Title: Birational King's Conjecture and Global Coherent Constructible Correspondence

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Collection/Series: Mathematical Physics

Subject: Mathematical physics

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Abstract:

In this talk, I will discuss a birational realization of King's conjecture which is indeed true, and its connections with noncommutative algebraic geometry and mirror symmetry. In particular, I will also establish the A-side analog of this result using constructible sheaves and promote the celebrated Coherent-Constructible Correspondence to a global family. The talk is based on recent joint work https://arxiv.org/abs/2501.00130 and work in preparation with David Favero.

Birational King's Conjecture & Coherent Constructible Correspondence

I. $\mathbb{P}^n$ (over $\mathbb{C}$)

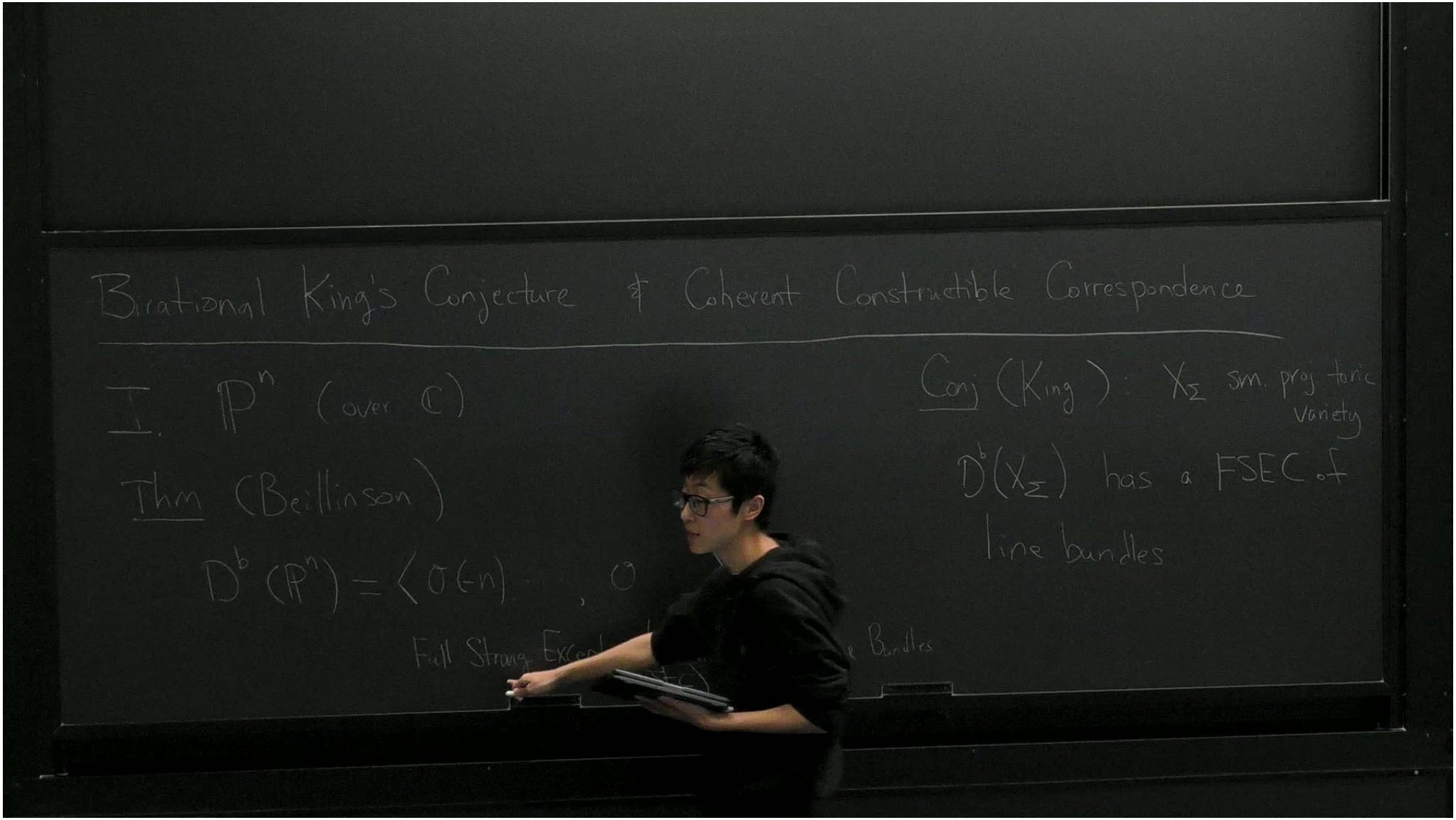
Thm (Beilinson)

$D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O} \rangle$

Full Strong Exceptional Collection of Line Bundles (FSECLB)

Conj (King): X_Σ sm. proj toric variety

$D^b(X_\Sigma)$ has a FSECLB


Birational King's Conjecture & Coherent Constructible Correspondence
I. $\mathbb{P}^n$ (over $\mathbb{C}$)
Thm (Beilinson)
$D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n)$, $\mathcal{O}$
Full Strong Exce
Bundles
Conj (King): X_Σ sm. proj toric variety
$D^b(X_\Sigma)$ has a FSEC of line bundles

Conj is false (Hille-Perling '06, Michalek '11, Efimov '24)

but inspired many works on $D^b(X_\Sigma)$

Thm (Kawamata '06)

$D^b(X_\Sigma)$ has a FEC.

generates

$\mathrm{Ext}^i(E,E) = \begin{cases} k & i=0 \\ 0 & i>1 \end{cases}$

Birational King's Conjecture & Coherent Constructible Correspondence

I. $\mathbb{P}^n$ (over $\mathbb{C}$)

Full: generate

Strong: $\mathrm{Ext}^i > 0 \Rightarrow$

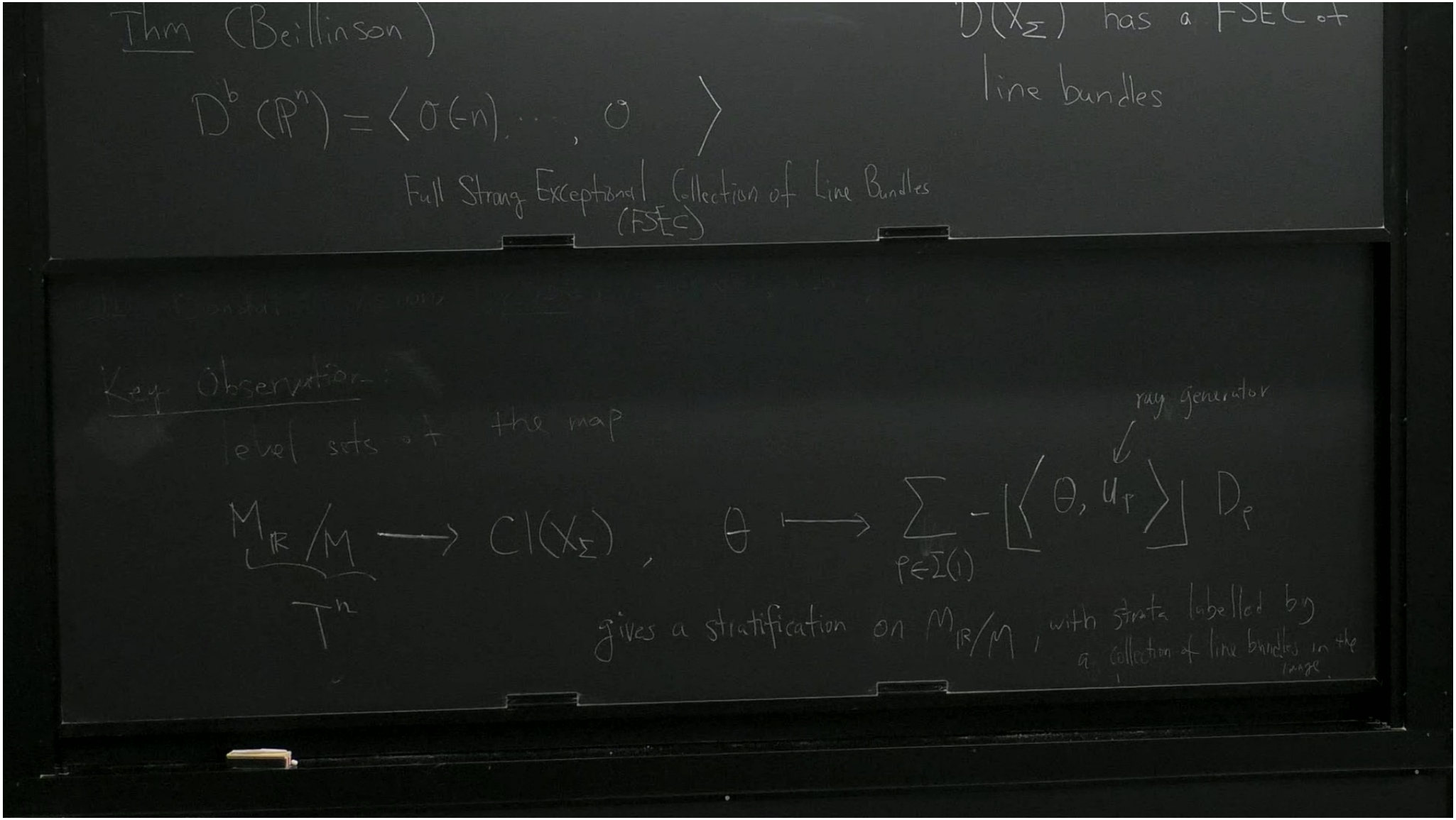
Conj (King): X_Σ sm. proj toric variety

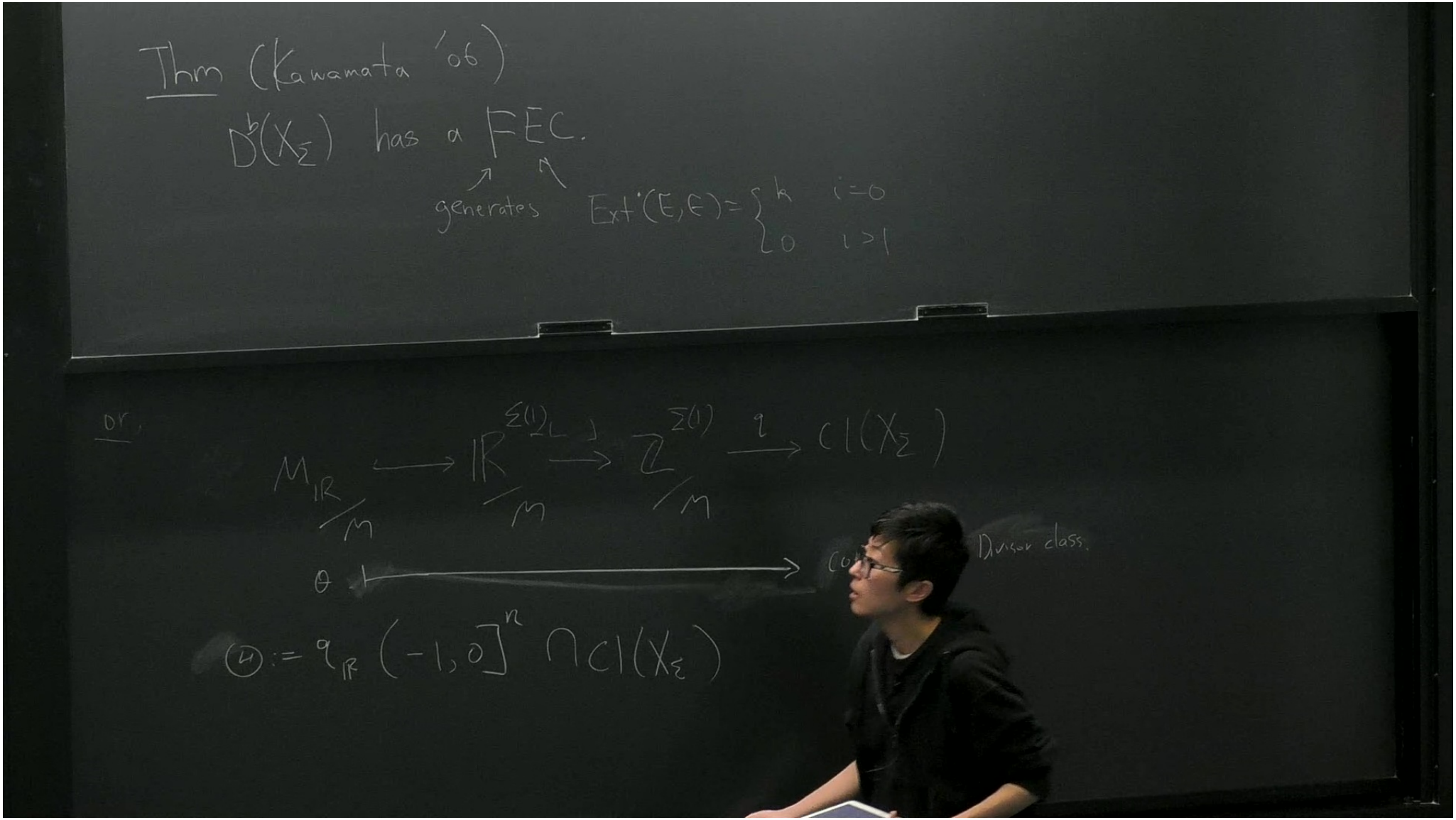
$D^b(X_\Sigma)$ has a FSEC of line bundles

Thm (Beilinson)

$D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O} \rangle$

Full Strong Exceptional Collection of Line Bundles (FSEC)


Thm (Beilinson)
$D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O} \rangle$
Full Strong Exceptional Collection of Line Bundles
(FSEC)
$D(X_\Sigma)$ has a FSEC of
line bundles
Key Observation:
level sets of the map
ray generator
$M_{\mathbb{R}}/M \longrightarrow Cl(X_\Sigma), \quad \theta \longmapsto \sum_{\rho \in \Sigma(1)} -\lfloor \langle \theta, u_\rho \rangle \rfloor D_\rho$
T^n
gives a stratification on $M_{\mathbb{R}}/M$, with strata labelled by
a collection of line bundles in the
image


Thm (Kawamata '06)
$D^b(X_\Sigma)$ has a FEC.
generates
$\mathrm{Ext}^i(E,E) = \begin{cases} k & i=0 \\ 0 & i>1 \end{cases}$
or,
$M_{\mathbb{R}} \longrightarrow \mathbb{R}^{\Sigma(1)} \longrightarrow \mathbb{Z}^{\Sigma(1)} \xrightarrow{q} Cl(X_\Sigma)$
M
M
M
θ
Divisor class.
$\textcircled{4} := q_{\mathbb{R}}(-1,0]^n \cap Cl(X_\Sigma)$

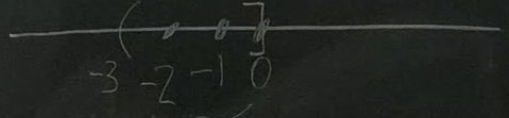
θ ⟶ corresponding Divisor class.

$$④ := q_{\mathbb{R}}(-1,0]^n \cap Cl(X_\Sigma)$$

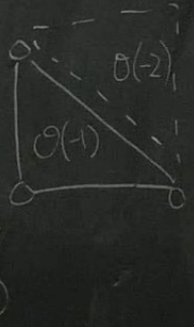
Ex: $\mathbb{P}^2$

Σ

$N_{\mathbb{R}}$ on $Cl(\mathbb{P}^2) = \mathbb{Z}$

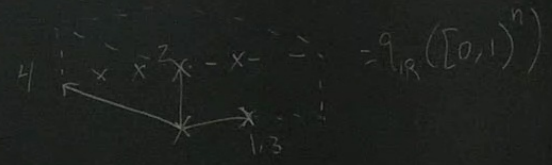


on $M_{\mathbb{R}} \cong \mathbb{R}^2$



Ex of ① for $\mathcal{H}_3$:

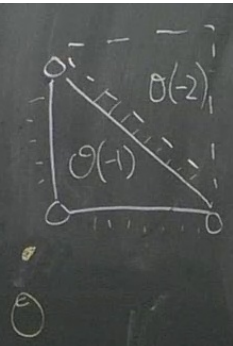
$$\mathbb{Z}^2 \to \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1 & 0 & 1 & -n \\ 0 & 1 & 0 & 1 \end{bmatrix}} \mathbb{Z}^n$$

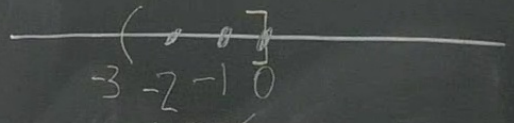


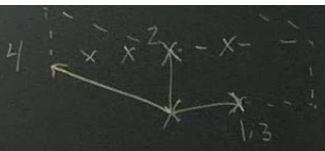
$= q_{\mathbb{R}}([0,1)^n)$

$Cl(\mathcal{H}_3) \cong \mathbb{R}^2$

on $M_{\mathbb{R}} \cong \mathbb{R}$







$= \mathcal{G}_{\mathbb{R}}([0,1])$

$Cl(\mathcal{H}_3) \cong \mathbb{R}^2$

Thm. (CCC) X_Σ any toric variety,

$$D^b(X_\Sigma) \xrightarrow{\simeq} Sh(M_{\mathbb{R}}/M, \Lambda_\Sigma) \subset \Lambda_\Sigma = \bigcup_{\sigma \in \Sigma} \sigma^\perp \times (-\sigma)$$

(Fang–Liu–Treumann–Zaslow '12

Kuwagaki '16)

Thm (Beilinson)

$D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O} \rangle$

Full Strong Exceptional Collection of Line Bundles (FSEC)

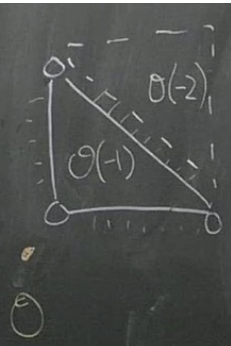
$D(X_\Sigma)$ has a FSEC of line bundles

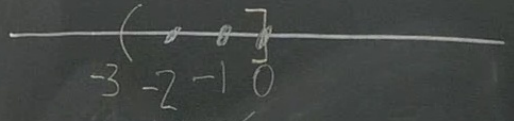
II. Resolution of Diagonal & Birational Modification of Kirwan's Conj.

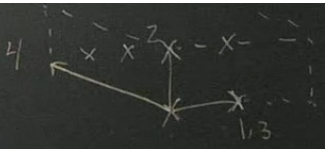
Thm (Favero–Hl. 2022, 2023): given a poset stratified cw cpx $X \xrightarrow[\text{surj}]{f} P$ whose strata $f^{-1}(p)$ is a contractible union of cells, gives $Q \hookrightarrow X$ $\Rightarrow A = kQ/\langle p-q : p \simeq_{\times} q \rangle$ have:

1. $Sh_{\Gamma}(X) \simeq A\text{-mod} \Rightarrow Sh_P(X)$ always has a FSEC
2. $BExit_P(X)$ gives a cellular resolution of diagonal A-bimodules of length $\leq$ length(X)

on $M_{\mathbb{R}} \cong \mathbb{R}$







$= g_{\mathbb{R}}([0,1])$

$\nearrow \quad Cl(\mathcal{H}_3) \cong \mathbb{R}^2$

$\Rightarrow$ gl.dim $(A) \leq \dim X$ for X regular cw

$\Rightarrow$ gl.dim $(A_{(k)}) = \dim(M_{\mathbb{R}}/M) = \dim(X_\Sigma) = r\dim(\tilde{D}^k(X_\Sigma))$

Thm (Hanlon-Hicks-Lazarev, Brown-Erman, Anderson)

$\exists$ resolution of $\mathcal{O}_{\Delta_{X_\Sigma}}$ by lin. bundles in $(-1) \times (1)$

of length $= \dim(X_\Sigma)$

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Observation: $\Lambda_{(1)} = \bigcup$ ss of generators $= \bigcup \Lambda_\Sigma$

X_Σ is the t.v. corres. to the chambers

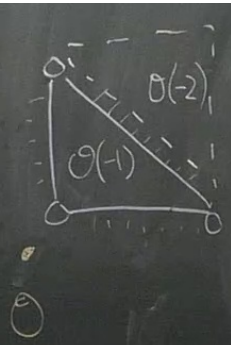
Defn: $\tilde{X} = X_{\tilde{\Sigma}}$, $\pi_\Gamma \downarrow$, X_{Σ_Γ}

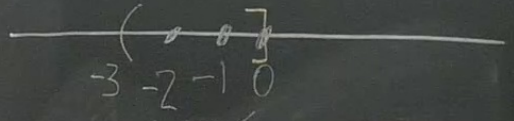
$\tilde{\Sigma}$ ↑ common refinement of fans from all chambers

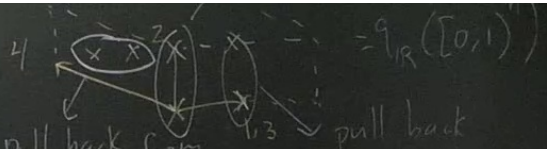
Define the **Cox Category** to be

$$D_{Cox} := \langle \pi_\Gamma^* D(X_{\Sigma_\Gamma}) \rangle_{\Gamma \subseteq \sum_{\text{codim}} \mathbb{Z}} \subseteq D^b(\tilde{X})$$

on $M_{\mathbb{R}} \simeq \mathbb{R}$





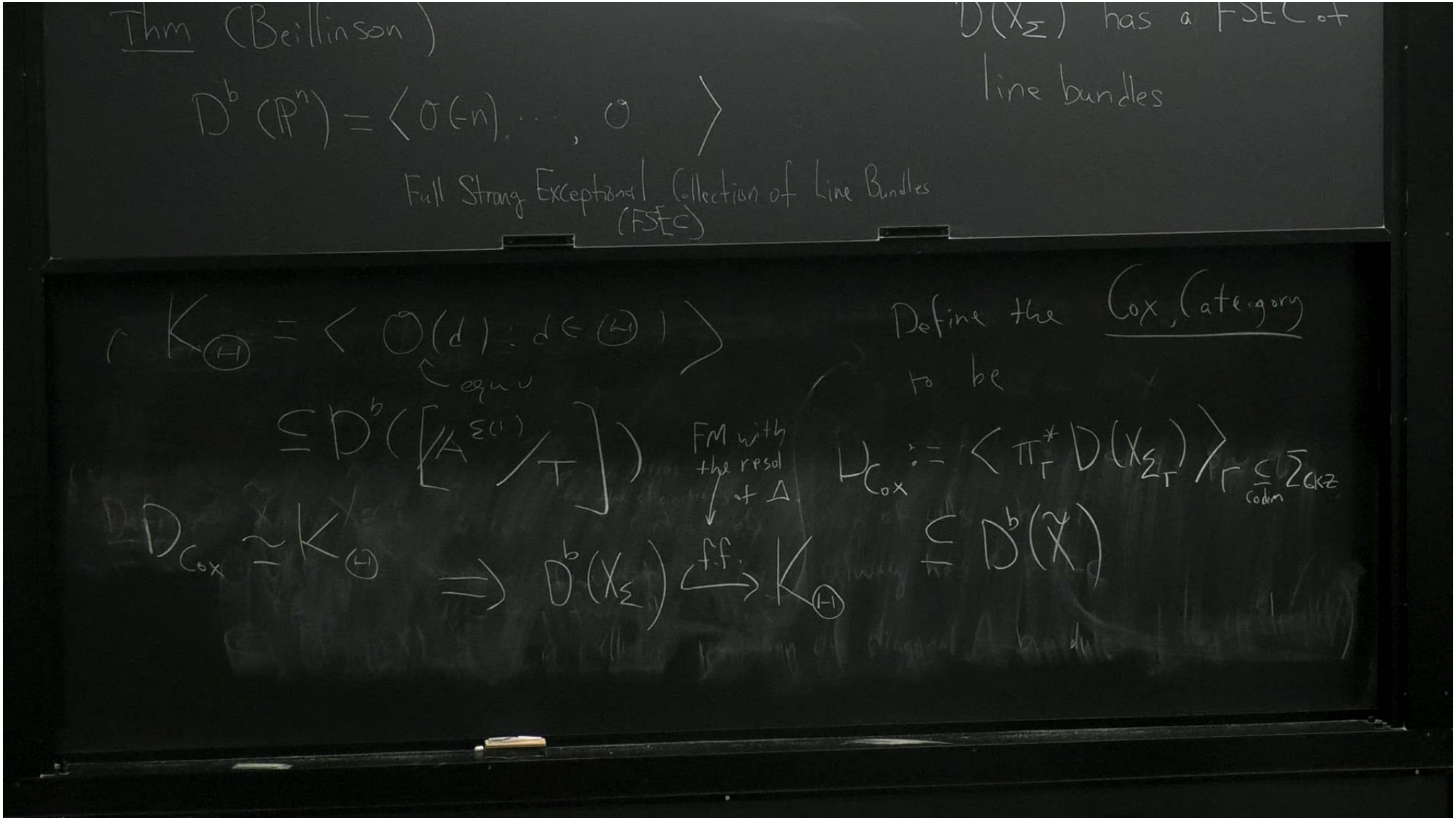


$\forall d \in \Theta, \mathcal{O}_{Cox}(d) := \pi_\Gamma^* \mathcal{O}_{X_\Gamma}(d)$ if $-d \in \Gamma$

Thm (B^3CEFGH^2)

$\bigoplus_{d \in \Theta} \mathcal{O}_{Cox}(d)$ is tilting in D_{Cox}

(FSEC in the proj case) → Birational King's ~~conj~~ Thm.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Full Strong Exceptional Collection of Line Bundles
(FSEC)
$D(X_\Sigma)$ has a FSEC of line bundles
$K_{(-1)} = \langle \mathcal{O}(d) : d \in (-1) \rangle$
equiv
$\subseteq D^b([\mathbb{A}^{\Sigma(1)}/T])$
FM with the resol of Δ
$D_{Cox} \simeq K_{(-1)} \Rightarrow D^b(X_\Sigma) \xhookrightarrow{f.f.} K_{(-1)}$
Define the Cox Category to be
$\mathcal{U}_{Cox} := \langle \pi_\Gamma^* D(X_{\Sigma_\Gamma}) \rangle_{\Gamma \subseteq \sum_{codim} \mathbb{Z}^k}$
$\subseteq D^b(X)$