Title: Quantum generators – from stochastic dynamics to Lindbladian extraction

Speakers: Emilio Onorati

Collection/Series: Quantum Information

Subject: Quantum Information

Date: April 14, 2025 - 3:00 PM

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Abstract:

Quantum generators are crucial objects for understanding quantum systems. They encode all the information required to predict the evolution of closed (Hamiltonians) and open memoryless dynamics (Lindbladians). Additionally, they offer structural insight into the noise affecting experimental implementations of quantum processes.

In this talk, we will discuss the definition of continuous-time random unitary evolutions characterized by stochastic Hamiltonians, showing mixing properties and efficient convergence to the uniform measure over the unitary group. We will then consider the problem of embedding a quantum map into a Markovian evolution, presenting a scheme to extrapolate the full description of the Lindbladian that is the best fit for the tomographic measurements of any (noisy) quantum channel.

Finally, we will introduce a new bound on quantum operators generated by arbitrary time-dependent Hamiltonians by leveraging a correspondence with binary trees structures.

References:

[1] E. Onorati et al., ‘Mixing properties of stochastic quantum Hamiltonians’, https://arxiv.org/abs/1606.01914

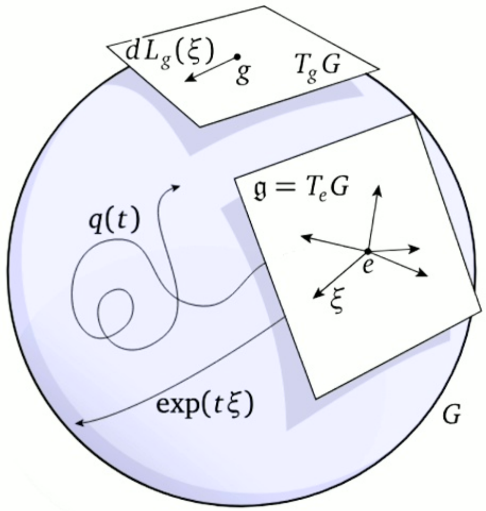
[2] E. Onorati et al., ‘Fitting quantum noise models to tomographic data’, https://arxiv.org/abs/2103.17243

Generators of quantum dynamics – structures, properties and learnability

Emilio Onorati

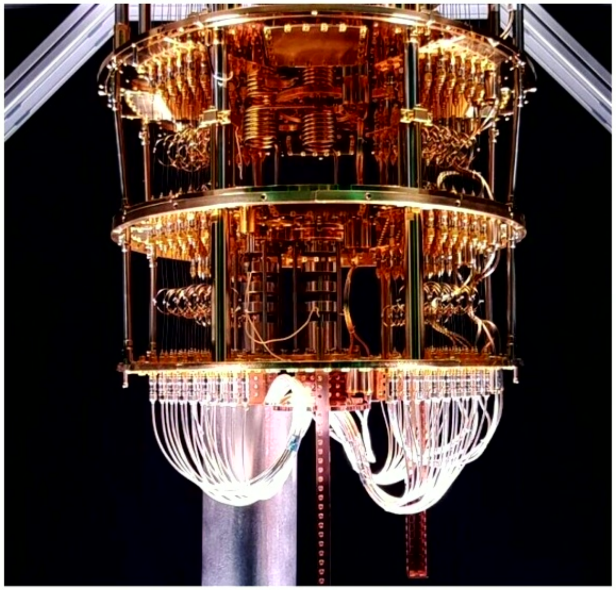
Technical University of Munich – Quantum Information Theory group

Technical University of Munich



Motivation – why studying quantum generators?

Black holes: information scramblers?



Implementing 'more accurate' quantum operations (gates)

2 /78

Motivation – why studying quantum generators?

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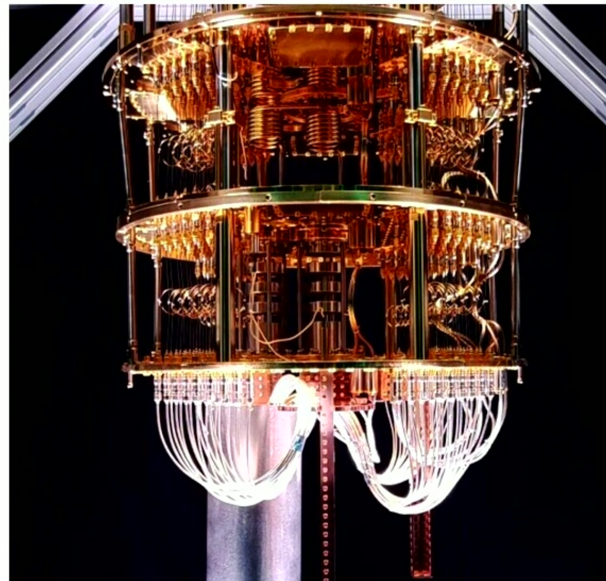
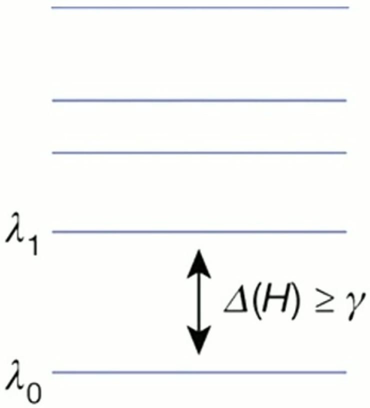


Closed system: Hamiltonians

- Hermitian operators
- Solution of the Schrödinger equation
- Encode everything about the physics of the system
- Many properties are derived from the **spectral gap**



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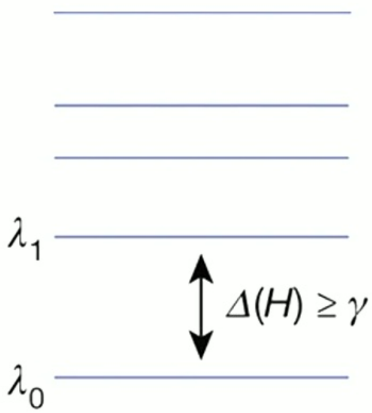
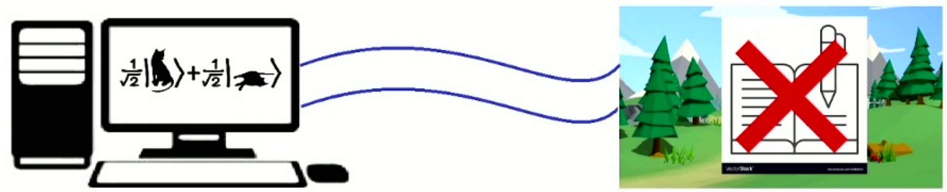


Open system: Lindbladians

memoryless process only



Markov property:

$$\mathbb{P}(X_{t_n+s} = y \mid X_{t_n} = y_n, \ldots, X_{t_1} = y_1) = \mathbb{P}(X_{t_n+s} = y \mid X_{t_n} = y_n)$$

quantum Markovian channel = quantum map forming a one-parameter semigroup

$$M(t+s) = M(t) \circ M(s)$$

Closed system: Hamiltonians

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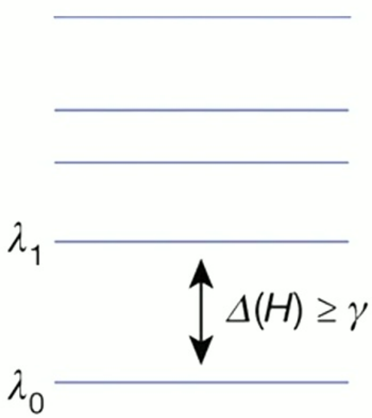

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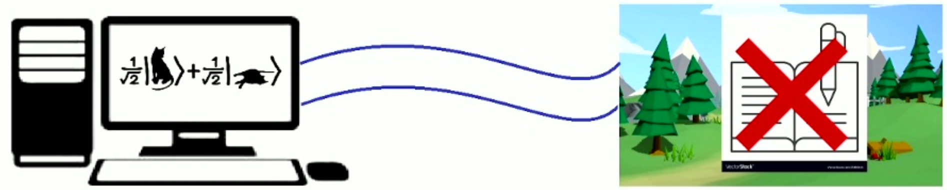


Lindblad master equation

$$\mathfrak{L}(\rho) := \mathrm{i}[\rho, H] + \sum_{\alpha} \left[J_\alpha \rho J_\alpha^\dagger - \frac{1}{2} \left(J_\alpha^\dagger J_\alpha \rho + \rho J_\alpha^\dagger J_\alpha \right) \right] .$$

J_α Jump operators

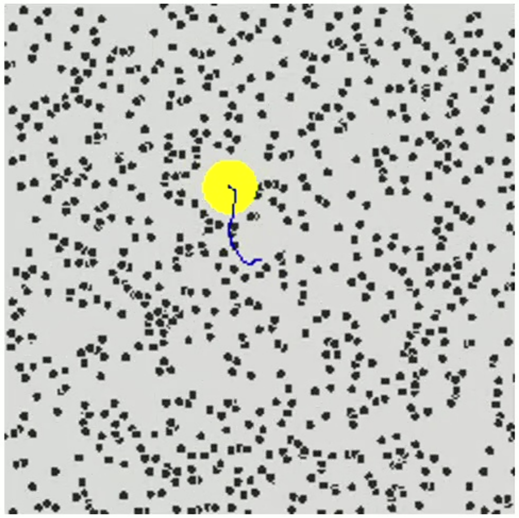
→ Then $M(t) = \mathrm{e}^{\mathfrak{L}t}$ is a quantum Markov process (and in particular at CPT map at any time t)

Outline of the talk

1. Continuous-time random Hamiltonians and mixing properties
2. Fitting data to Lindbladian and noise characterisation
3. A new bound on Magnus expansion with binary trees

Continuous-time random evolution of a closed system

Brownian motion



A real-valued stochastic process B_t, with $t \in [0, \infty)$, is called *Brownian motion* if

(1) $B_0 = 0$,

(2) B has *stationary* increments, i.e., for $s \leq t$, $B_t - B_s \stackrel{d}{=} B_{t-s}$,

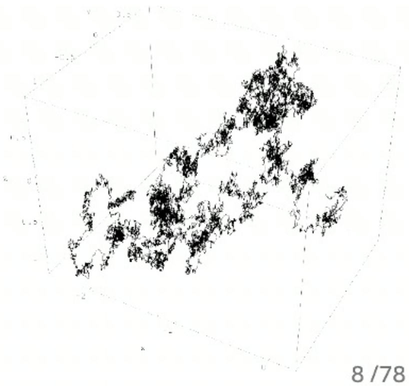
(3) B has *independent* increments, i.e., for all $0 < t_1 < \cdots < t_n$, the increments

$$B_{t_1} - B_0, \quad B_{t_2} - B_{t_1}, \quad \ldots, \quad B_{t_n} - B_{t_{n-1}}$$

are independent,

(4) for all $0 \leq s < t$, $B_t - B_s \sim \mathcal{N}(0, t-s)$,

(5) the paths $t \mapsto B_t$ are continuous *almost surely.*



Brownian motion... on the unitary group

A process U_t on the unitary group $\mathcal{U}(N)$ is called *Brownian motion* if the following conditions are satisfied:

(1) $U_0 = \mathbb{1}$,

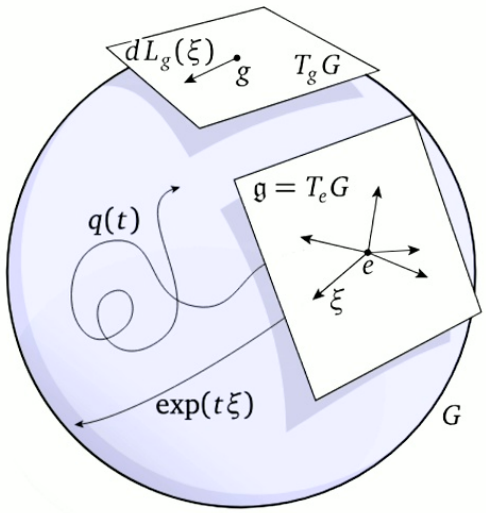
(2) For any time $t \geq 0$, the increments are stationary, i.e., for any $\Delta t > 0$, the increment $U_{t+\Delta t}U_t^\dagger$ is equal in distribution to $U_{\Delta t}U_0^\dagger$,

(3) For all $0 < t_1 < t_2 < \cdots < t_n$, the (left) increments

$$U_{t_1}U_0^\dagger, \quad U_{t_2}U_{t_1}^\dagger, \quad \ldots, \quad U_{t_n}U_{t_{n-1}}^\dagger$$

are independent,

(4) The paths $t \mapsto U_t$ are continuous almost surely.



Brownian motion... on the unitary group

$$\mathfrak{u}(N) := \{X \in \mathbb{C}^{N\times N} :\ X = -X^\dagger\}.$$

Construction on $\mathbb{U}(N)$

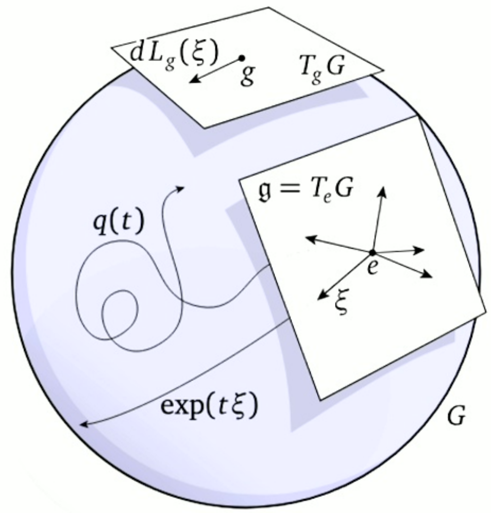
Inject the differential of Brownian motion from $\mathfrak{u}(N)$ via product integral of the exponential map

$$U_t = \lim_{\Delta t \to 0} \prod_{\ell = t/\Delta t}^{1} \exp\left\{ \frac{\mathrm{i}}{\Delta t} (H_{\ell \Delta t} - H_{(\ell-1)\Delta t} \right\} U_0 \,.$$

In more physical terms:

$$H_{\ell,\Delta t} = \sum_{e \in E} \left(h_0^{(e)} + \mathrm{i} \sum_{\mu} A_{\mu}^{(e)} \, \xi_{\ell,\Delta t}^{(e,\mu)} \right)$$

Some interaction Graph



Brownian motion... on the unitary group

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$$H_{\ell,\Delta t} = \sum_{e \in E} \left(h_0^{(e)} + \mathrm{i} \sum_{\mu} A_\mu^{(e)} \xi_{\ell,\Delta t}^{(e,\mu)} \right)$$

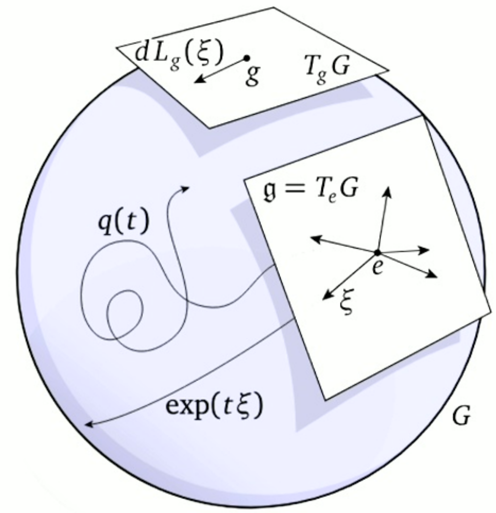
We assume the increments satisfies

$$\mathbb{E}\left[\xi_{\ell,\Delta t}^{(e,\mu)} \right] = 0,$$

$$\mathbb{E}\left[\xi_{\ell,\Delta t}^{(e,\mu)} \, \xi_{\ell',\Delta t}^{(e',\mu')} \right] = -\frac{a}{\Delta t} \, \delta_{\ell,\ell'} \, \delta_{e,e'} \, \kappa_{\mu,\mu'}^{-1},$$

where $a > 0$ and κ is the Killing metric tensor

$$\kappa_{\mu,\nu} := -2d^2 \mathrm{Tr}(A_\mu^\dagger A_\nu).$$



Home > Communications in Mathematical Physics > Article

Mixing Properties of Stochastic Quantum Hamiltonians

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Volume 355, pages 905–947, (2017) Cite this article

E. Onorati, O. Buerschaper, M. Kliesch, W. Brown, A. H. Werner & J. Eisert

Mixing properties – what do we mean?

In the context of stochastic processes,

mixing = reaching the fixed point/stationary state, "forgetting " the initial point

→ In our context, it means approximating the uniform Haar measure over the unitary group.

Definition: Haar measure

The Haar measure is the unique measure that is is left- and right-invariant,

$$\mu_{\mathrm{Haar}}(B) = \mu_{\mathrm{Haar}}(uB) = \mu_{\mathrm{Haar}}(Bu),$$

for any $u \in \mathbb{U}$ and Borel set B of $\mathbb{U}$

Mixing properties – what do we mean?

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→ In our context, it means approximating the uniform Haar measure over the unitary group.

Definition: *k*-th moment operator

The *k-th moment operator* M_μ^k on $\mathcal{L}(\mathcal{H}^{\otimes k})$ with respect to a distribution μ on $\mathbb{U}(N)$ is given by

$$X \mapsto M_\mu^k(X) := \mathbb{E}_\mu \left[U^{\otimes k} X (U^\dagger)^{\otimes k} \right]$$

Definition: approximate unitary *k*-design

Let μ be a distribution over the unitary group $\mathbb{U}(N)$. Then μ is an *ε-approximate unitary k-design* if

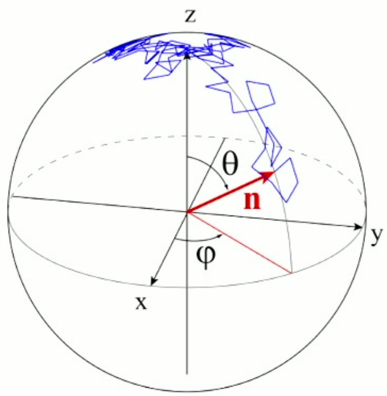
$$\left\| M_\mu^k - M_{\text{Haar}}^k \right\|_\diamond \leq \varepsilon.$$

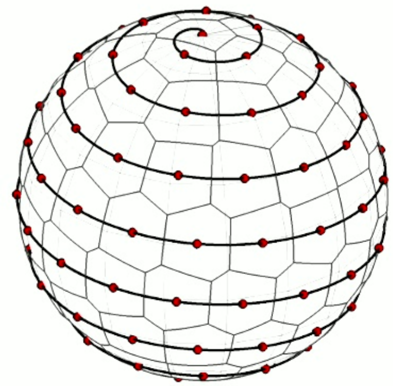
Main result: mixing properties of stochastic Hamiltonians

Let U_t be a unitary Brownian motion. Then, for any run time $t \geq T$ with

$$T = 850\lceil \log_d(4k) \rceil^2 d^2 k^5 k^{3.1/\ln(d)}\, a^{-1} nk \ln(d/\varepsilon)\,,$$

U_t is an ε-approximate unitary k-design.

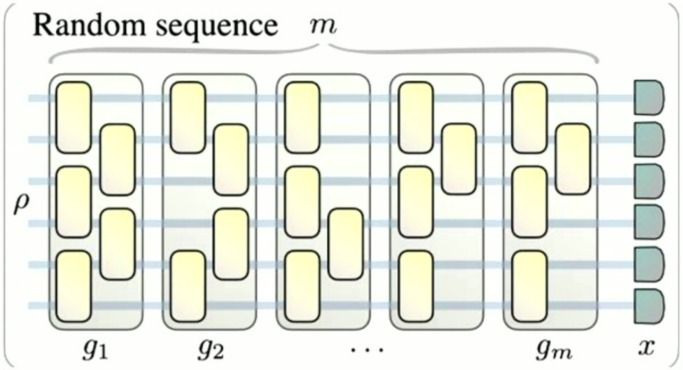




Proof ingredient 1 – relate to random quantum circuits

Relating to circuits and to local spectral gap

$$\left\| M^k_{\mathrm{BM}(\Delta t)} - M^k_{\mathrm{Haar}} \right\|_\infty \leq 1 - \Delta\left(m^k_{\mathrm{BM}(\Delta t)}\right)\left(1 - \left\| M^k_{\mathrm{circuit(Haar)}} - M^k_{\mathrm{Haar}} \right\|_\infty\right)$$



Then, use previous result[1] for random circuits:

$$\left(1 - \left\| M^k_{\mathrm{circuit(Haar)}} - M^k_{\mathrm{Haar}} \right\|_\infty\right) \geq \frac{1}{425\, n \lceil \log_d(4k) \rceil^2\, d^2\, k^5\, k^{3.1/\ln(d)}}$$

[1]F. G. S. L. Brandao, A. W. Harrow, and M. Horodecki. *Local random quantum circuits are approximate polynomial-designs*. Comm. Math. Phys

Proof ingredient 2 – identify the k-independent local gap

We assume the increments satisfies

$$\mathbb{E}\left[\xi^{(e,\mu)}_{\ell,\Delta t}\right] = 0,$$
$$\mathbb{E}\left[\xi^{(e,\mu)}_{\ell,\Delta t}\,\xi^{(e',\mu')}_{\ell',\Delta t}\right] = -\frac{a}{\Delta t}\,\delta_{\ell,\ell'}\,\delta_{e,e'}\,\kappa^{-1}_{\mu,\mu'},$$

where $a > 0$ and κ is the Killing metric tensor

$$\kappa_{\mu,\nu} := -2d^2\mathrm{Tr}(A^\dagger_\mu A_\nu).$$

Lemma: relating local and global gap

$$\Delta\left(m^k_{\mathrm{BM}(\Delta t)}\right) \sim \frac{1}{n}\mathbb{E}\left[\left(\sqrt{n}\,A_\mu\,\xi^\mu\right)^{\otimes k,k}\right]^2 \sim \left[\sum_{\mu,\nu=1}^{N^2-1}\kappa^{-1}_{\mu,\nu}\,(A_\mu A_\nu)\right]^{\otimes k,k} = C(\pi_{k,k})$$

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$C(\pi_{k,k})$ is the Casimir element, the object at the center of the universal enveloping algebra of u(N) in the k-tensor representation

Proof ingredient 2 – identify the k-independent <u>local gap</u>

Lemma: Casimir gap

Let $\mathcal{I}_k$ be the set of irreducible representations occurring in $\pi_{k,k}$ and let $v(\pi)$ denote the multiplicity of each such representation π. Then

$$C(\pi_{k,k}) \simeq \bigoplus_{\pi \in \mathcal{I}_k} c(\pi) \mathbb{1}_{\dim(\pi)} \otimes \mathbb{1}_{v(\pi)},$$

where

$$c(\pi) = \frac{1}{N} \sum_{i,j}^{N-1} (\lambda_i + 2)(A^{-1})_{i,j} \lambda_j = \begin{cases} = 0 & \text{if } \pi \simeq \pi_1, \\ = 1 & \text{if } \pi \simeq \pi_{\text{ad}}, \\ > 1 & \text{otherwise.} \end{cases}$$

In particular, the spectral gap of $C(\pi_{k,k})$ is independent of k.

Proof ingredient 2 – identify the k-independent <u>local gap</u>

Lemma: Casimir gap

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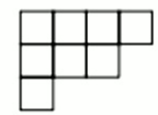
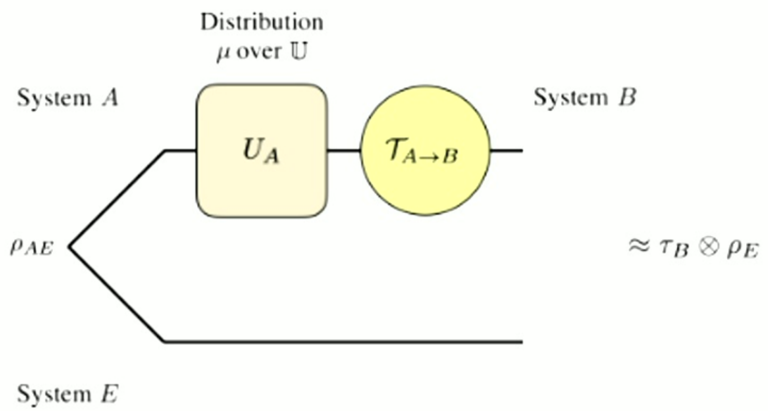
Cartan matrix

Dynkin label of a Young diagram



$\lambda = (1,2,1)$

Decoupling of quantum systems = "the task of approximately bringing a quantum mechanical system into a tensor product state with its environment"



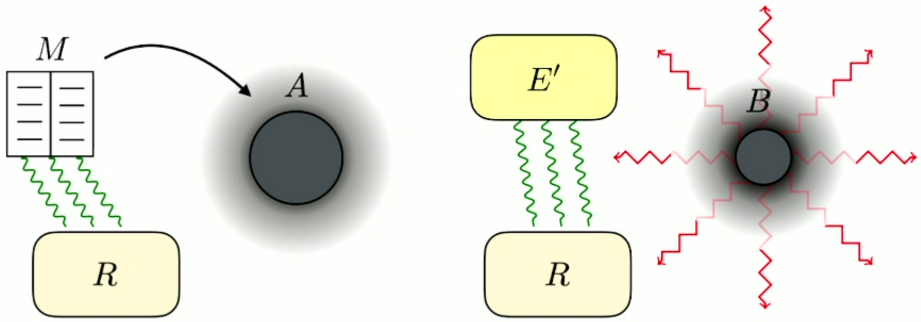
Main result 2: fast decoupling with stochastic Hamiltonians

$\mathcal{T} : \mathcal{S}_A \to \mathcal{S}_B$ CPT map. $\tau_{A'B}$ Choi-Jamiolkowski isomorph of $\mathcal{T}$.
Then, for run times $t \geq \varsigma\, n \log^2 n$ and for large enough n

$$\mathbb{E}\left\{ \left\| \mathcal{T}\left(U_t\, \rho_{AE} U_t^\dagger\right) - \tau_B \otimes \rho_E \right\|_1 \right\} \leq \left(\frac{1}{\mathrm{poly}(n)} + 5^{\delta n} \cdot 2^{-H_2(A|B)_\tau - H_2(A|E)_\rho} \right)^{1/2}$$

Application: black holes information paradox

Black holes are the fastest scramblers in nature and take logarithmic time to scramble information[2] (or non cloning-theorem is violated)



[2]Patrick Hayden and John Preskill. Black holes as mirrors: quantum information in random subsystems. Journal of High Energy Physics, 2007

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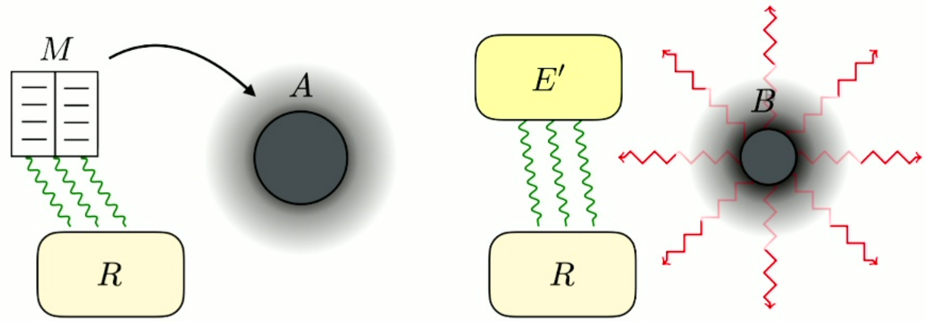


Black holes are the fastest scramblers in nature and take logarithmic time to scramble information[2] (or non cloning-theorem is violated)

Our work:

(i) Model block holes naturally with a random, continuous-time evolution.

(ii) Aligns with the prediction of log time scrambling, since we attain the Pauli mixing condition.

ρ_{AR} quantum state where subsystem E shares m Bell pairs with A, and A is otherwise mixed

$\gamma =$ #qubits emitted as Hawking radiation $-$ #qubits of system M.

The scrambling statement

$$\mathbb{E}_\omega \left\{ \left\| \mathrm{Tr}_{A \setminus B} \left(U_A \, \rho_{AR} U_A^\dagger \right) - \frac{\mathbb{1}_B}{|B|} \otimes \rho_R \right\|_1 \right\} \leq \sqrt{4^{-\gamma} + 4^m \varepsilon}$$

is satisfied in time $t = \mathcal{O}(\log n)$.

[2]Patrick Hayden and John Preskill. Black holes as mirrors: quantum information in random subsystems. Journal of High Energy Physics, 2007

Outline of the talk

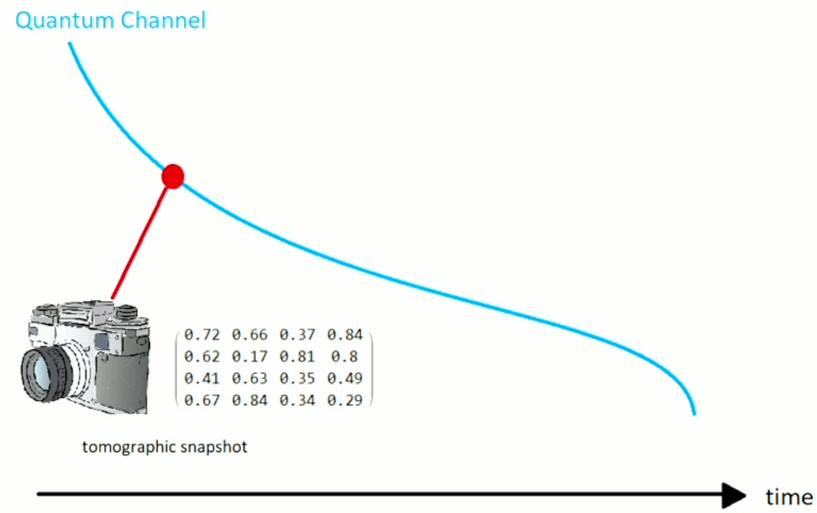
1. Continuous-time random Hamiltonians and mixing properties

2. Fitting data to Lindbladian and noise characterisation

3. A new bound on Magnus expansion with binary trees

The quantum embedding problem

Given a quantum map, is it compatible with a Markovian evolution?
That is, does some Lindbladian L exist so that $M \approx e^{L}$



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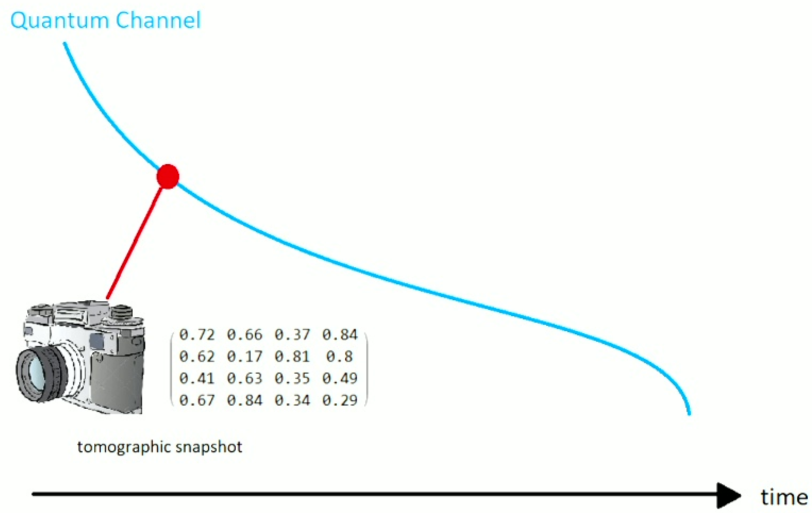

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The Complexity of Relating Quantum Channels to Master Equations

Published: 04 January 2012

Volume 310, pages 383–418, (2012) Cite this article

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Toby S. Cubitt, Jens Eisert & Michael M. Wolf

The problem is NP-hard in the system dimension!

... but still efficient in the required precision of the embedding!

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Fitting quantum noise models to tomography data

Emilio Onorati[1,2], Tamara Kohler[1,3], and Toby S. Cubitt[1]

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[2]Technische Universität München, Fakultät für Mathematik, DE
[3]Instituto de Ciencias Matemáticas, Madrid, ES

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Doi: https://doi.org/10.22
Citation: Quantum 7, 1197 (2

Design a scheme that:

- does not require any knowledge of the master equation nor interaction system <--> environment
- low resource-demanding – a single snapshot suffices
- returns full description of the Lindbladian

Quantum
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Fitting quantum noise models to tomography data

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Doi: https://doi.org/10.2233
Citation: Quantum 7, 1197 (202

key idea

- given: one or a few snapshots of a quantum channel, $M_1, \ldots, M_q$
- retrieve the Lindbladian that best approximates them
- ...or alternatively measure the non-Markovian component in terms of **white noise addition**
- Do so by casting a with a convex-optimisation task whose constraints are the necessary and sufficient conditions for a Lindbladian generator[4].

Ingredient 1: **convex optimization**

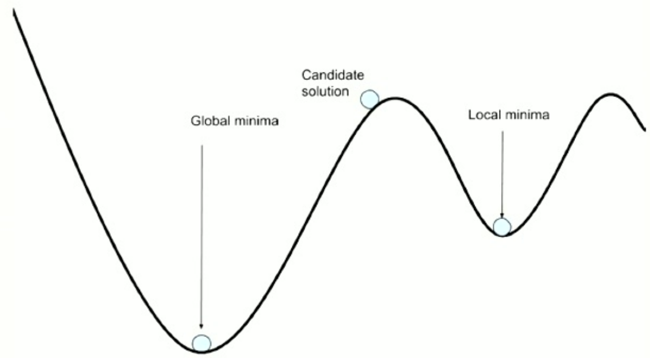
standard form

$$\begin{array}{lll} \textit{minimize} & f_0(x) & \\ \textit{subject to} & f_j(x) \leq 0, & j = 1, \dots, n \\ & \langle a_k, x \rangle = b_k & k = 1, \dots, m \end{array}$$

where $f_0, f_1, \dots, f_n$ are convex functions.

A fundamental property of convex optimisation problems is that

local minimum = global minimum



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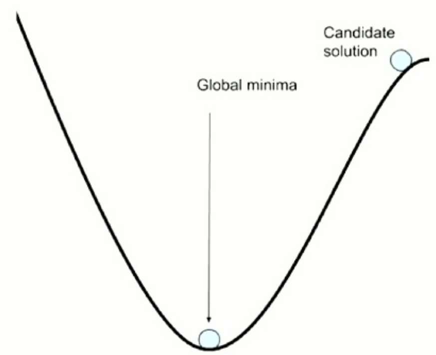

Ingredient 1: **convex optimization**

→ allows us to include an **error tolerance parameter ε**

.

→ the scheme can handle tomographic inaccuracies

→ the scheme is efficient in ε for **fixed** Hilbert space dimensions

Ingredient 2: **necessary and sufficient conditions for Lindbladian generator**

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Assessing Non-Markovian Quantum Dynamics

M. M. Wolf[1,2], J. Eisert[3,4], T. S. Cubitt[5], and J. I. Cirac[1]

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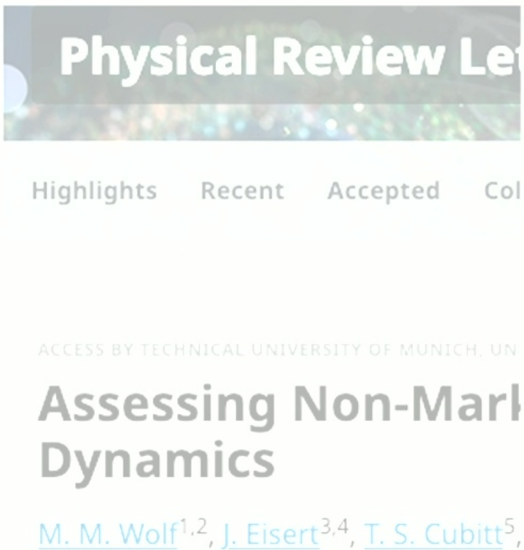
Ingredient 2: necessary and sufficient conditions for Lindbladian generator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Assessing Non-Marl
Dynamics
M. M. Wolf[1,2], J. Eisert[3,4], T. S. Cubitt[5],

The **natural basis representation** of $\mathcal{L}$ is the $d^2 \times d^2$ matrix defined as

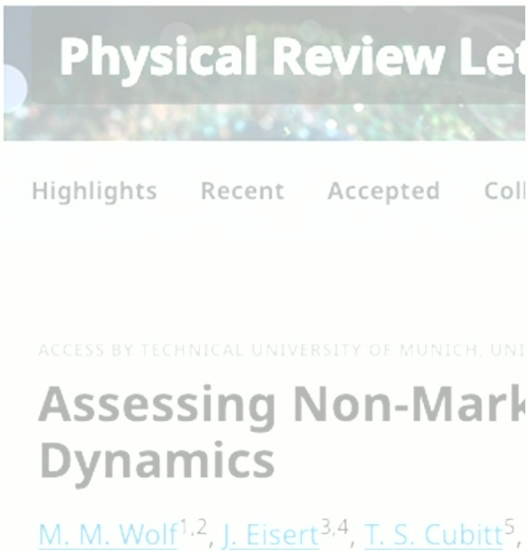
$$L_{(j,k),(\ell,m)} := \mathrm{Tr}\left[\,|e_k\rangle\langle e_j|\,\mathcal{L}(|e_\ell\rangle\langle e_m|)\right].$$

1. L' is hermiticity-preserving, that is, $L'|v^\dagger\rangle = (L'|v\rangle)^\dagger$ for all $|v\rangle$.
2. $(L')^\Gamma$ is **conditionally completely positive**, that is,

$$\omega_\perp\,(L')^\Gamma\,\omega_\perp \geq 0,$$

   where $\omega_\perp = (\mathbb{1} - |\omega\rangle\langle\omega|)$.
3. $\langle\omega|\,L' = \langle 0|$, which corresponds to the trace-preserving property.

Ingredient 2: **necessary and sufficient conditions for Lindbladian generator**



rewritten in terms of the **Choi-representation**

$$\tau(\mathcal{L}) := d\big(\mathcal{L} \otimes \mathcal{I}\big)(|\omega\rangle\langle\omega|) = (L')^{\Gamma}$$

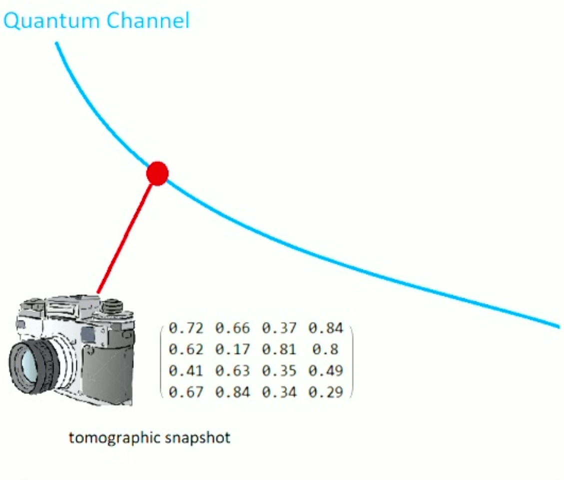
1. τ is hermitian
2. conditionally completely positive condition:

$$\omega_{\perp}\, \tau\, \omega_{\perp} \geq 0,$$

   where $\omega_{\perp} = (\mathbb{1} - |\omega\rangle\langle\omega|)$.

3. trace preserving condition: $\big\| \mathrm{Tr}_1 \left[\tau\right] \big\| = 0$ in any matrix norm

Core algorithm: single snapshot



Algorithm 1: Retrieve best-fit Lindbladian

Input : matrix M positive real number ε, positive integer $m_{\max}$

Output: L closest Lindbladian to $\vec{m}$-branch of $\log M$ such that $\|M - \exp L\|_{\mathrm{F}} < \varepsilon$ is minimal over all $\vec{m} \in \{-m_{\max}, \dots, 0, \dots, m_{\max}\}^{\times d^2}$

$G_0 \leftarrow \log(M)$

for $G_{\vec{m}} \leftarrow G_0 + 2\pi \mathrm{i} \sum_{j=1}^{d^2} m_j P_j$ *(branches of G_0)* **do**

  Run convex optimisation programme on variable $X(\vec{m})$:

  minimise $\left\| X(\vec{m}) - L^{\Gamma}_{\vec{m}} \right\|_{\mathrm{F}}$

  subject to $X(\vec{m})$ hermitian

  $\omega_\perp X(\vec{m}) \omega_\perp \geq 0$

  $\|\mathrm{Tr}_1[X(\vec{m})]\|_1 = 0$

  if $\left\| M - \exp X^{\Gamma}(\vec{m}) \right\|_{\mathrm{F}} < \varepsilon$ **then**

    Store $X(\vec{m})$

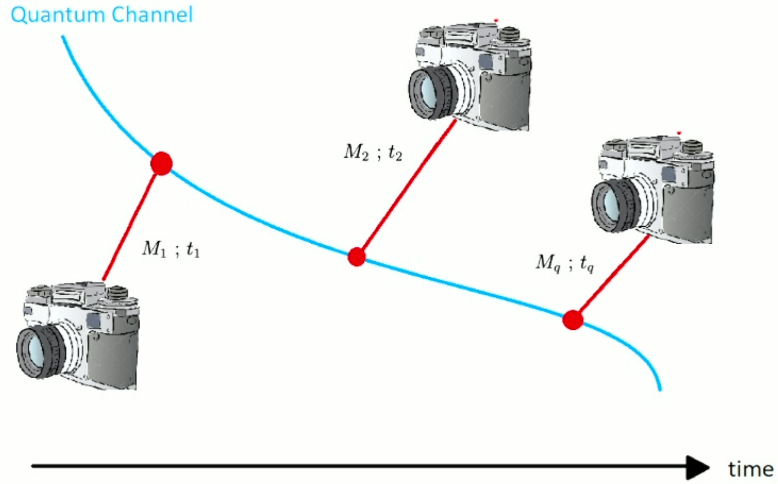
    distance$(\vec{m}) \leftarrow \left\| M - \exp X^{\Gamma}(\vec{m}) \right\|_{\mathrm{F}}$

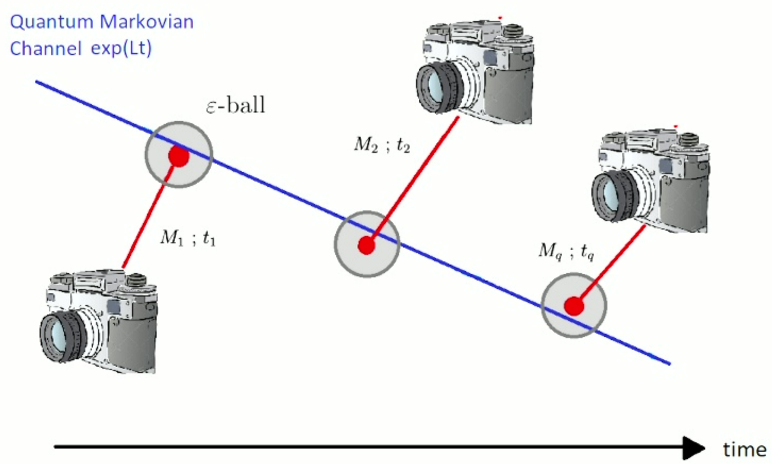
  end

end

return $L = X^{\Gamma}(\vec{m}')$ for $\vec{m}' = \mathrm{argmin}\,\{\mathrm{distance}(\vec{m})\}$

Condition for a Lindbladian in Choi representation *X*





Core algorithm: multiple snapshots

Algorithm 3: Retrieve best-fit Lindbladian for multiple snapshot

Input : $(d^2 \times d^2)$-dimensional matrices $M_1 \ldots, M_N$, positive real nu
positive real number ε, positive integer $m_{\max}$

Output: L Lindbladian minimising $\sum_{c=1}^{N} \|t_c L - \log M_c\|_F$ such that $\|M_c - \exp t_c L\|_F < \varepsilon$ for all c

for $G^c_{\vec{m}} \leftarrow G^c_0 + 2\pi i \sum_{j=1}^{d^2} m_j P^c_j$ *(branches of G_0)* **do**

Run convex optimisation programme on variable $X(\vec{m})$:

minimise $\sum_c \|t_c X(\vec{m}) - (G^c_{\vec{m}})^\Gamma\|_F$

subject to $X(\vec{m})$ hermitian

$\omega_\perp X(\vec{m}) \omega_\perp \geq 0$

$\|\mathrm{Tr}_1[X(\vec{m})]\|_1 = 0$

if $\|M_c - \exp t_c X^\Gamma(\vec{m})\|_F < \varepsilon$ *for all* $c = 1, \ldots, N$ **then**

Store $X(\vec{m})$

distance$(\vec{m}) \leftarrow \sum_c \|M_c - \exp t_c X^\Gamma(\vec{m})\|_F$

end

end

return $L = X^\Gamma(\vec{m}')$ for $\vec{m}' = \mathrm{argmin}\,\{\mathrm{distance}(\vec{m})\}$

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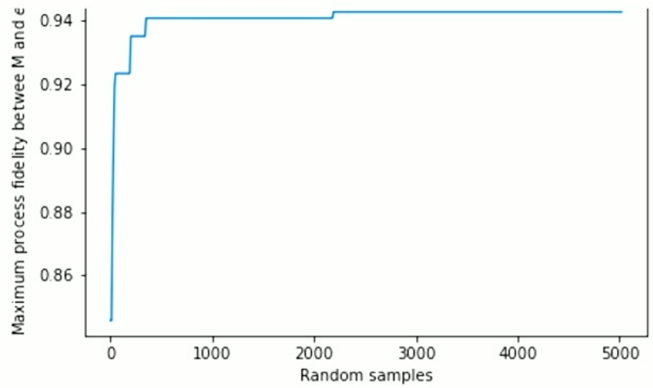
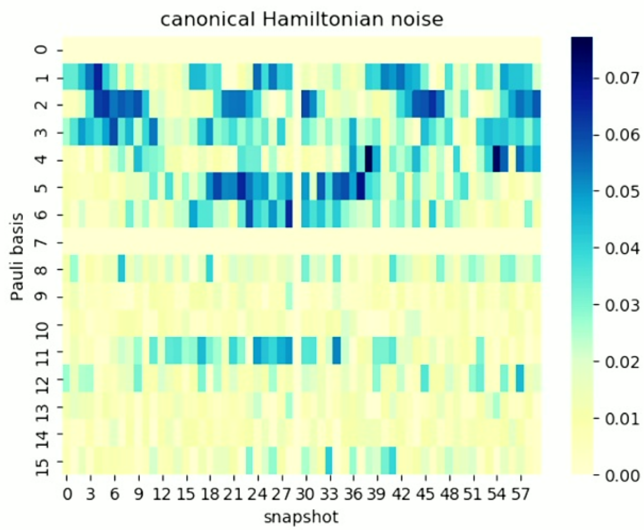
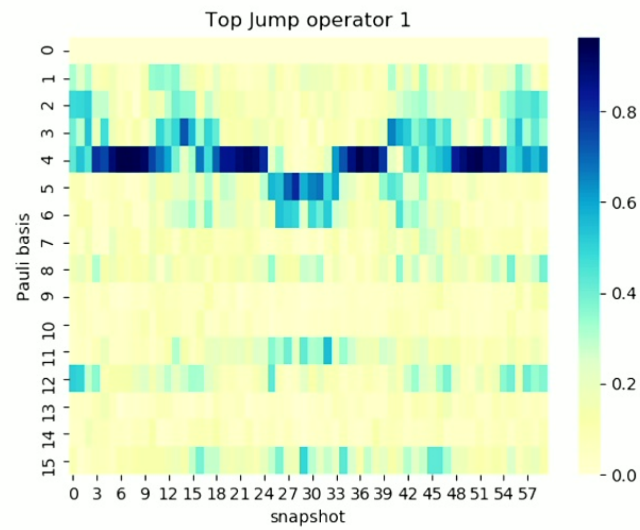
Simulation with synthetic data



Fidelity algorithm vs ground truth
CZ gate

Noise breakdwon in Pauli basis





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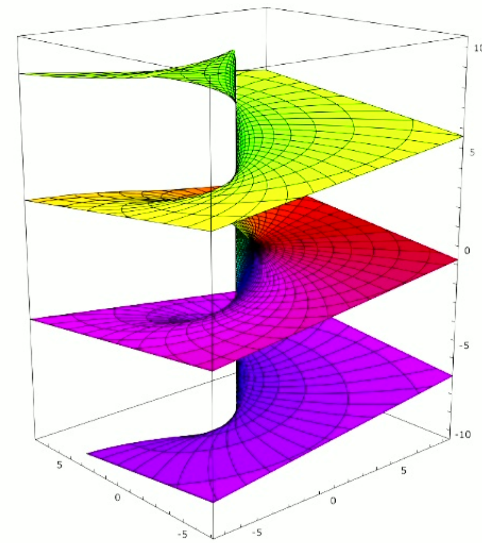
Math challenge 1: the matrix logarithm

- let T be a $d^2 \times d^2$ matrix
- let $\{\lambda_1, \dots \lambda_{d^2}\}$ denote its eigenvalues (all different and non-degenerate)
- let ℓ_j, r_j the respective left and right eigenvectors of λ_j such that $\langle \ell_j | r_k \rangle = \delta_{jk}$



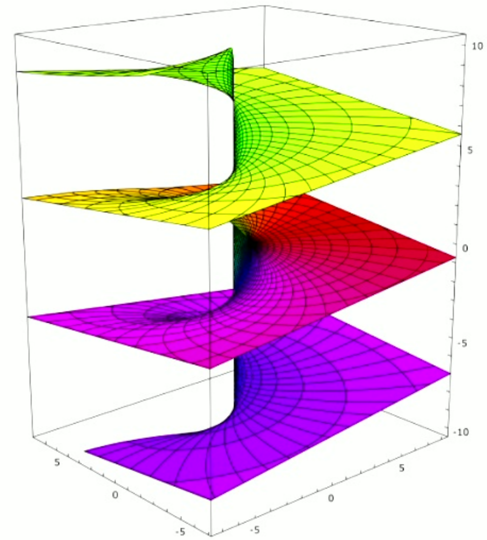
Then the 0-branch of the matrix logarithm of T is **uniquely defined** as

$$L_0 := \log(T) = \sum_{j=1}^{d^2} \log \lambda_j \, |r_j\rangle\langle \ell_j| \, ;$$

the $\vec{m}$-branch for $\vec{m} \in \mathbb{Z}^{d^2}$ is then

$$L_{\vec{m}} := L_0 + \sum_{j=1}^{d^2} m_j \, 2\pi \mathrm{i} \, |r_j\rangle\langle \ell_j| \, .$$

Math challenge 1: the matrix logarithm



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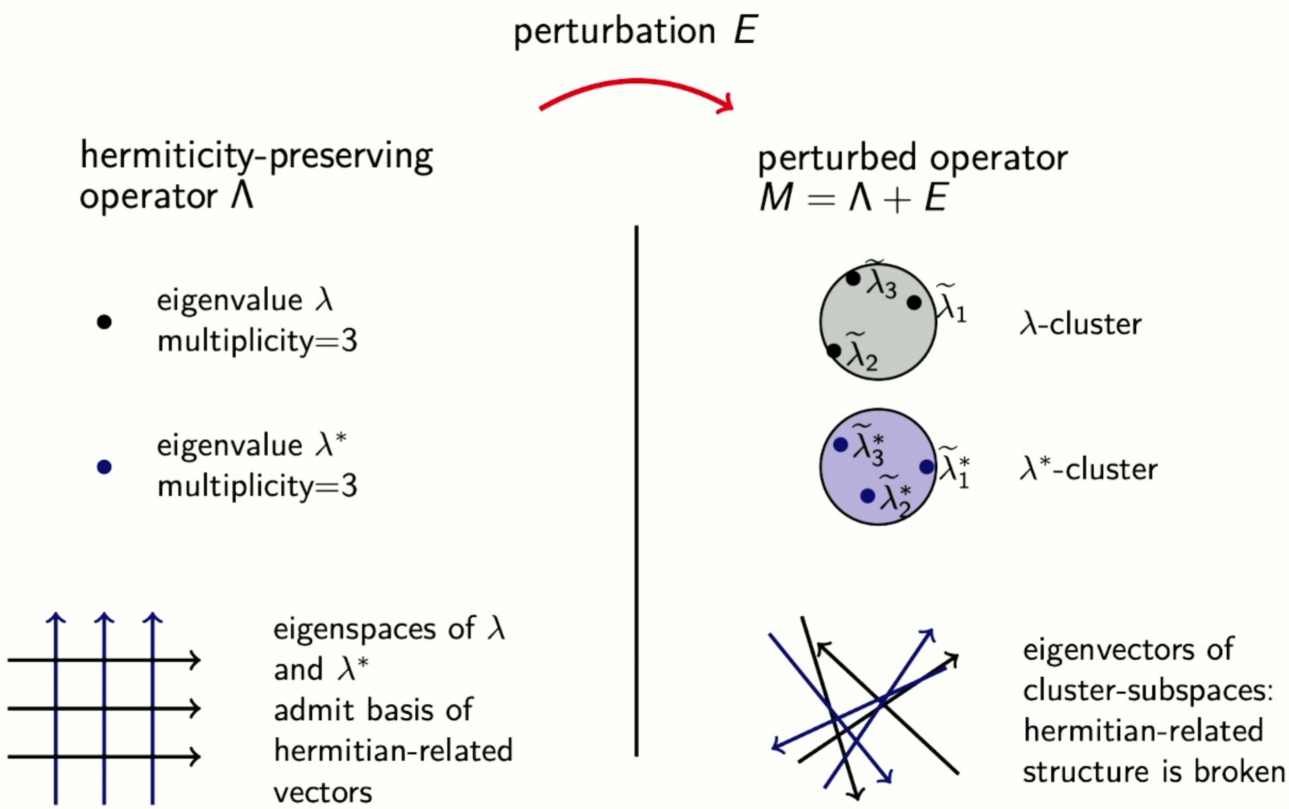
$$L_{\vec{m}} := L_0 + \sum_{j=1}^{d^2} m_j \, 2\pi \mathrm{i} \, |r_j\rangle\langle \ell_j| \, .$$

$m_j = O\left(2^{2^{poly(d)}}\right)$ suffices to reach the optimal in convex optimization[3]

Heuristically, $m = \pm 1$ suffices (low noise frequencies)

[3]Leonid Khachiyan and Lorant Porkolab. “Computing integral points in convex semi-algebraic sets”. In: Proceedings 38° Annual Symposium on Foundations of ComputerScience. IEEE. 1997,

Math challenge 2: Perturbations and degenerate spectrum



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Math challenge 2: Perturbations and degenerate spectrum

Scenario 1: consider $E = \varepsilon\big(|w_1\rangle\langle w_1| - |w_2\rangle\langle w_2| + |w_3\rangle\langle w_3| - |w_4\rangle\langle w_4| \big)$

where w_1, w_2, w_3, w_4 are all self-adjoint vectors with w_1 and w_2 spanning the eigenspace of 1 and w_3, w_4 spanning the eigenspace of -1

$\log(X + E)$ has eigenvalues $\varepsilon, -\varepsilon, \mathrm{i}\pi - \varepsilon, \mathrm{i}\pi + \varepsilon$ (up to first order in ε) with respect to eigenvectors w_1, w_2, w_3, w_4

➡ Lindbladian retrieved from convex-optimisation approach is $L' = \varepsilon\, |w_1\rangle\langle w_1| - \varepsilon\, |w_2\rangle\langle w_2| + \varepsilon\, |w_3\rangle\langle w_3| - \varepsilon\, |w_4\rangle\langle w_4|$

➡ $e^{L'} \approx \mathcal{I}$ ✗

Scenario 2: consider $E = \varepsilon\big(|w_1\rangle\langle w_1| - |w_2\rangle\langle w_2| + |w_5\rangle\langle w_5| - |w_6\rangle\langle w_6| \big)$

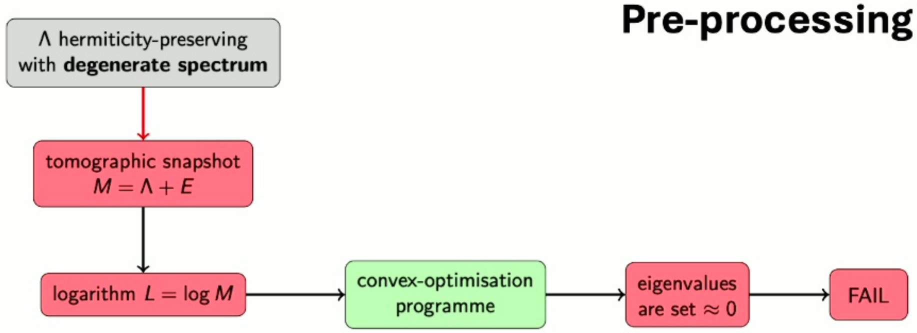
where w_5 and w_6 are hermitian-related vectors spanning the eigenspace of -1

$\log(X + E)$ has eigenvalues $\varepsilon, -\varepsilon, \mathrm{i}\pi - \varepsilon, \mathrm{i}\pi + \varepsilon$ (up to first order in ε) with respect to eigenvectors w_1, w_2, w_5, w_6

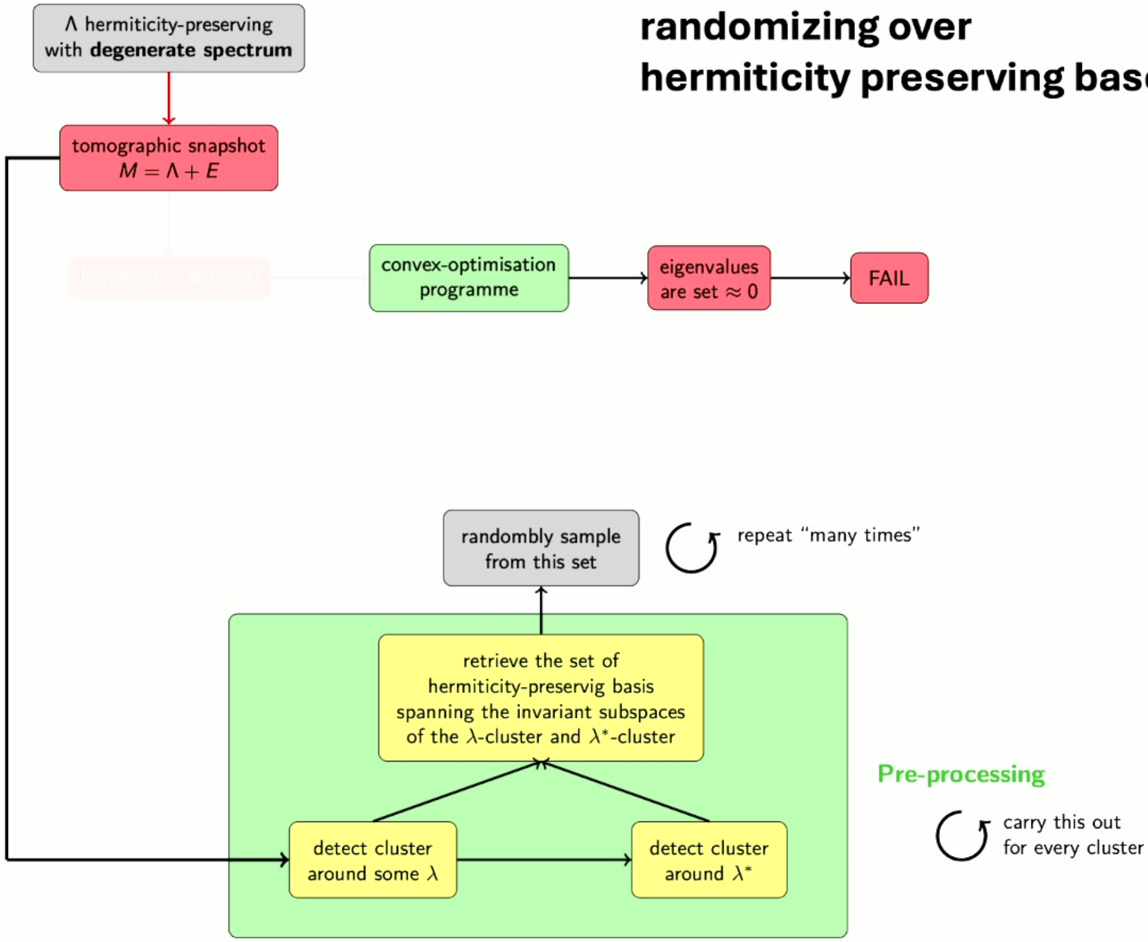
➡ Lindbladian retrieved from convex-optimisation approach on $\vec{m} = (0, 0, 0, -1)$-branch is $L' = \varepsilon\, |w_1\rangle\langle w_1| - \varepsilon\, |w_2\rangle\langle w_2| + \mathrm{i}\pi\, |w_5\rangle\langle w_5| - \mathrm{i}\pi\, |w_6\rangle\langle w_6|$

➡ $e^{L'} \approx X$ ✔

Pre-processing algorithm



Pre-processing algorithm – randomizing over hermiticity preserving bases



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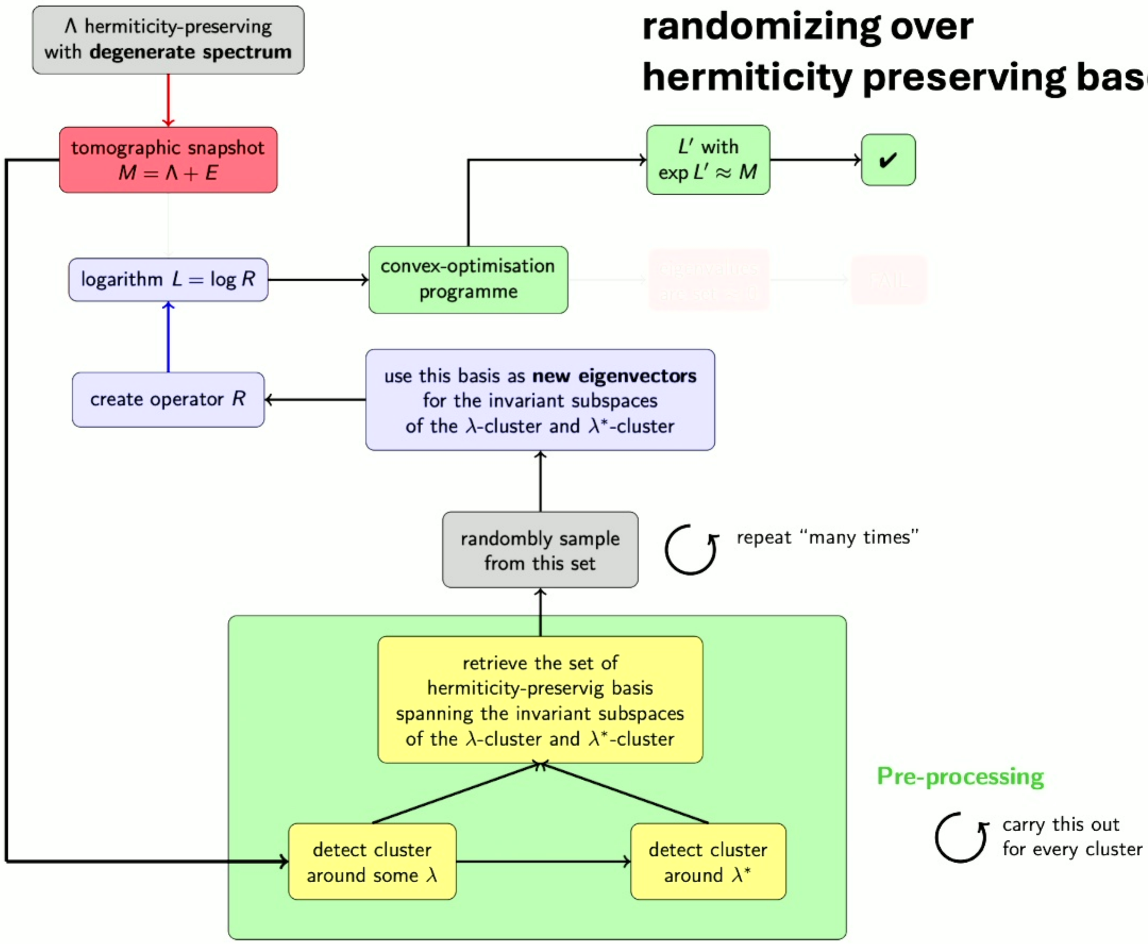

The theoretical guarantee under the hood

Theorem: Stability of the hermiticity-preserving structure

Let Λ be an hermiticity-preserving map and $M = \Lambda + E$ its perturbed version.

If λ is a complex and n-degenerate eigenvalue of Λ, then there exist a set of basis vectors $\{w_i\}_{i=1}^{n}$ spanning the right invariant subspace of M with respect to the λ-cluster and a set of basis vectors $\{w_i\}_{i=n+1}^{2n}$ spanning the right invariant subspace of M with respect to the λ^*-cluster such that

$$\left\| w_i^\dagger - w_{i+n} \right\| = \mathcal{O}\big(\|X_2\|_1 \|E_{21}\|_1 + \|Y_2\|_1 \|E'_{21}\|_1\big) \qquad \text{for } i = 1, \ldots, n$$

→ E_{21} is a **submatrix of** E under a basis transformation embedding X_2 into a unitary for the ***spectral resolution*** of M, 'mixing' the invariant subpaces of the λ-clusters with its complement

→ $\mathcal{O}\big(\|X_2\|_1 \|E_{21}\|_1 + \|Y_2\|_1 \|E'_{21}\|_1\big)$ is "small" ✔

Outline of the talk

1. Continuous-time random Hamiltonians and mixing properties
2. Fitting data to Lindbladian and noise characterisation
3. A new bound on Magnus expansion with binary trees

Can we leverage recursions and fractals to express

time-dependent Hamiltonian evolutions?

Raffaello – *The School of Athens*

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Recap: time-independent Hamiltonian evolution

Schrödinger equation: $i\,\hbar\,\psi(t) = H\,\psi(t)$.

Solution: $\psi(t) = U(t)\psi(0)$ with $U(t) = e^{-\frac{i}{\hbar}Ht}$

Recap: time-*dependent* Hamiltonian evolution

Schrödinger equation: $i\,\hbar\,\psi(t) = H\,(t)\,\psi(t)$.

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<u>Recap:</u> time-*dependent* Hamiltonian evolution

Schrödinger equation: $i\,\hbar\,\psi(t) = H\,(t)\,\psi(t)$.

Solution: $\psi(t) = U(t)\psi(t)$ with $U(t) = e^{-\frac{i}{\hbar}\int H(t)dt}$ ✗ → $U(t) = \mathsf{T}\exp\left(-\frac{i}{\hbar}\int_0^t H(\tau)d\tau\right)$

Recap: time-*dependent* Hamiltonian evolution

Time ordering operator $\mathcal{T}\,[H(t_1)H(t_2)\cdots H(t_n)] = H(t_{j_1})H(t_{j_2})\cdots H(t_{j_n})$
where $t_{j_1} > t_{j_2} > \cdots > t_{j_n}$

Schrödinger equation: $i\,\hbar\,\dot\psi(t) = H\,(t)\,\psi(t)$.

Solution: $\psi(t) = U(t)\psi(t)$ with $U(t) = e^{-\frac{i}{\hbar}\int H(t)dt}$ ✗ → $U(t) = \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_0^t H(\tau)d\tau\right)$

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Recap: time-*dependent* Hamiltonian evolution

Schrödinger equation: $i\,\hbar\,\dot\psi(t) = H\,(t)\,\psi(t)$.

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$U(t) = \exp(\boldsymbol{M}(t))$ where ***M***(t) is the **Magnus Expansion**

The Magnus expansion can be written as a series $\boldsymbol{M}(t) = \sum_{n=1}^{\infty} M_n(t)$

$$M_1(t) = \frac{1}{(i\hbar)}\int_0^t H(t_1)dt_1,$$

$$M_2(t) = \frac{1}{2(i\hbar)^2}\int_0^t dt_1 \int_0^{t_1} dt_2\,[H(t_1), H(t_2)],$$

$$M_3(t) = \frac{1}{6(i\hbar)^3}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \Big([H(t_1),[H(t_2),H(t_3)]] + [H(t_3),[H(t_2),H(t_1)]]\Big),$$

$$M_4(t) = \frac{1}{12(i\hbar)^4}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \Big([[[H(t_1),H(t_2)],H(t_3)],H(t_4)] + [H(t_1),[[H(t_2),H(t_3)],H(t_4)]] + [H(t_1),[H(t_2),[H(t_3),H(t_4)]]] + [H(t_2),[H(t_3),[H(t_4),H(t_1)]]]\Big).$$

→ The expression for higher *n* becomes increasingly complex

Recursion formula for the Magnus expansion (D. P. Burum, Phys. Rev. B, 1981)

Define the *Dyson term*

$$D_n = \frac{1}{(i\hbar)^n}\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, H(t_1)H(t_2)\cdots H(t_n)$$

Then, the *n*-th Magnus term is defined as

$$M_n = D_n - \sum_{k=1}^{n} \frac{1}{j} Q_{n,k}$$

where $Q_{n,k} = \sum_{j_1+j_2+\cdots+j_k} M_{j_1} M_{j_2} \cdots M_{j_k}$ with $j_1 + j_2 + \cdots + j_k = n$

k-composition of n

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k-composition of n

The number of composition grows **exponentially**, #compositions = 2^{n-1} → Can we stop at some point ("truncate") and still obtain a "good" approximation?

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Mathematical layout: full binary trees

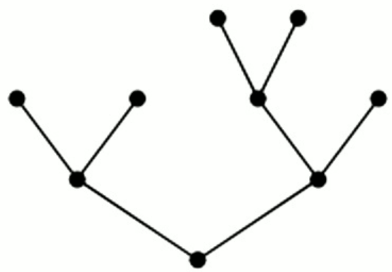

Figure 1: A full binary tree with 5 leaves

Mathematical layout: full binary trees

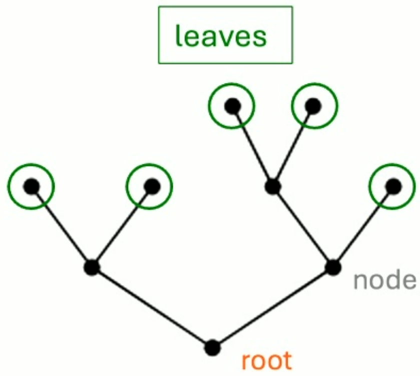

Figure 1: A full binary tree with 5 leaves

A full binary tree has

- a single root
- nodes in hierarchical structure
- every node has either 0 or 2 successors
- every node has one single parent node (except the root)
- a node without successors is called a *leaf*

  $\mathcal{T}_n$: set of all trees with n leaves

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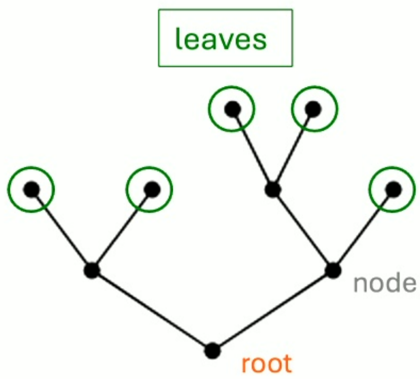

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Binary tree naturally connects to nested commutators!

$\sim [[l_1, l_2], [[l_3, l_4], l_5]]$

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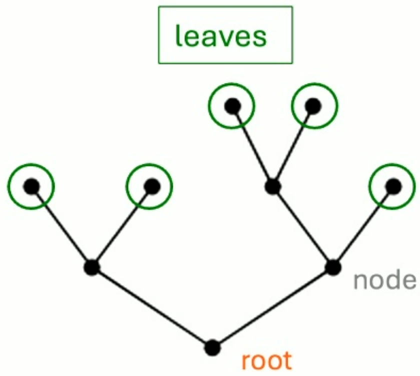

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$\mathcal{T}_n$: set of all trees with n leaves

Binary tree naturally connects to nested commutators!

$\sim [[l_1, l_2], [[l_3, l_4], l_5]]$

...but we still have to account for the integrals!

$$\int_0^T [[H(t_1), \int_0^{t_1}, H(t_2)], \int_0^{t_1} [[H(t_3), \int_0^{t_3} H(t_4)], \int_0^{t_3} H(t_5)]]]$$

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Integration with binary trees

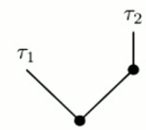
(A. Iserles, S. Nørsett, *Phylosophical Transactions of Royal Society,* 1999)

Additional rules for *subtrees*:

1) Integration of a subtree: appending a singular node below the root of the associated tree.

$\sim \int_0^\kappa H_{\tau_1} dt_1$

2) Commutation: integrate the right sub-tree as per rule 1), then join the two roots with a new node which becomes the new root.



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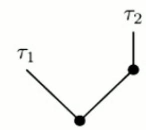

$$\sim \quad \left[H_{\tau_1}, \int_0^\kappa H_{\tau_2} dt_2\right]$$

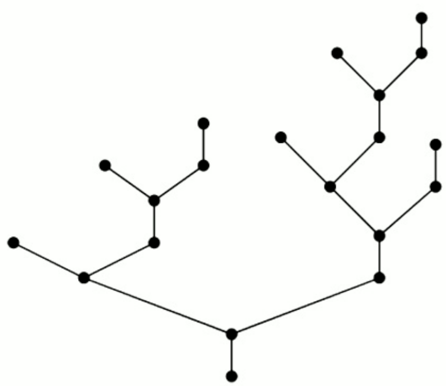

Figure 4: A 7-leaves binary tree.

$$= \int_0^t H_\tau(\kappa)d\kappa = \int_0^t d\kappa\Big[[h(\kappa), \int_0^\kappa dt_2[h(t_2), \int_0^{t_2} dt_3 h(t_3)]], \int_0^\kappa dt_4[[h(t_4), \int_0^{t_4} dt_5[h(t_5), \int_0^{t_5} dt_6 h(t_6)]], \int_0^{t_4} dt_7 h(t_7)]\Big]$$

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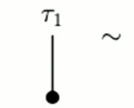

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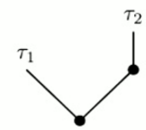

$\sim [H_{\tau_1}, \int_0^\kappa H_{\tau_2} dt_2]$

→ yields expression for the Magnus expansion

$$\mathcal{M}(t) = \sum_{n=1}^{\infty} M_n(t) = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_n} \alpha_\tau \int_0^t H_\tau(\kappa) d\kappa,$$

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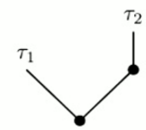


$\sim \quad [H_{\tau_1}, \int_0^\kappa H_{\tau_2} dt_2]$

yields expression for the Magnus expansion

$$\mathcal{M}(t) = \sum_{n=1}^{\infty} M_n(t) = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_n} \alpha_\tau \int_0^t H_\tau(\kappa) d\kappa,$$

$$H_{\tau=(\tau_1,\tau_2,\ldots,\tau_r)} = [\cdots[H(\kappa), \int_0^\kappa H_{\tau_1}], \int_0^\kappa H_{\tau_2}], \ldots, \int_0^\kappa H_{\tau_r}]$$

recursively constructed

<u>Integration with binary trees</u>

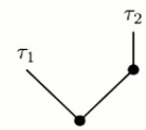
(A. Iserles, S. Nørsett, *Phylosophical Transactions of Royal Society,* 1999)

Additional rules for *subtrees*:

1) Integration of a subtree: appending a singular node below the root of the associated tree.

$\sim \quad \int_0^\kappa H_{\tau_1} dt_1$

2) Commutation: integrate the right sub-tree as per rule 1), then join the two roots with a new node which becomes the new root.



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→ yields expression for the Magnus expansion

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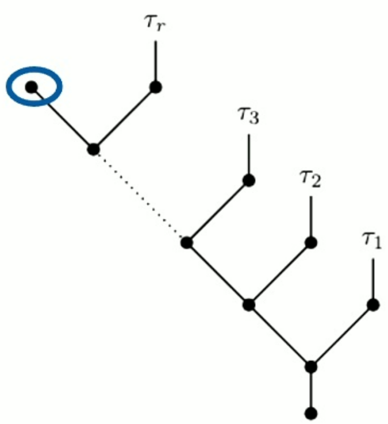
Unique representation through left-ordered subtrees structure



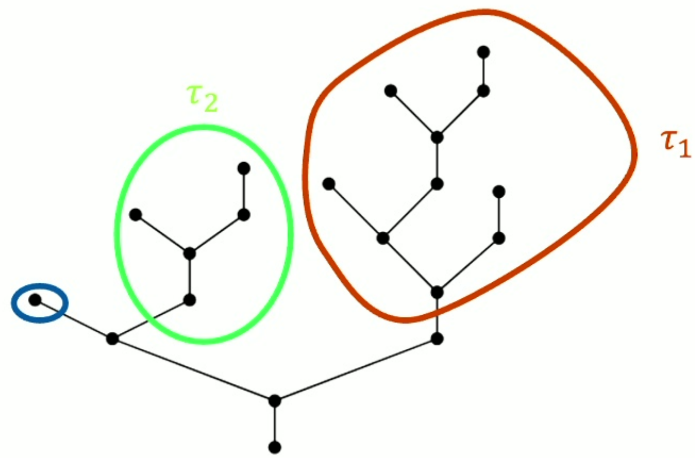
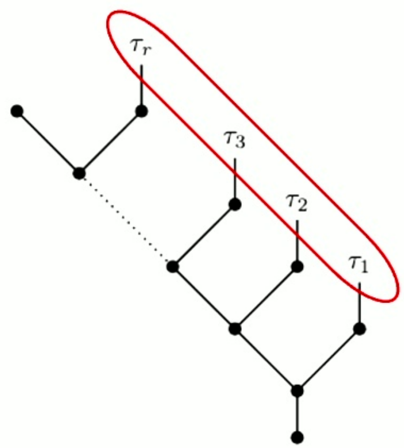


Figure 4: A 7-leaves binary tree.

Unique representation through left-ordered subtrees structure



Iserles, Nørsett

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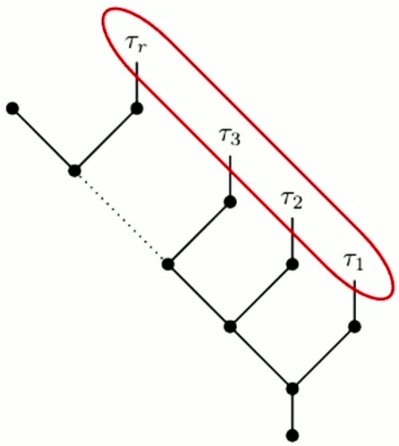
$$\alpha_{(\tau_1,\ldots,\tau_\ell)} = \frac{B_\ell^+}{\ell!} \prod_{i=1}^{\ell} \alpha_{\tau_i} \;,$$

Bernoulli numbers $B_n^{\pm}$

n	fraction	decimal
0	1	+1.000000000
1	$\pm\frac{1}{2}$	±0.500000000
2	$\frac{1}{6}$	+0.166666666
3	0	+0.000000000
4	$-\frac{1}{30}$	−0.033333333
5	0	+0.000000000
6	$\frac{1}{42}$	+0.023809523

Unique representation through left-ordered subtrees structure



Iserles, Nørsett

Can we exploit the same subtree structure to compute **simultaneously** α_τ *and* the integrals?

$$\mathcal{M}(t) = \sum_{n=1}^{\infty} M_n(t) = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_n} \alpha_\tau \int_0^t H_\tau(\kappa) d\kappa,$$

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A general truncation bound of the Magnus expansion

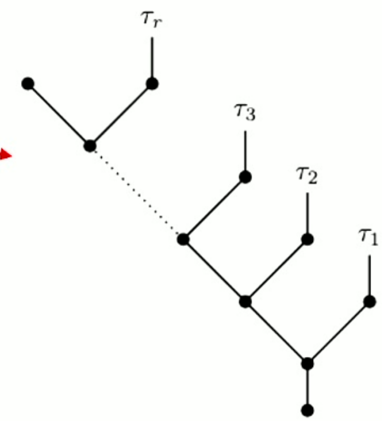
Harriet Apel [*,1], Emilio Onorati [§,2], and Toby Cubitt[1]

[1]Department of Computer Science, University College London, UK
[2]Zentrum Mathematatik, Technische Universität München, DE

Let $\mu(\tau = (\tau_r, \dots, \tau_r))$ be the crude integral defined by the structure of τ, namely,

$$\mu(\tau = (\tau_1, \dots, \tau_r))\, t^n := \int_0^t d\kappa \int_0^{\star} dt_1 \int_0^{\star} dt_2 \dots \int_0^{\star} dt_n$$

Integration end points depend on subtrees structure



Lemma 2 (Integral coefficient recursion formula). *The* integral coefficient *of a binary tree $\tau \in \mathcal{T}_n$ with the sub-trees structure $(\tau_1, \dots, \tau_r)$ is obtained by*

$$\mu(\tau) = \frac{1}{n}\mu(\tau_1)\mu(\tau_2)\cdots\mu(\tau_r)\ .$$

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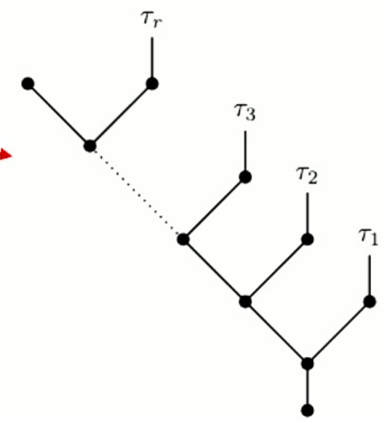

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$$\nu_n := \sum_{\tau \in T_n} |\alpha_\tau| \cdot \mu(\tau)$$

Sum over all trees with N leaves

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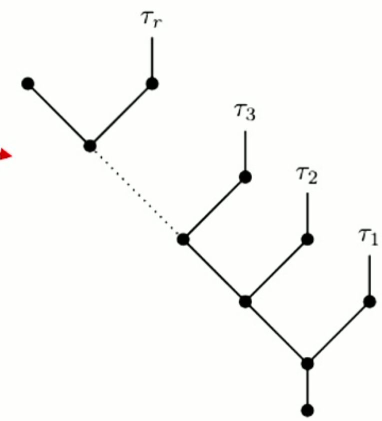

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Sum over all trees with N leaves

Theorem 4. *Let $\nu_0 = 0$. Then the recursion formula for the tree coefficients ν_j is given by*

$$(n+1)\cdot\nu_{n+1} = \sum_{r=1}^{n}\frac{|B_r|}{r!}\sum_{\substack{j_1,\dots,j_r\\ \text{composition}(n,r)}}\prod_{i=1}^{r}\nu_{j_i}\ .$$

Theorem: for all n $\nu_n \leq 8 \cdot \delta_\xi^n \, n^{-2} \, 2^{-n}$

δ_ξ = 0.920075 Convergence radius

S Blanes, F Casas, J A Oteo, and J Ros.
Journal of Physics A (1998)

P. C. Moan, Techincal report, (1998)

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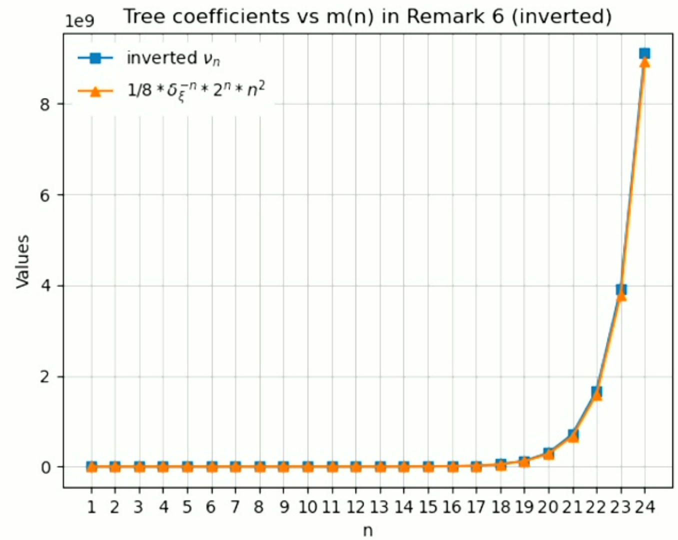
Tree coefficient	Exact value	Approximate value
ν_1	1	1.00000000×10^{0}
ν_2	$\frac{1}{4}$	$2.50000000 \times 10^{-1}$
ν_3	$\frac{5}{72}$	$6.94444444 \times 10^{-2}$
ν_4	$\frac{16513198633691819}{864691128455135232}$	$1.90972222 \times 10^{-2}$
ν_5	$\frac{719074740503489183}{129703669268270284800}$	$5.54398148 \times 10^{-3}$
ν_6	$\frac{531124516054560481}{311288806243848683520}$	$1.70621142 \times 10^{-3}$
ν_7	$\frac{1066195936782552629}{1906643938243573186560}$	$5.59200339 \times 10^{-4}$
ν_8	$\frac{8994169792961911357}{46485795065748070072320}$	$1.93482112 \times 10^{-4}$
ν_9	$\frac{21947865537739751284 9}{313779116693799472988 1600}$	$6.99468651 \times 10^{-5}$
ν_{10}	$\frac{1824179791028592169 31}{6972869259862210510848000}$	$2.61611070 \times 10^{-5}$



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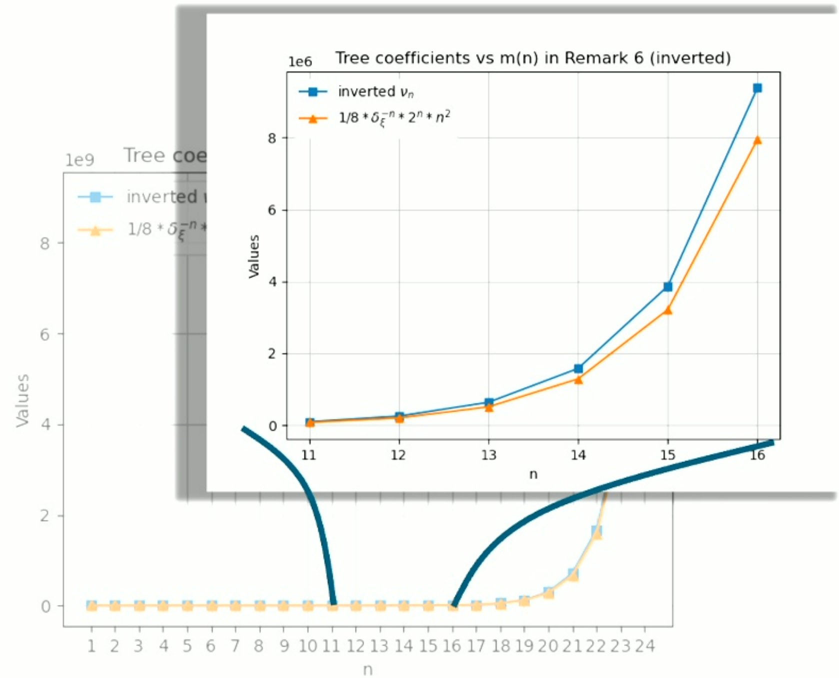

... now back to the main result for **arbitrary Hamiltonians**!

Lemma 7 (Upper bound for Magnus term). *Let $H(t)$ be a general time-dependent Hamiltonian with time evolution operator $U(t) = \exp(\mathcal{M}(t))$ generated by the Magnus expansion $\mathcal{M}(t) = \sum_{n=1}^{\infty} M_n(t)$. Then operator norm of the n-th term in the Magnus expansion is upper bounded by,*

$$\|M_n(t)\| = \left\| \sum_{\tau \in T_n} \alpha_\tau \int_0^t H_\tau(\kappa) d\kappa \right\| \leq 2^{n-1} (h_{\max} t)^n \nu_n$$

for all $n \geq 1$, where $h_{\max} := \max_{t' \in [0,t]} \{\|H(t')\|_\infty\}$.

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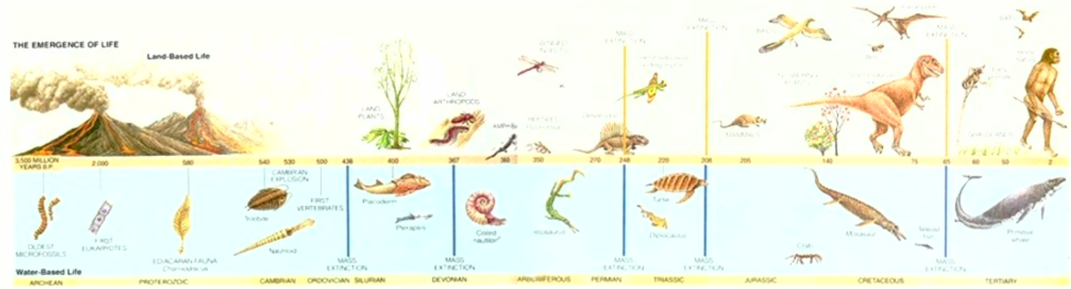
Error truncation of the Magnus expansion at the *N* term

$$\left\|\mathcal{M}(t) - \mathcal{M}^{(N)}(t)\right\| \leq \frac{4}{(N+1)^2}\,\frac{(\delta_\xi\, h_{\max} t)^{N+1}}{1 - \delta_\xi\, h_{\max} t}\;,$$

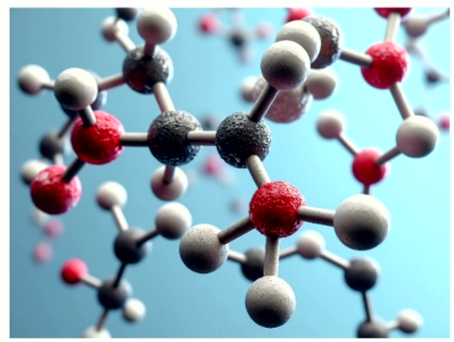
Exponential decay within the convergence radius δ_ξ coupled with $1/N^2$ factor

Approximate time evolutions



Applications?

Efficient prediction
for
materials behavior



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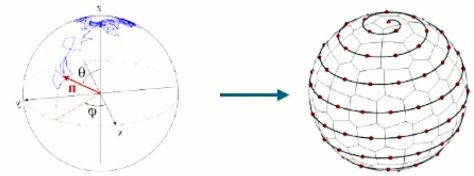
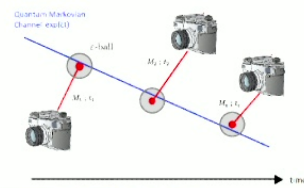
Wrap-up



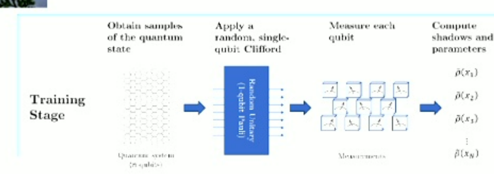
- Explored mixing properties of random generators and spectral gaps



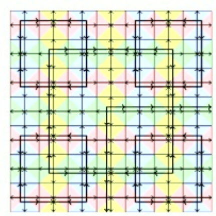
- Presented a working algorithm for the quantum embedding problem – overcoming the problem of degeneracies

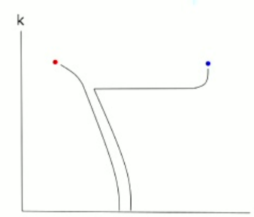
- A new bound on the Magnus expansion with binary trees

... and there is more



- Quantum learning: predicting properties within dissipative phases of matters





- Uncomputably complex Renormalization Group flows

Thank you!

Harriet Apel
PsiQuantum

Tamara Kohler
Stanford

Toby Cubitt
UCL

Technical University of Munich

Albert H. Werner
University of Copenhagen

Martin Kliesch
Hamburg University

Jens Eisert
FU Berlin

Oliver Buerschaper   Winton Brown