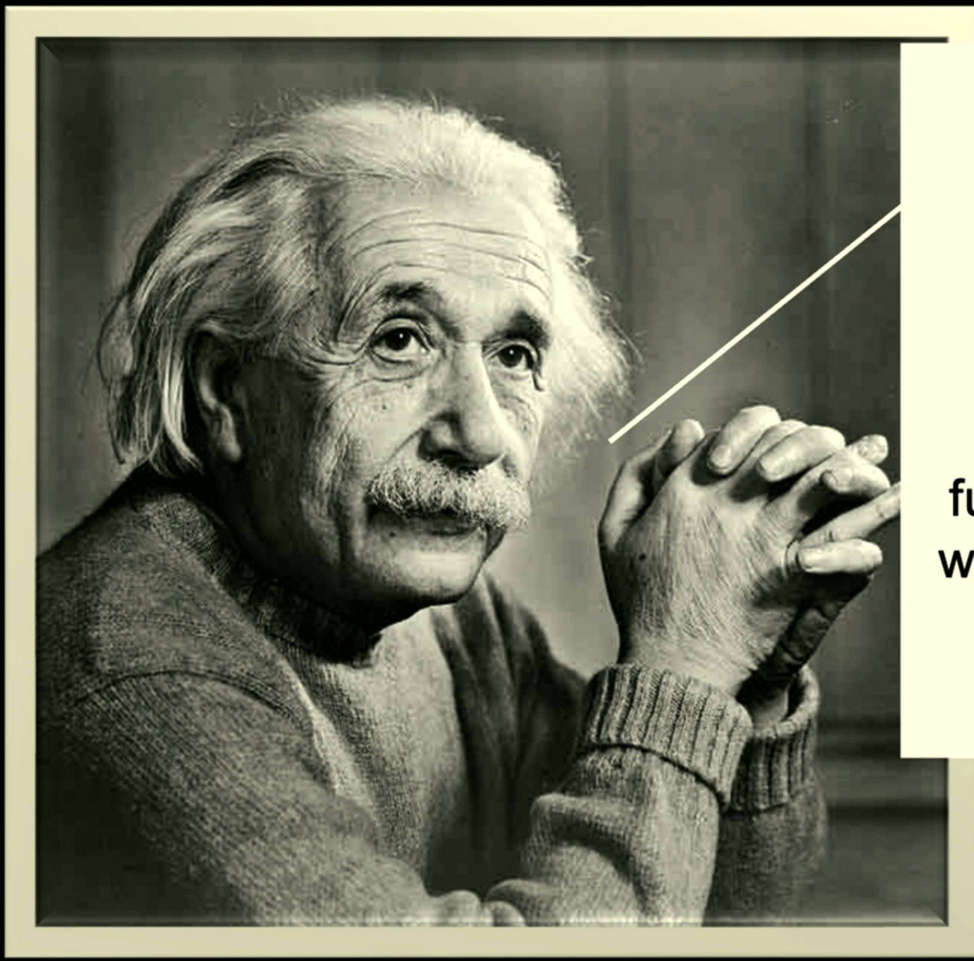
Post-select on outcome



ψ is epistemic



Given post-selection, your posterior about C tracks your prior about A



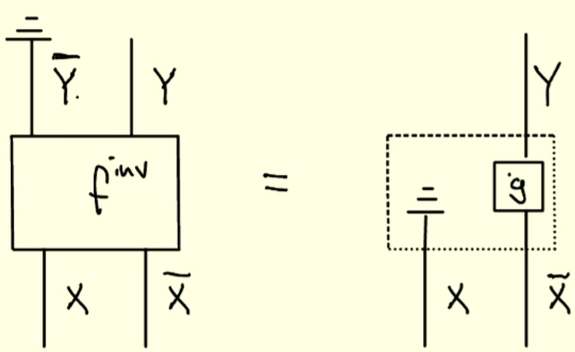
"ψ_2 does not describe the totality of what "really" pertains to the partial system 2, rather only what we know about it in this particular case."

"I incline to the opinion that the wave function does not (completely) describe what is real, but only a to-us-empirically-accessible maximal knowledge regarding that which really exists."

Classical

variable X has no influence on variable Y if Y has a trivial dependence on X

for an invertible function f^{inv}

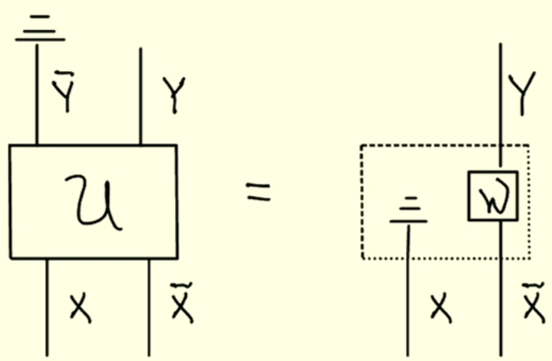


$$f^{\mathrm{inv}}|_Y(X, \bar{X}) = g(\bar{X})$$

Quantum

system X has no influence on system Y if Y has a trivial dependence on X

for a unitary channel U



$$\mathrm{Tr}_{\bar{Y}} \circ \mathcal{U}_{\bar{Y}Y|X\bar{X}} = \mathcal{W}_{Y|\bar{X}} \otimes \mathrm{Tr}_X$$

A circuit decomposition is **causally faithful** if influences are given by structure of diagram

It is an open question whether every multipartite unitary channel admits a causally faithful decomposition

Known to be true for:
3 inputs and N outputs
N inputs and 3 outputs
Some other special cases

Classical

$$P_B = \Gamma_{B|A}[P_A]$$

Stochastic map preserving positivity and normalization

In terms of a conditional

$$P_B = \sum_A P_{B|A} P_A$$

$$\sum_B P_{B|A} = 1$$

$$P_{B|A} \geq 0$$

Quantum

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

Completely positive trace-preserving map

$\mathcal{E}_{B|A} \in \mathrm{CP}$ (map is CP)

In terms of a conditional

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$$\mathrm{Tr}_B(\rho_{B|A}) = I_A$$

$\rho_{B|A}^{T_A} \geq 0$ (conditional is PPT)

	Classical	Quantum
Relation of conditional to joint	$P_{B\|A} = \frac{P_{AB}}{P_A}$	$\rho_{B\|A} = \rho_A^{-1/2} \rho_{AB} \rho_A^{-1/2}$ $\rho_{B\|A} \geq 0 \quad \text{(positive)}$
	$P_{AB} = P_{B\|A} P_A$	$\rho_{AB} = \rho_A^{1/2} \rho_{B\|A} \rho_A^{1/2}$
Normalization condition	$\sum_B P_{B\|A} = 1$	$\mathrm{Tr}_B(\rho_{B\|A}) = I_A$
Belief propagation	$P_B = \sum_A P_{B\|A} P_A$ $P_B = \Gamma_{B\|A}[P_A]$	$\rho_B = \mathrm{Tr}_A(\rho_{B\|A} \rho_A)$ $\rho_B = \mathcal{E}_{B\|A}(\rho_A)$ $\mathcal{E}_{B\|A} \circ T_A \in \mathrm{CP} \quad \text{(co-CP)}$
Bayesian inversion	$P_{B\|A} = \frac{P_{A\|B} P_B}{P_A}$	$\rho_{B\|A} = \rho_A^{-1/2} \rho_B^{1/2} \rho_{A\|B} \rho_B^{1/2} \rho_A^{-1/2}$

Still unclear how to do this in general

Problems with various proposals are highlighted here:
Horsman et al., Proc. R. Soc. A **473**, 20170395 (2017)

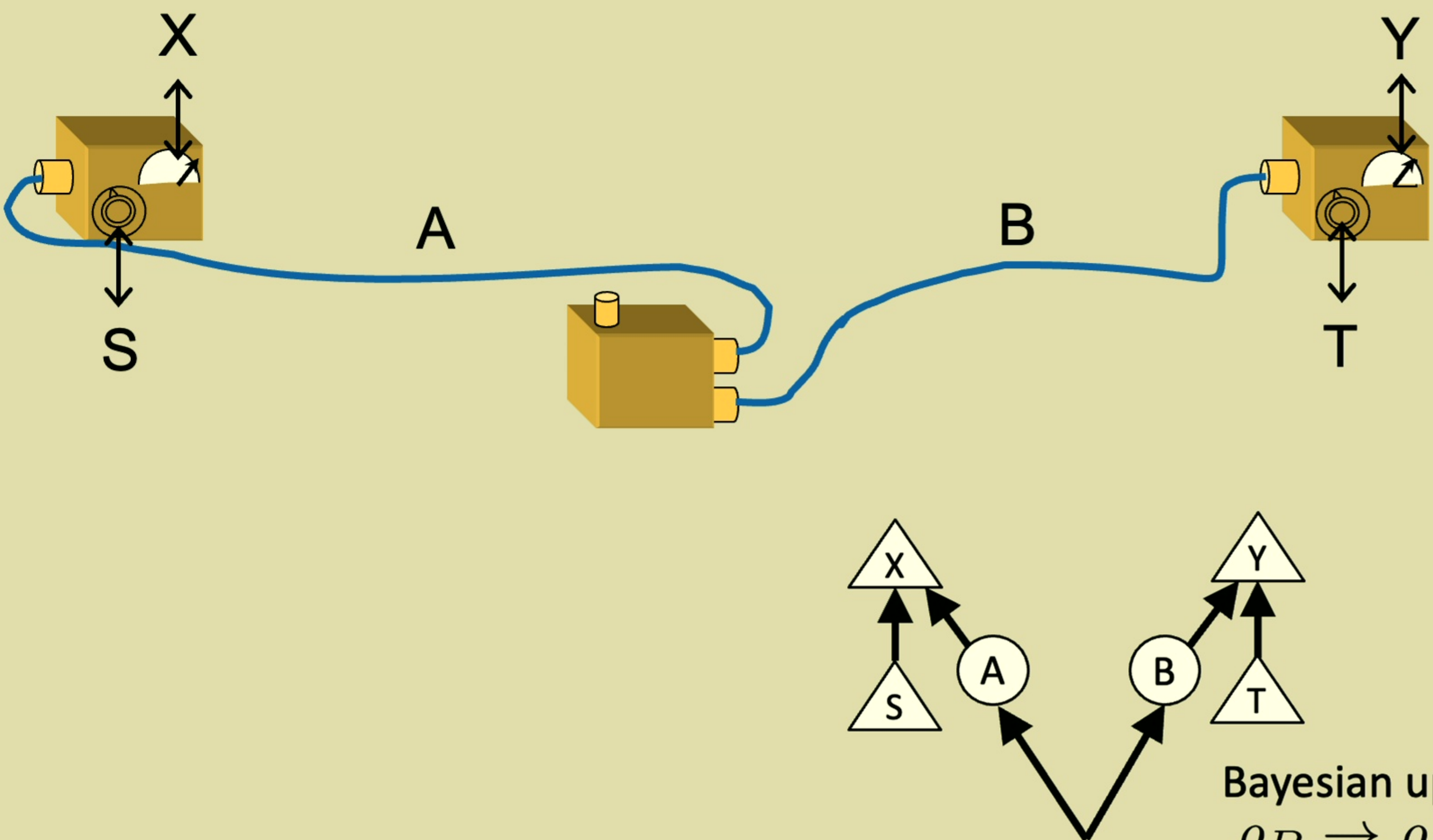
Quantum
Causation and Inference

Classical
Causation and Inference

≈

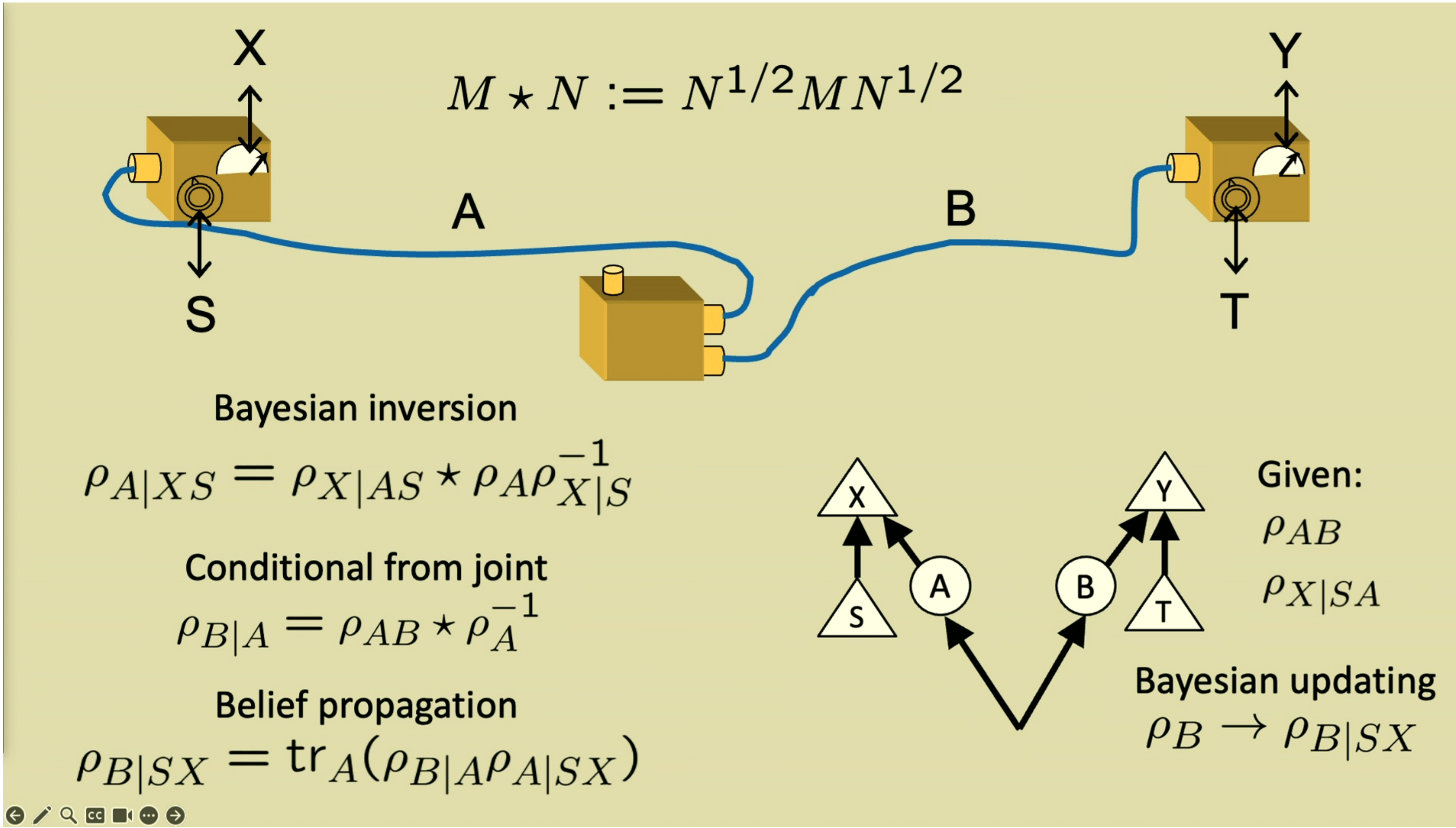
Relativistic
Notions of Space
and Time

PreRelativistic
Notions of Space
and Time

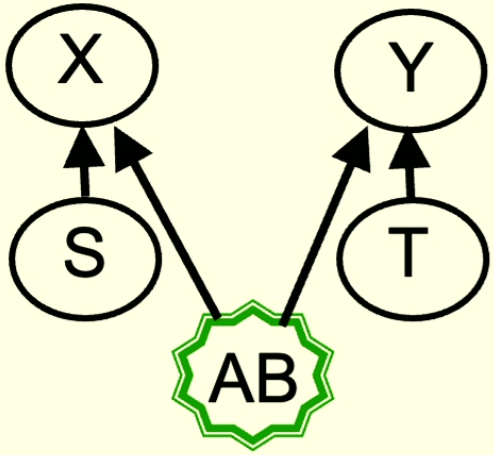


Bayesian updating

$$\rho_B \to \rho_{B|SX}$$


$M \star N := N^{1/2} M N^{1/2}$
X
S
A
B
Y
T
Bayesian inversion
$\rho_{A|XS} = \rho_{X|AS} \star \rho_A \rho_{X|S}^{-1}$
Conditional from joint
$\rho_{B|A} = \rho_{AB} \star \rho_A^{-1}$
Belief propagation
$\rho_{B|SX} = \mathrm{tr}_A(\rho_{B|A}\rho_{A|SX})$
X
S
A
B
Y
T
Given:
ρ_{AB}
$\rho_{X|SA}$
Bayesian updating
$\rho_B \rightarrow \rho_{B|SX}$

Causal structure



Parameters

$\rho_{X|SA}$

$\rho_{Y|TB}$

ρ_{AB}

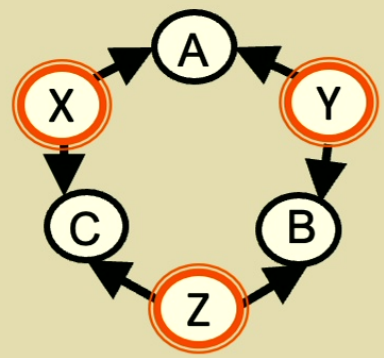
$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Causal compatibility constraints:

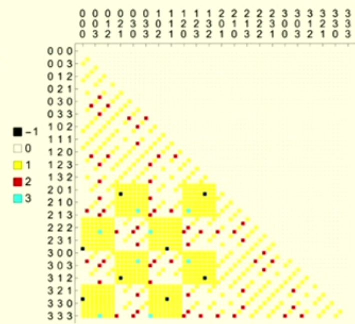
$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \underline{0.85}$$

Tsirelson, Lett. Math. Phys. 4, 93 (1980)



$$\sum_{a,b,c,a',b',c'} y_{abca'b'c'} P_{ABC}(abc) P_{ABC}(a'b'c') \geq 0$$

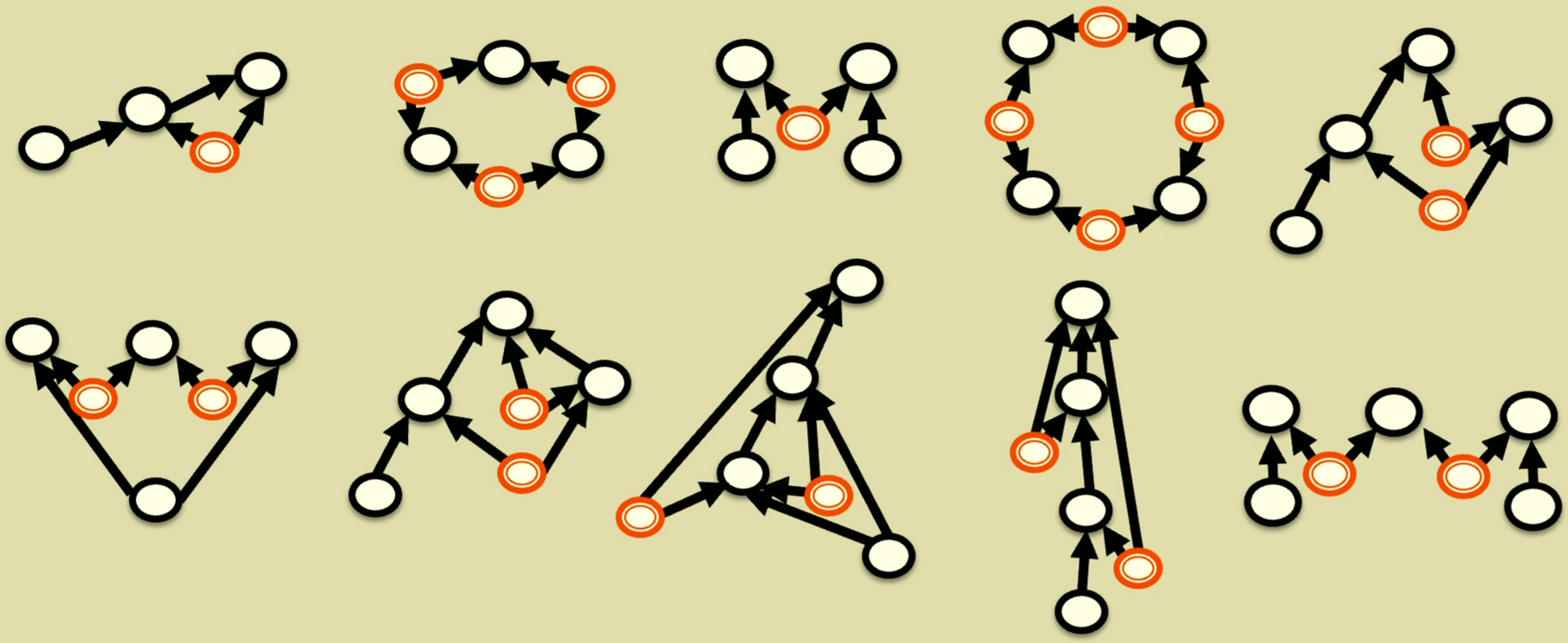


(b) The coefficients of a quadratic inequality

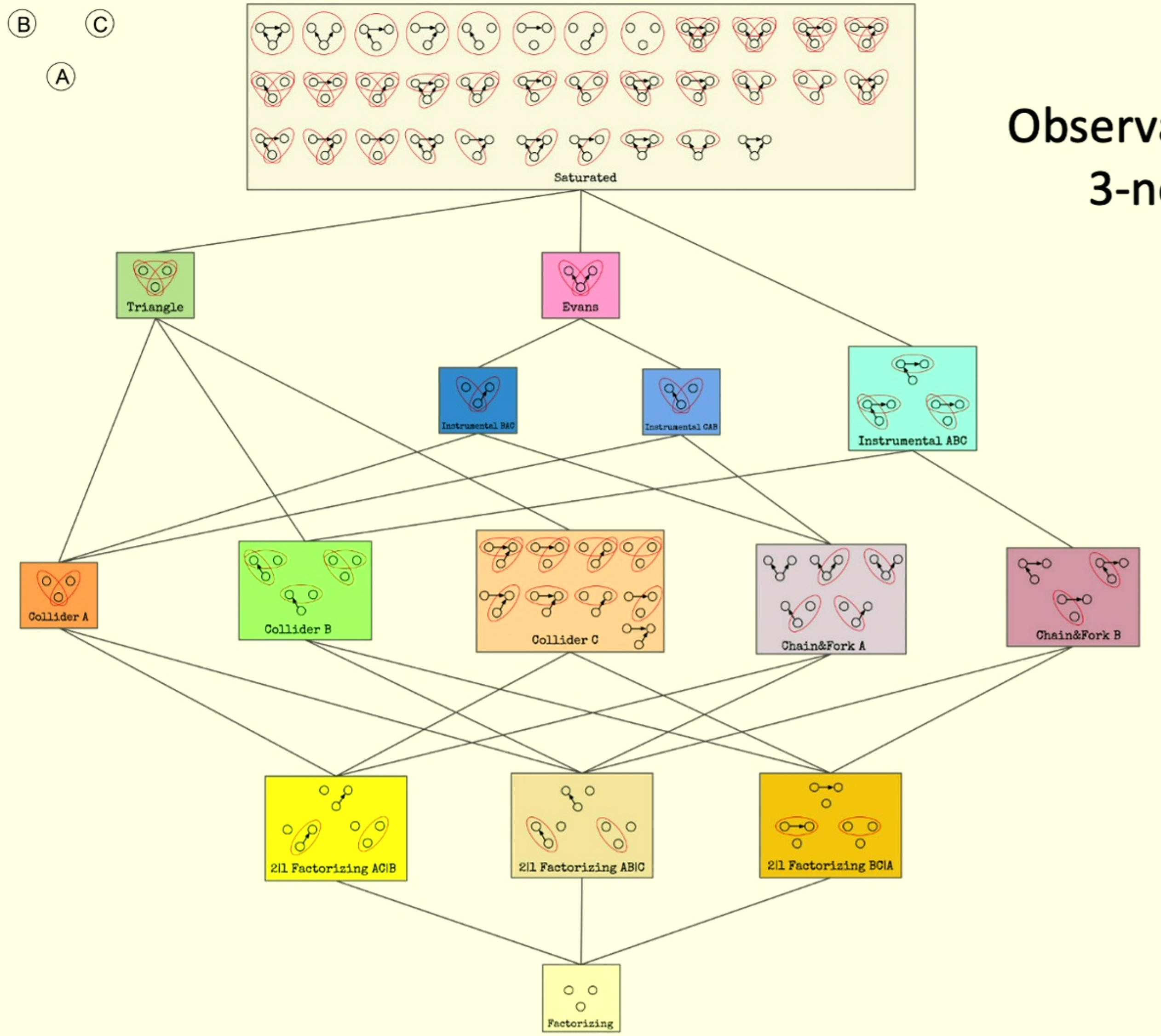
Polino et al., Nat. Comm. 14, 909 (2023)

This inequality was obtained from the web inflation, which is fanout, and therefore might be quantumly violated

Some causal structures that admit of inequality constraints



Observational order of 3-node mDAGs



Fraction of classes that are nonalgebraic

2-node mDAGs: 0%

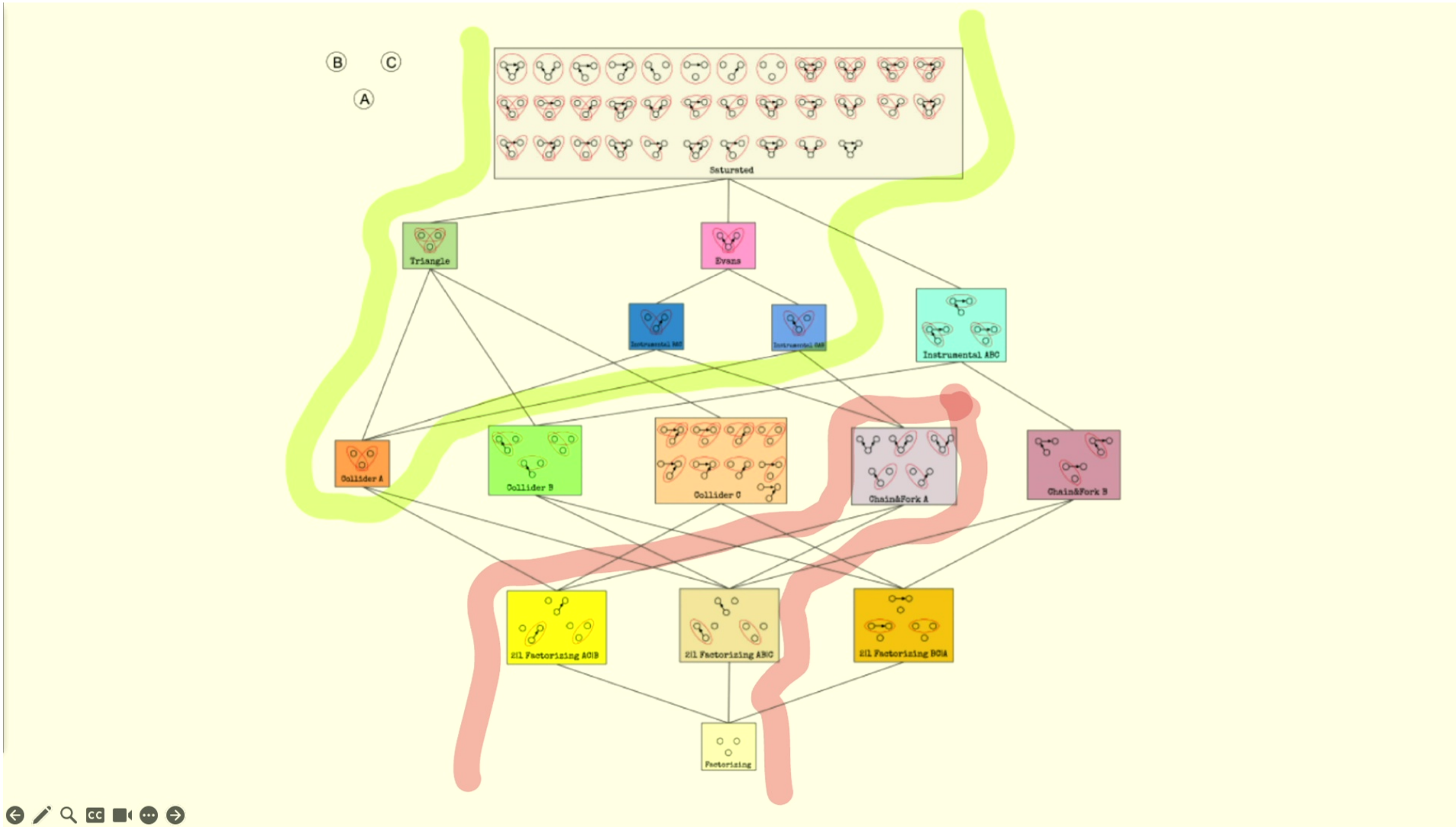
3-node mDAGs: 33%

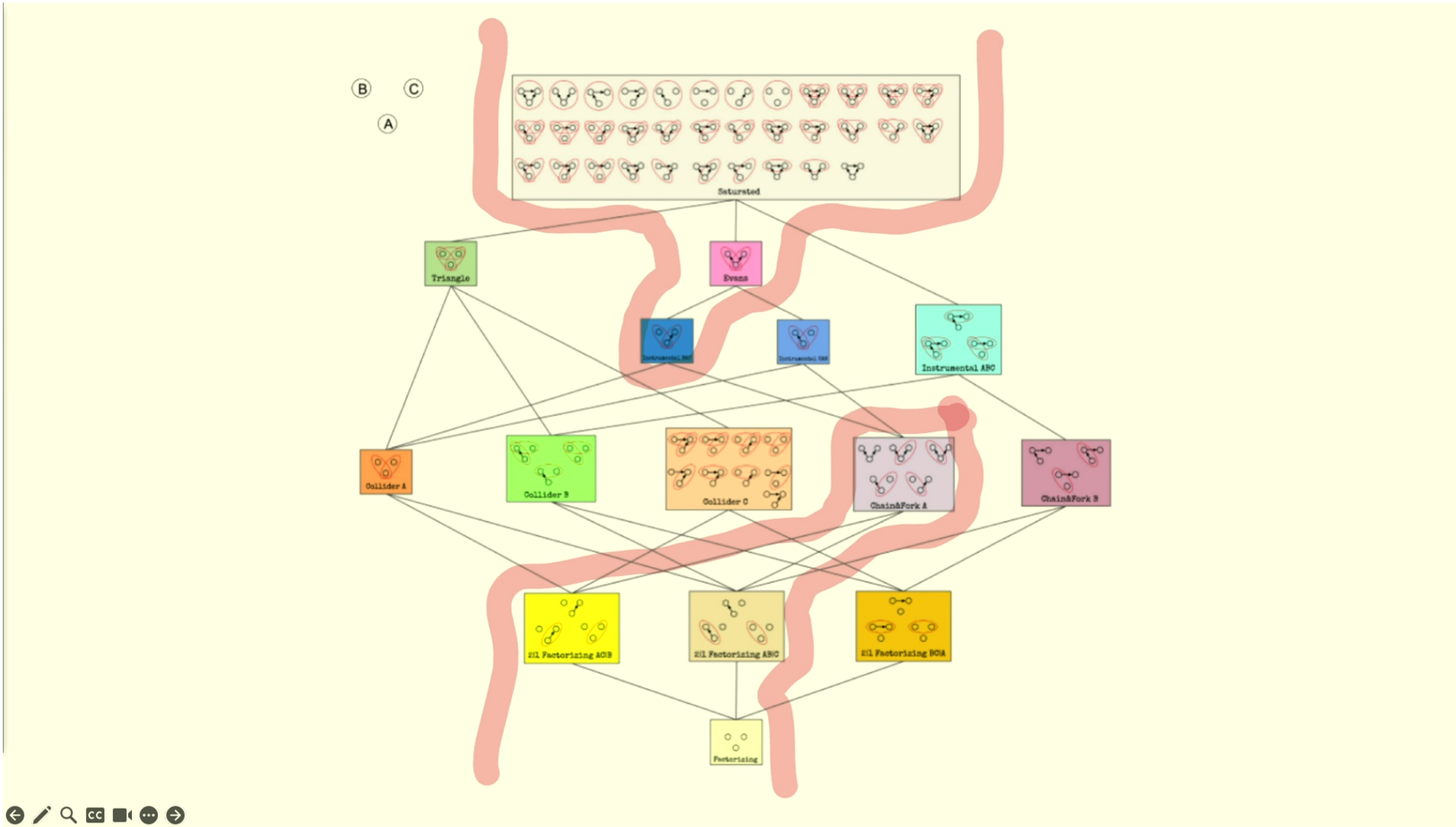
3-node mDAGs: >85.2%

Could inequality constraints still be unimportant relative to conditional independence relations?
For instance, could most equivalence classes still be singled out by their set of conditional independence constraints alone?

The fraction of equivalence classes can be identified by d-separation relations alone:

For 2-node mDAGs: 2 of 2, i.e., 100%
For 3-node mDAGs: 9 of 15, i.e., 60%
For 4-node mDAGs: 115 of N>1253, i.e., less than 10%


B
C
A
Saturated
Triangle
Evans
Instrumental ABC
Collider A
Collider B
Collider C
Chain&Fork A
Chain&Fork B
2|1 Factorizing AC|B
2|1 Factorizing AB|C
2|1 Factorizing BC|A
Factorizing


B
C
A
Saturated
Triangle
Evans
Instrumental ABC
Collider A
Collider B
Collider C
Chain&Fork A
Chain&Fork B
2|1 Factorizing AC|B
2|1 Factorizing AB|C
2|1 Factorizing BC|A
Factorizing

Estimating causal effects

The evidence

Violates the independence constraint:

$$P_{Y|X} \neq P_Y$$

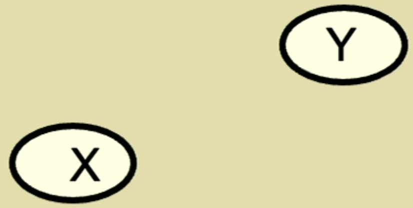
$$P_{Y|X}(1|1) \neq P_{Y|X}(1|0)$$

$$P_{F_{X\to Y}}(\mathbb{I}) \geq \max\{0, P_{Y|X}(1|1) - P_{Y|X}(1|0)\}$$

$$P_{F_{X\to Y}}(\mathbb{F}) \geq \max\{0, P_{Y|X}(1|0) - P_{Y|X}(1|1)\}$$

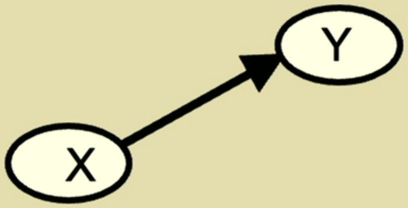
$$P_{F_{X\to Y}}(\texttt{1-bit}) \geq |P_{Y|X}(1|1) - P_{Y|X}(1|0)|$$

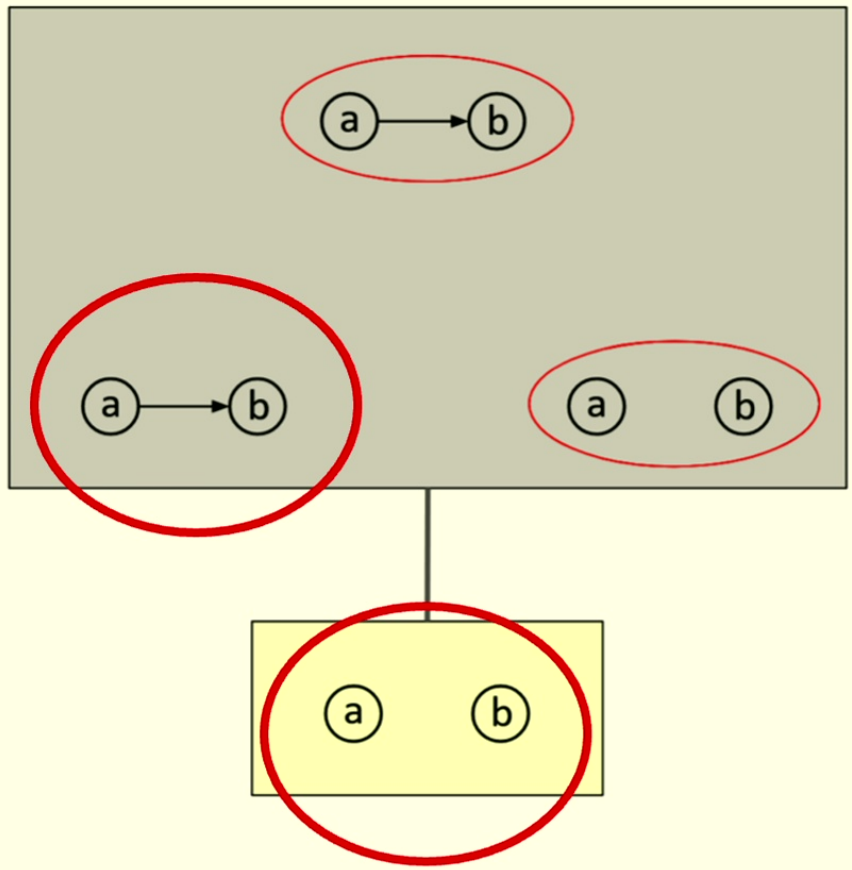
The hypotheses



Implies an independence constraint:

$$P_{Y|X} = P_Y$$




a
b
a
b
a
b
a
b

The evidence

Violates the independence constraint:

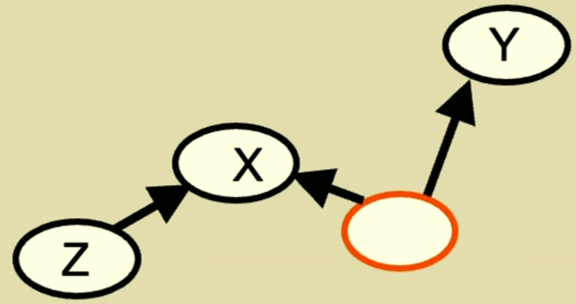
$$P_{Y|Z} \neq P_Y$$

$$P_{Y|Z}(1|1) \neq P_{Y|Z}(1|0)$$

$$P_{F_{X\to Y}}(\texttt{1-bit}) \geq |P_{Y|Z}(1|1) - P_{Y|Z}(1|0)|$$

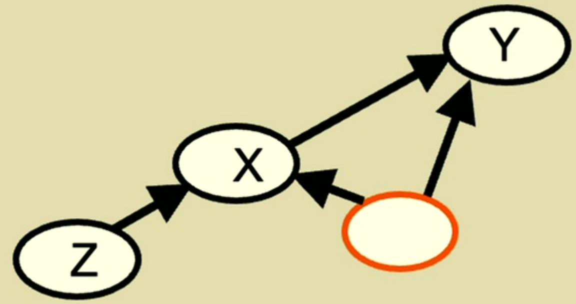
$$P_{F_{Z\to X}}(\texttt{1-bit}) \geq |P_{Y|Z}(1|1) - P_{Y|Z}(1|0)|$$

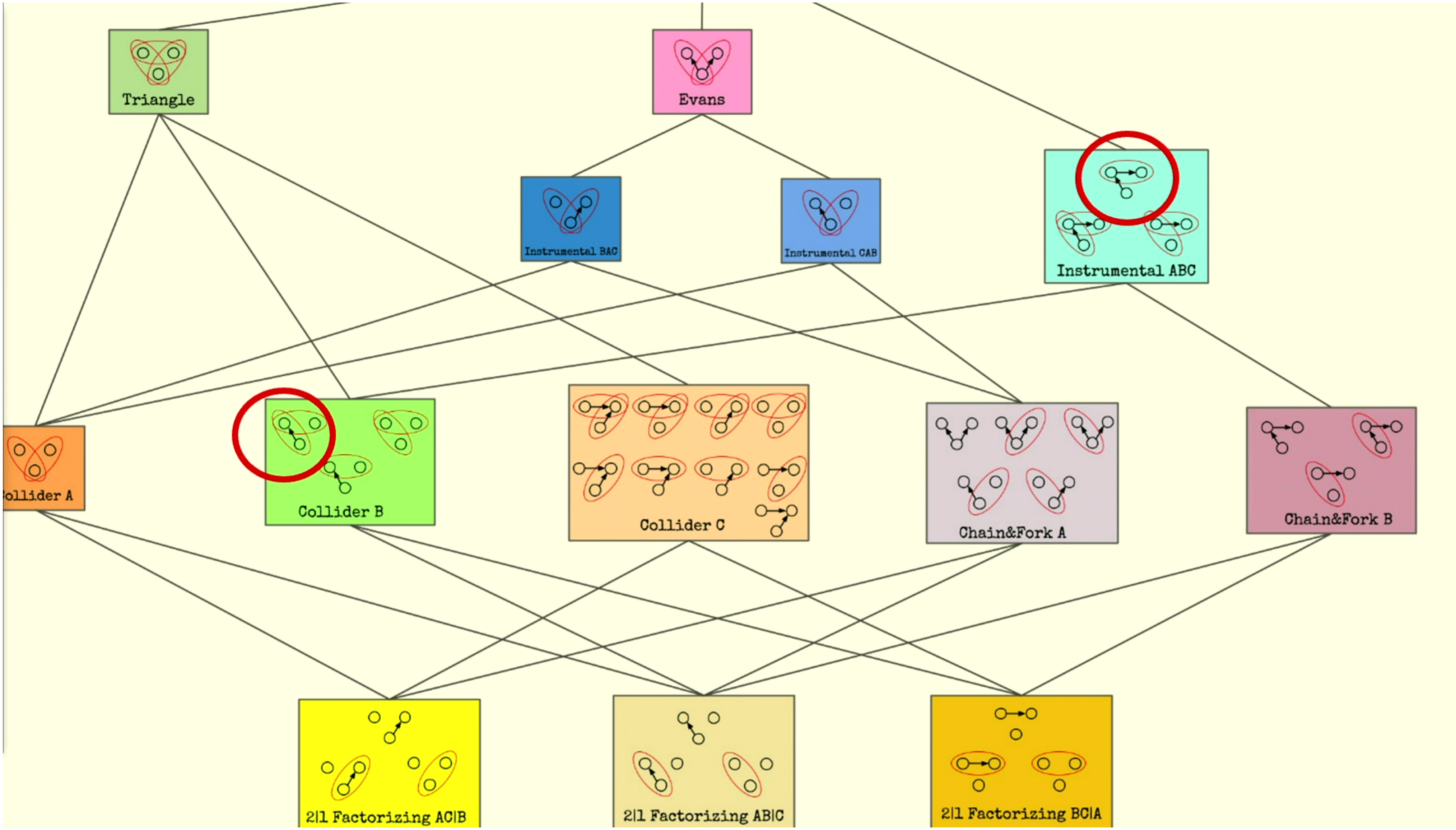
The hypotheses



Implies an independence constraint:

$$P_{Y|Z} = P_Y$$





Triangle
Evans
Instrumental BAC
Instrumental CAB
Instrumental ABC
ollider A
Collider B
Collider C
Chain&Fork A
Chain&Fork B
2|1 Factorizing AC|B
2|1 Factorizing AB|C
2|1 Factorizing BC|A

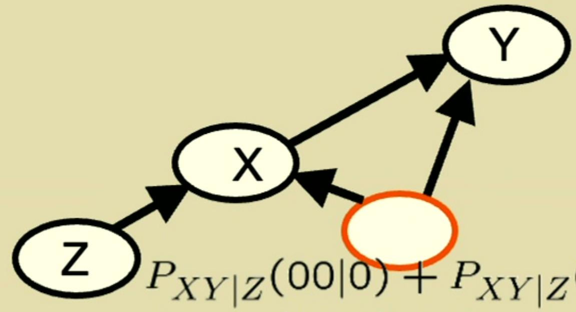
The evidence

Violates one or more inequality constraint

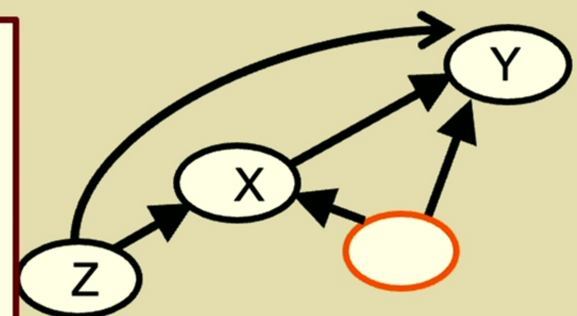
$$P_{XY|Z}(00|0) + P_{XY|Z}(01|1) - 1 \geq 0$$
$$P_{XY|Z}(01|0) + P_{XY|Z}(00|1) - 1 \geq 0$$
$$P_{XY|Z}(10|0) + P_{XY|Z}(11|1) - 1 \geq 0$$
$$P_{XY|Z}(10|0) + P_{XY|Z}(11|1) - 1 \geq 0$$

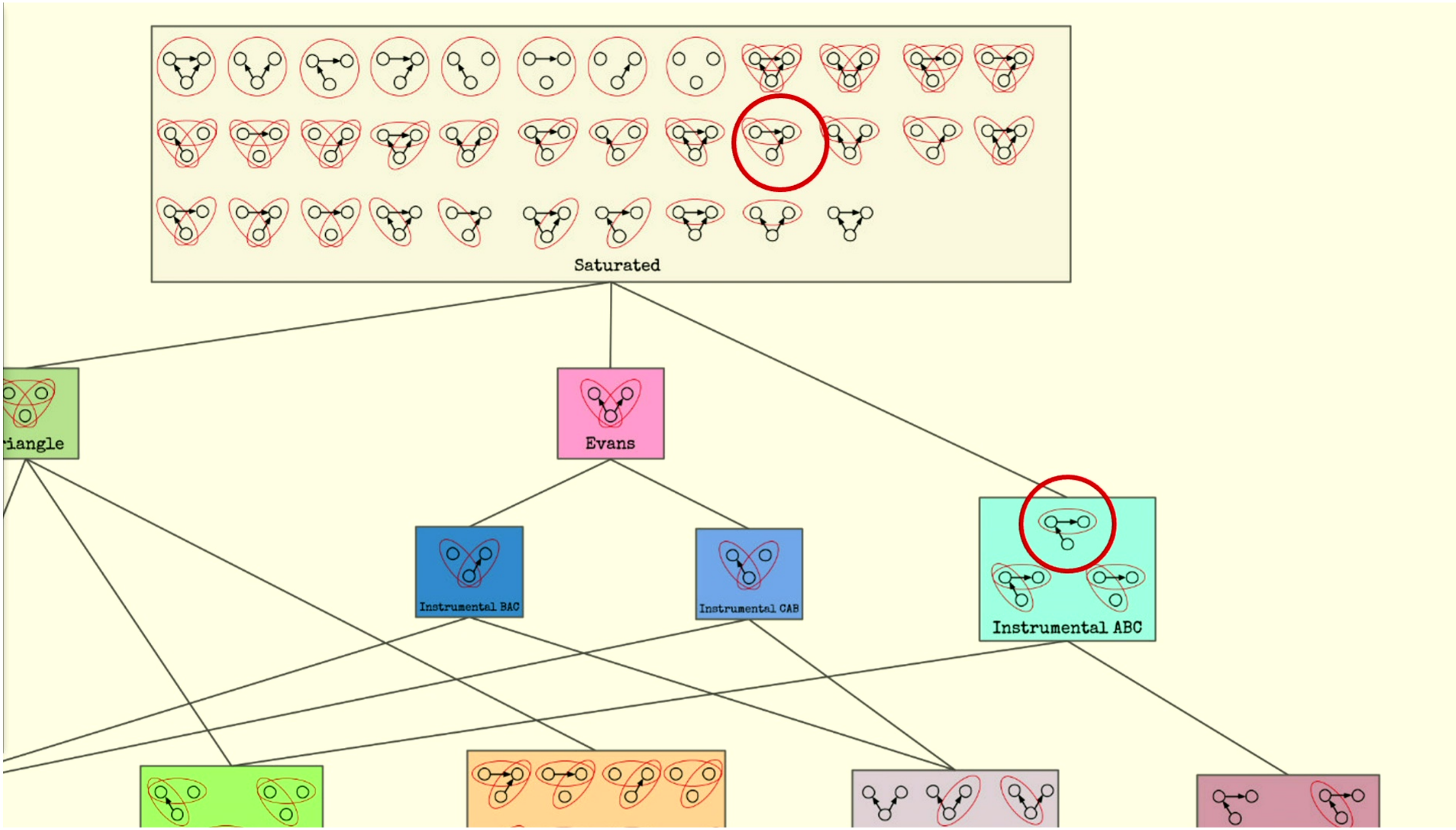
$$P_{F^{X=0}_{Z\to Y}}(\mathbb{I}) \geq \max\{0, P_{XY|Z}(00|0) + P_{XY|Z}(01|1) - 1\}$$
$$P_{F^{X=0}_{Z\to Y}}(\mathbb{F}) \geq \max\{0, P_{XY|Z}(01|0) + P_{XY|Z}(00|1) - 1\}$$
$$P_{F^{X=1}_{Z\to Y}}(\mathbb{I}) \geq \max\{0, P_{XY|Z}(10|0) + P_{XY|Z}(11|1) - 1\}$$
$$P_{F^{X=1}_{Z\to Y}}(\mathbb{F}) \geq \max\{0, P_{XY|Z}(11|0) + P_{XY|Z}(10|1) - 1\}$$

The hypotheses



$$P_{XY|Z}(00|0) + P_{XY|Z}(01|1) \leq 1$$
$$P_{XY|Z}(01|0) + P_{XY|Z}(00|1) \leq 1$$
$$P_{XY|Z}(10|0) + P_{XY|Z}(11|1) \leq 1$$
$$P_{XY|Z}(11|0) + P_{XY|Z}(10|1) \leq 1$$

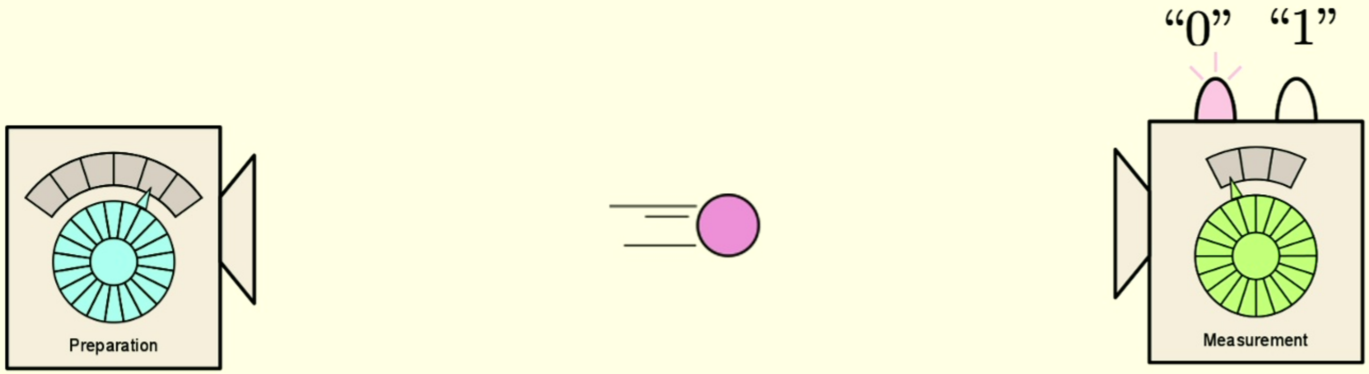




Saturated
iangle
Evans
Instrumental BAC
Instrumental CAB
Instrumental ABC

GPT-latent-permitting causal models (Generalized Probabilistic Theories)

Henson, Lal and Pusey, New J. Phys. 16, 113043 (2014)
Fritz, Comm. Math. Phys. 341, 391 (2016)



$P_1, \ldots, P_m$

$M_1, \ldots, M_n$

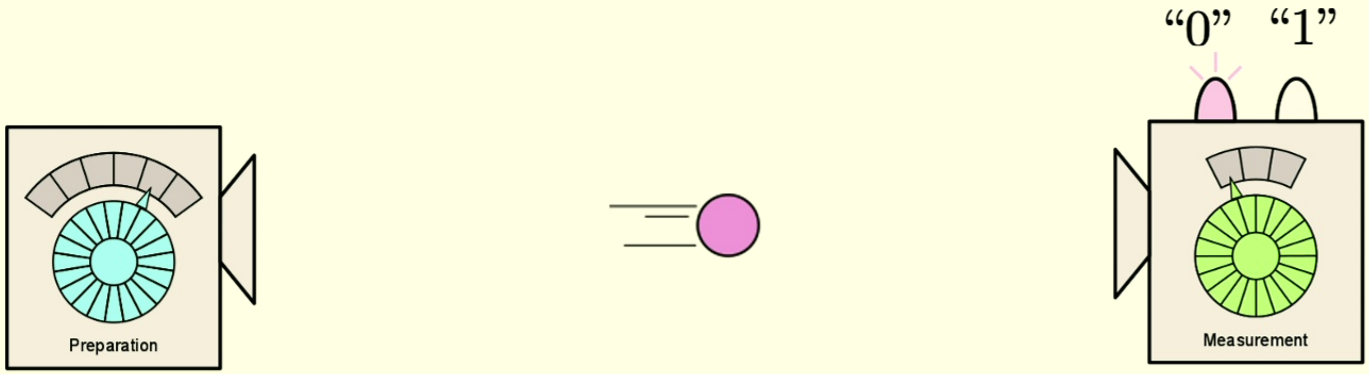
$$\begin{pmatrix} 1 & p(0|P_1, M_2) & p(0|P_1, M_3) & p(0|P_1, M_4) & p(0|P_1, M_5) & \cdots \\ 1 & p(0|P_2, M_2) & p(0|P_2, M_3) & p(0|P_2, M_4) & p(0|P_2, M_5) & \cdots \\ 1 & p(0|P_3, M_2) & p(0|P_3, M_3) & p(0|P_3, M_4) & p(0|P_3, M_5) & \cdots \\ 1 & p(0|P_4, M_2) & p(0|P_4, M_3) & p(0|P_4, M_4) & p(0|P_4, M_5) & \cdots \\ 1 & p(0|P_5, M_2) & p(0|P_5, M_3) & p(0|P_5, M_4) & p(0|P_5, M_5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$
\begin{pmatrix}
1 & s_1^{(1)} & \cdots & s_{k-1}^{(1)} \\
1 & s_1^{(2)} & \cdots & s_{k-1}^{(2)} \\
1 & s_1^{(3)} & \cdots & s_{k-1}^{(3)} \\
1 & s_1^{(4)} & \cdots & s_{k-1}^{(4)} \\
1 & s_1^{(5)} & \cdots & s_{k-1}^{(5)} \\
\vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\begin{pmatrix}
1 & e_0^{(2,0)} & e_0^{(3,0)} & e_0^{(4,0)} & e_0^{(5,0)} & \cdots \\
0 & e_1^{(2,0)} & e_1^{(3,0)} & e_1^{(4,0)} & e_1^{(5,0)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
0 & e_{k-1}^{(2,0)} & e_{k-1}^{(3,0)} & e_{k-1}^{(4,0)} & e_{k-1}^{(5,0)} & \cdots
\end{pmatrix}
$$

$$
p\left(0 \middle| P_i, M_j\right) = \left(1, s_1^{(i)}, \ldots, s_{k-1}^{(i)}\right) \cdot \left(e_0^{(j,0)}, \ldots, e_{k-1}^{(j,0)}\right) = \mathbf{s}^{(i)} \cdot \mathbf{e}^{(j,0)}
$$

GPT state

GPT effect



$P_1, \ldots, P_m$ $\qquad$ $M_1, \ldots, M_n$

$$
\begin{pmatrix}
1 & s_1^{(1)} & \cdots & s_{k-1}^{(1)} \\
1 & s_1^{(2)} & \cdots & s_{k-1}^{(2)} \\
1 & s_1^{(3)} & \cdots & s_{k-1}^{(3)} \\
1 & s_1^{(4)} & \cdots & s_{k-1}^{(4)} \\
1 & s_1^{(5)} & \cdots & s_{k-1}^{(5)} \\
\vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\begin{pmatrix}
1 & e_0^{(2,0)} & e_0^{(3,0)} & e_0^{(4,0)} & e_0^{(5,0)} & \cdots \\
0 & e_1^{(2,0)} & e_1^{(3,0)} & e_1^{(4,0)} & e_1^{(5,0)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
0 & e_{k-1}^{(2,0)} & e_{k-1}^{(3,0)} & e_{k-1}^{(4,0)} & e_{k-1}^{(5,0)} & \cdots
\end{pmatrix}
$$

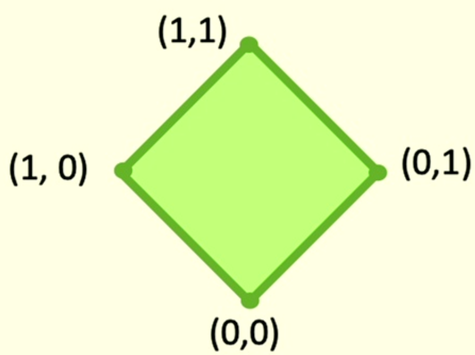
2-level classical system (bit)

k=2

$$p(0) = p(b=0)p(0|b=0) + p(b=1)p(0|b=1)$$

$$(p(b=0), p(b=1))$$

$$(p(0|b=0), p(0|b=1))$$



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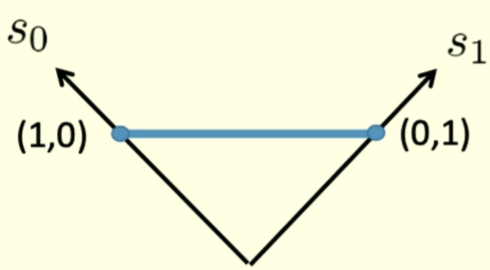
2-level classical system (bit)

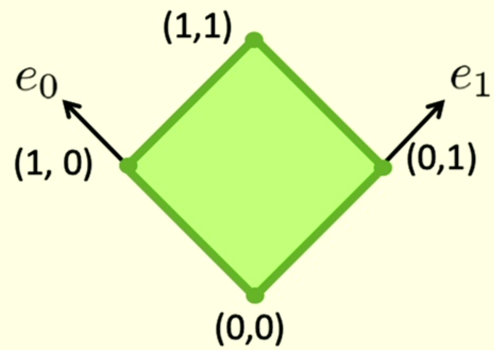
k=2

$$p(0) = p(b=0)p(0|b=0) + p(b=1)p(0|b=1)$$
$$= s_0 e_0 + s_1 e_1$$

$$\mathbf{s} = (s_0, s_1)$$

$$\mathbf{e} = (e_0, e_1)$$





$$s_0 = p(b=0)$$
$$s_1 = p(b=1)$$

$$e_0 = p(0|b=0)$$
$$e_1 = p(0|b=1)$$

2-level classical system (bit)

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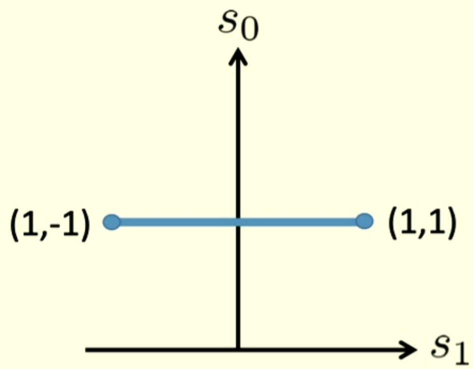
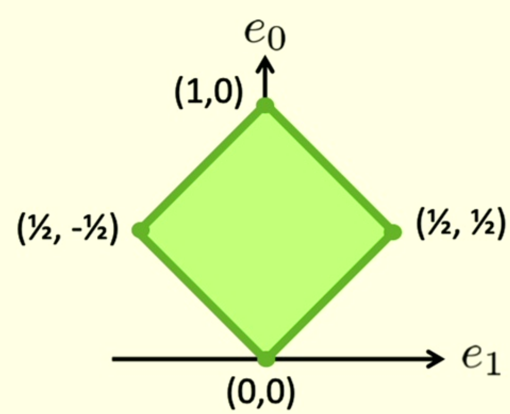


$$\mathbf{e} = (e_0, e_1)$$



$$s_1 = p(b=0) - p(b=1)$$

$$e_0 = \frac{1}{2}(p(0|b=1) + p(0|b=0))$$
$$e_1 = \frac{1}{2}(p(0|b=1) - p(0|b=0))$$

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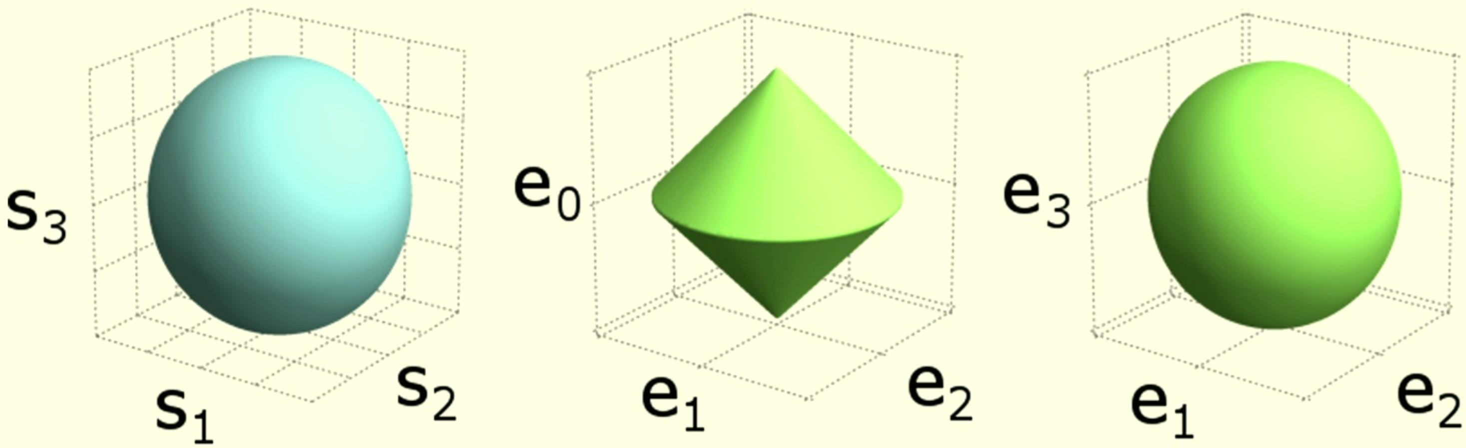
Qubit

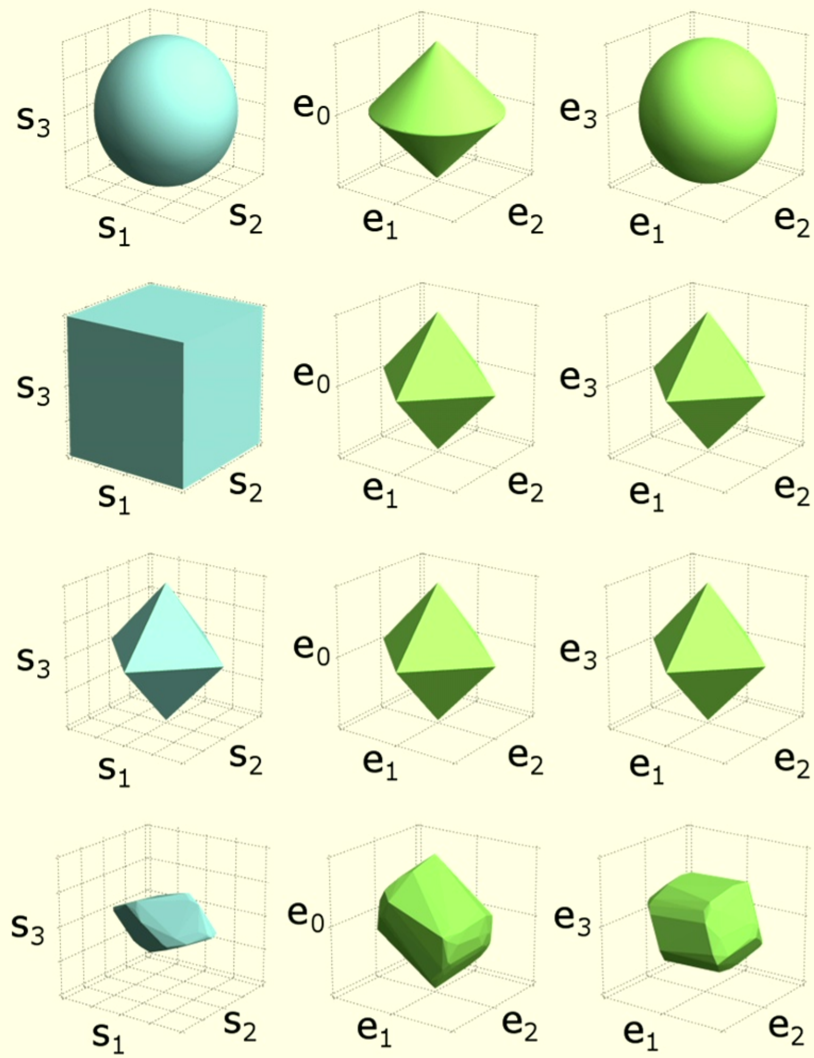
$$\rho = \tfrac{1}{2}\left(\mathbb{I} + s_X\sigma_X + s_Y\sigma_Y + s_Z\sigma_Z\right)$$

$$E = \tfrac{1}{2}\left(e_0\mathbb{I} + e_X\sigma_X + e_Y\sigma_Y + e_Z\sigma_Z\right)$$

$$\mathrm{Tr}(\rho E) = \tfrac{1}{2}(e_o + s_\mathrm{X}e_\mathrm{X} + s_\mathrm{Y}e_\mathrm{Y} + s_\mathrm{Z}e_\mathrm{Z})$$
$$= \underbrace{(1, s_\mathrm{X}, s_\mathrm{Y}, s_\mathrm{Z})}_{\mathbf{s}} \cdot \underbrace{\frac{1}{2}(e_0, e_\mathrm{X}, e_\mathrm{Y}, e_\mathrm{Z})}_{\mathbf{e}}$$



Examples of GPTs for systems with k=4



Qubit

Generalized No-signalling Theory (Boxworld)

Convex hull of my toy theory

Randomly generated example

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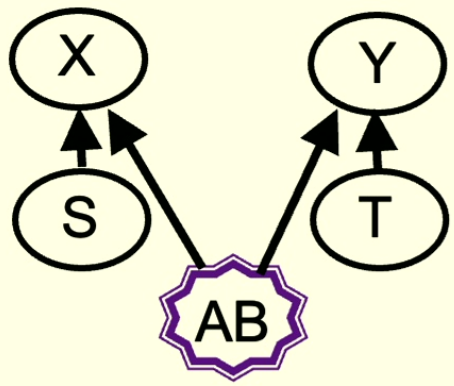
Boxworld-latent Bell model



$$\mathbf{r}^{A}_{x|s} \in \mathbb{R}^{d_A}$$

$$\mathbf{r}^{B}_{y|t} \in \mathbb{R}^{d_B}$$

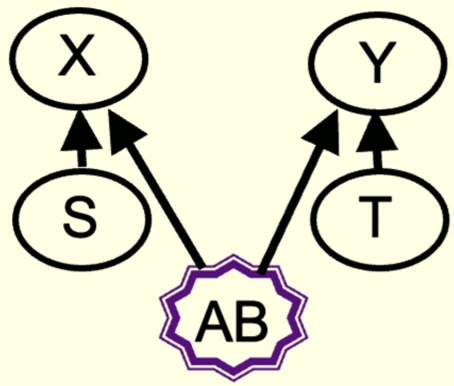
$$\mathbf{s}^{AB} \in \mathbb{R}^{d_A} \otimes \mathbb{R}^{d_B}$$

Boxworld

- Only product effects
- All states that are positive on product effects

$$P_{XY|ST}(xy|st) = (\mathbf{r}^{A}_{x|s} \otimes \mathbf{r}^{B}_{y|t}) \cdot \mathbf{s}^{AB}$$

Boxworld-latent Bell model



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- Only product effects
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$$P_{XY|ST}(xy|st) = (\mathbf{r}^A_{x|s} \otimes \mathbf{r}^B_{y|t}) \cdot \mathbf{s}^{AB}$$

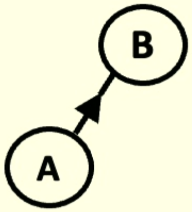
$X \perp T|S$

$Y \perp S|T$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq 1$$

No nontrivial inequality constraints

J. Barrett, Phys. Rev. A 75, 032304 (2005)



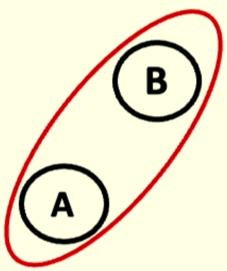
$\rho_{B|A} =$ image of $\mathcal{E}_{B|A}$ under CJ-isomorphism

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$\rho_{B|A}^{T_A} \geq 0$ (conditional is PPT)

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

$\mathcal{E}_{B|A} \in \mathrm{CP}$ (map is CP)



$$\rho_{B|A} = \rho_A^{-1/2}\rho_{AB}\rho_A^{-1/2}$$

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$\rho_{B|A} \geq 0$ (conditional is positive)

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

$\mathcal{E}_{B|A} \circ T_A \in \mathrm{CP}$ (map is co-CP)

	Classical	Quantum
Relation of conditional to joint	$P_{B\|A} = \frac{P_{AB}}{P_A}$	$\rho_{B\|A} = \rho_A^{-1/2} \rho_{AB} \rho_A^{-1/2}$ $\rho_{B\|A} \geq 0 \quad \text{(positive)}$
	$P_{AB} = P_{B\|A} P_A$	$\rho_{AB} = \rho_A^{1/2} \rho_{B\|A} \rho_A^{1/2}$
Normalization condition	$\sum_B P_{B\|A} = 1$	$\mathrm{Tr}_B(\rho_{B\|A}) = I_A$
Belief propagation	$P_B = \sum_A P_{B\|A} P_A$ $P_B = \Gamma_{B\|A}[P_A]$	$\rho_B = \mathrm{Tr}_A(\rho_{B\|A} \rho_A)$ $\rho_B = \mathcal{E}_{B\|A}(\rho_A)$ $\mathcal{E}_{B\|A} \circ T_A \in \mathrm{CP} \quad \text{(co-CP)}$
Bayesian inversion	$P_{B\|A} = \frac{P_{A\|B} P_B}{P_A}$	$\rho_{B\|A} = \rho_A^{-1/2} \rho_B^{1/2} \rho_{A\|B} \rho_B^{1/2} \rho_A^{-1/2}$

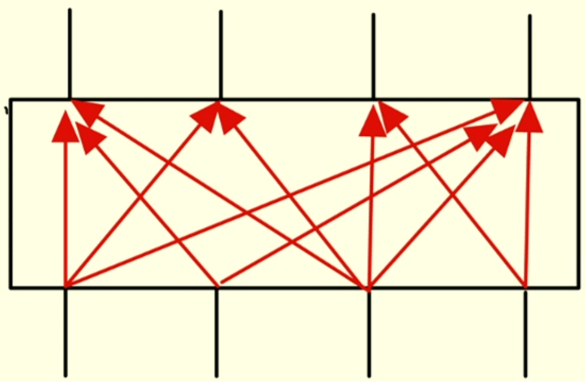
A circuit decomposition is **causally faithful** if influences are given by structure of diagram

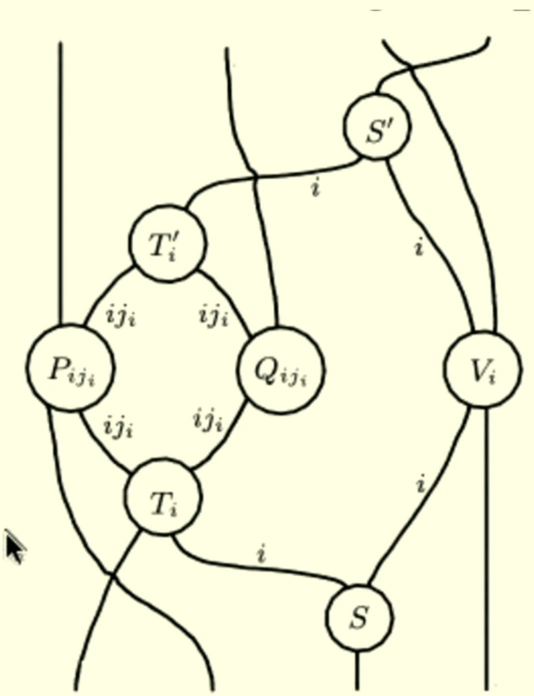
It is an open question whether every multipartite unitary channel admits a causally faithful decomposition

Known to be true for:
3 inputs and N outputs
N inputs and 3 outputs
Some other special cases

A circuit decomposition is **causally faithful** if influences are given by structure of diagram

unitary





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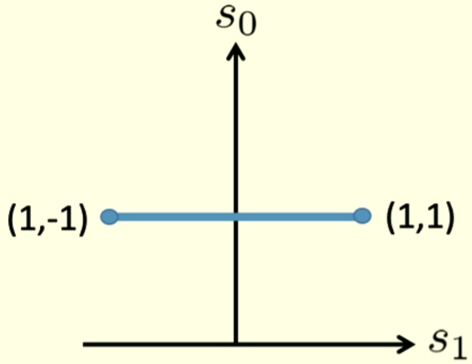
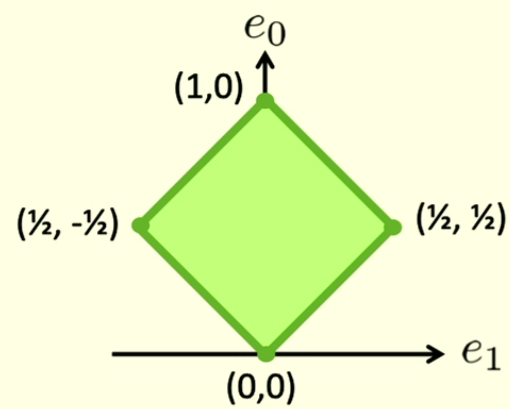


$$\mathbf{e} = (e_0, e_1)$$



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