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Quantum groups and cohomological Donaldson-Thomas theory

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Cohomological DT theory

- Given any projective 3 CY variety X, indivisible Chern class $\alpha \in \mathrm{H}^{\mathrm{even}}(X, \mathbb{Z})$ and a generic stability condition ζ, the Donaldson-Thomas invariants DT_α^ζ defined by Thomas in 2000 "counts" semistable sheaves on X of given Chern class α.
- They are defined by taking the degree of the virtual fundamental class of the moduli space of sheaves $\mathcal{M}_\alpha^\zeta(X)$. Given any scheme Y, Behrend in 2005, defined a constructible function $\nu_Y : Y \to \mathbb{C}$ and showed that DT_α^ζ is precisely the weighted Euler characteristic $\chi(\mathcal{M}_\alpha^\zeta(X), \nu_{\mathcal{M}_\alpha^\zeta(X)})$.
- When Y can be written as the critical locus of a function $f : \widetilde{Y} \to \mathbb{C}$ where $\widetilde{Y}$ is smooth, then there exists the vanishing cycle sheaf φ_f which satisfies

$$\sum_{i \geq 0} (-1)^i \dim \mathrm{H}^i(\widetilde{Y}, \varphi_f) = \chi(Y, \nu_Y).$$

- In cohomolgical DT theory, inititated by Konstevich-Soibelman; one studies the cohomology $\mathrm{H}(Y, \varphi_f)$ itself. Jacobi algebra provides local

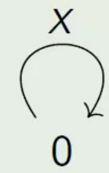
Quivers

- A Quiver $Q = (Q_0, Q_1)$ is a directed graph with vertices Q_0 and edges Q_1.

Quivers

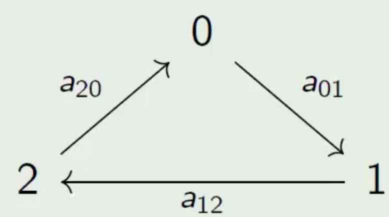
Example

Jordan Quiver

$Q_{\text{Jor}} =$ 

Example

Cyclic quiver of length 3:

$Q^2 =$ 

Q^K: cyclic quiver of length $K + 1$.

Quivers

- A Quiver $Q = (Q_0, Q_1)$ is a directed graph with vertices Q_0 and edges Q_1.
- Let $\mathbb{C}Q$ be it's path algebra.

Example

$\mathbb{C}Q_{\mathrm{Jor}} = \mathbb{C}[X]$

- The moduli stack of $\mathbf{d} = (\mathbf{d}_i)_{i \in Q_0}$ dimensional representation of $\mathbb{C}Q$ is

$$\mathfrak{M}_{\mathbf{d}}(Q) = \mathrm{Rep}_{\mathbf{d}}(\mathbb{C}Q) / \prod_{i \in Q_0} \mathrm{GL}(\mathbf{d}_i) = \prod_{\alpha : i \to j} \mathrm{Hom}(\mathbb{C}^{\mathbf{d}_i}, \mathbb{C}^{\mathbf{d}_j}) / \prod_{i \in Q_0} \mathrm{GL}(\mathbf{d}_i)$$

Example

$$\mathfrak{M}_d(Q_{\mathrm{Jor}}) = [\mathrm{End}(\mathbb{C}^d) / \mathrm{GL}(d)]$$

where $g \cdot X = gXg^{-1}$.

Jacobi Algebras

- Given any quiver $Q = (Q_0, Q_1)$, let $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ be the potential, i.e a linear combination of cyclic elements in the path algebra.
- Let $a \in Q_1$ then for any cycle W we define the derivative

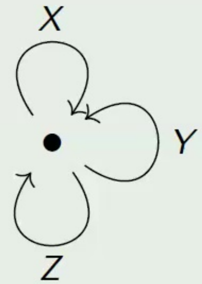
$$\frac{\partial W}{\partial a} = \sum_{W=cab} bc$$

and we extend the definition to any potential.

- Then the Jacobi algebra is defined as

$$\mathrm{Jac}(Q, W) := \mathbb{C}Q/\langle \frac{\partial W}{\partial a}, a \in Q_1 \rangle$$

Example



For $Q = \mathrm{Jor}_3 = \widetilde{Q_{\mathrm{Jor}}}$ and $\widetilde{W} = X[Y, Z]$, $\mathrm{Jac}(\widetilde{Q_{\mathrm{Jor}}}, \widetilde{W}) = \mathbb{C}[X, Y, Z]$.
Note that $\mathfrak{M}_d(\mathrm{Jac}(\widetilde{Q_{\mathrm{Jor}}}, \widetilde{W})) \simeq \mathrm{Coh}_n(\mathbb{C}^3)$.

Example (Tripled quiver with Canonical Potential)

For any quiver Q, we may consider the quiver $\widetilde{Q}$ formed by doubling $\overline{Q}$ and then adding loops on each vertex. We choose $\widetilde{W} = (\sum_{i \in Q_0} \omega_i)(\sum_{a \in Q_1} [a, a^*])$. Then we have isomorphism $\mathrm{Jac}(\widetilde{Q}, \widetilde{W}) \simeq \Pi_Q[\omega]$ where $\omega = \sum_i \omega_i$.

Example (Tripled cyclic quiver)

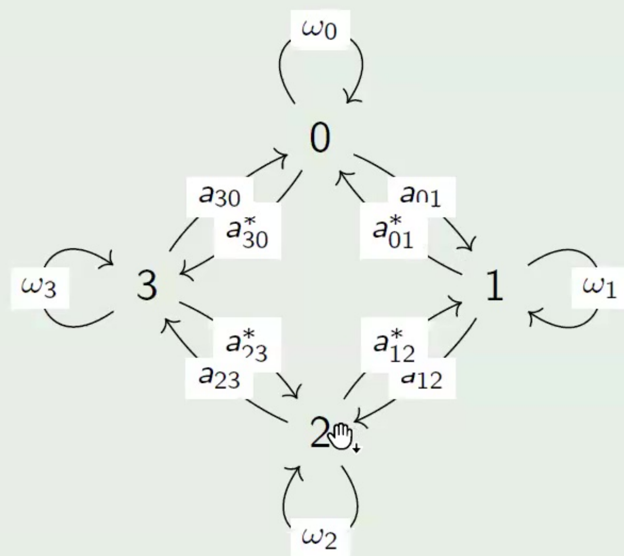


Figure: $\widetilde{Q^3}, \widetilde{W^3} = \sum_{i \in \mathbb{Z}/4\mathbb{Z}} \omega_i (a_{i,i+1}^* a_{i,i+1} - a_{i,i-1} a_{i,i-1}^*)$

Example (ALE surface)

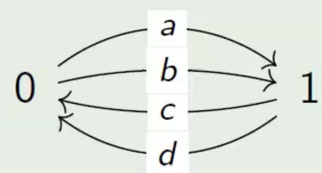
Let $\mathbb{Z}_{K+1} \subset \mathrm{SL}_2(\mathbb{C})$, where $(X, Y) \mapsto (\omega X, \omega^{-1} Y)$. Let $S_K \to \mathbb{C}^2/\mathbb{Z}_{K+1}$ be the unique minimal resolution. Then we have derived McKay correspondence due to [Kapranov-Vasserot 1998]

$$\Phi : D^{\mathrm{b}}(\mathrm{Coh}_c(S_K \times \mathbb{C})) \simeq D^{\mathrm{b}}(\mathrm{Jac}(\widetilde{Q^K}, \widetilde{W}) - \mathrm{mod}).$$

Example (Resolved conifold)

Let

$Q^{\mathrm{con}} =$



be a quiver with the Klebanov–Witten potential given by $W_{\mathrm{KW}} = acbd - adbc$. Then there is a derived equivalence

$$D^{\mathrm{b}}(\mathrm{Jac}(Q^{\mathrm{con}}, W_{\mathrm{KW}})) \simeq D^{\mathrm{b}}(\mathrm{Coh}_c(Y_{\mathrm{con}})).$$

where $Y_{\mathrm{con}} = \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ is the resolved conifold.

Kontsevich-Soibelman cohomological Hall algebra

- For any smooth X with a function $f : X \to \mathbb{C}$, there exist perverse sheaf φ_f, such that $\mathrm{Supp}(\varphi_f) = \mathrm{Crit}(f)$. Vanishing cycle measures singularity of $\mathrm{crit}(f)$.
- Given a quiver Q with potential W, note that $\mathrm{Tr}(W) : \mathfrak{M}_{\mathbf{d}}(Q) \to \mathbb{C}$, $\mathrm{Crit}(\mathrm{Tr}(W)) = \mathfrak{M}_{\mathbf{d}}(\mathrm{Jac}(Q, W))$.
- Kontsevich and Soibelman [2008] constructed an algebra product on the critical cohomology, as the algebra of BPS states

$$\mathcal{A}_{Q,W} = \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathrm{H}(\mathfrak{M}_{\mathbf{d}}(Q), \varphi_{\mathrm{Tr}(W)})$$

- When $Q = \widetilde{Q_{\mathrm{Jor}}}$ and $W = X[Y, Z]$ then

$$\mathrm{H}(\mathfrak{M}_n(\widetilde{Q_{\mathrm{Jor}}}), \varphi_{\mathrm{Tr}(W)}) = \mathrm{H}(\mathrm{Coh}_n(\mathbb{C}^3), \varphi_{\mathrm{Tr}(W)})$$

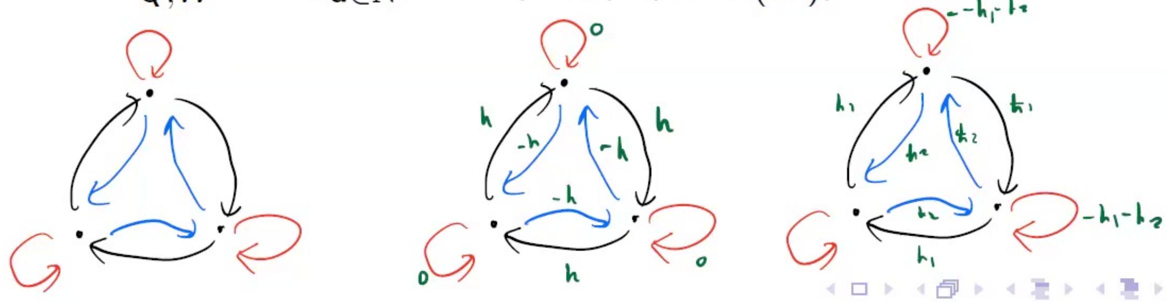
studies Cohomological DT theory of $\mathbb{A}^3$.

Algebra structure

- Defined by Thom-Sebastiani isomorphism, pullback and pushforward in the vanishing cycle cohomology

$$\mathfrak{M}_{\mathbf{d}}(Q) \times \mathfrak{M}_{\mathbf{e}}(Q) \xleftarrow{p_1 \times p_3} \mathrm{Ext}_{\mathbf{d},\mathbf{e}}(\mathfrak{M}(Q)) \xrightarrow{p_2} \mathfrak{M}_{\mathbf{d}+\mathbf{e}}(Q)$$

- For any torus T, which leaves potential invariant, one can similarly define $\mathcal{A}^T_{Q,W} = \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}^T(\mathfrak{M}_{\mathbf{d}}(Q), \varphi_{\mathrm{Tr}(W)})$.



Relation to preprojective CoHA and MO Yangian

- We have a dimension reduction isomorphism due to Davison[2015] and Konstevich-Soibelman (as vector spaces), giving an algebra isomorphism:

$$\mathcal{A}^T_{\tilde{Q},\widetilde{W}} \simeq \mathcal{A}^T_{\Pi_Q} := \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}^{BM}_T(\mathfrak{M}_{\mathbf{d}}(\Pi_Q), \mathbb{Q}^{\mathrm{vir}})$$

where $\mathcal{A}^T_{\Pi_Q}$ also has an algebra structure, defined by Schiffmann-Vasserot for Jordan quivers in the proof of AGT conjecture.

Theorem (Botta-Davison 2023, Schiffmann-Vasserot 2023)

For any quiver Q, there is an isomorphism of algebras

$$\mathcal{A}^T_{\Pi_Q} \simeq \mathbf{Y}^{\mathrm{MO},+}_Q$$

where T is any torus acaling the symplectic form.

- Only known for type ADE and Jordan quiver.

Affine Yangian of $\mathfrak{gl}_K$: $\mathbf{Y}^{(K)}_{\hbar_1,\hbar_2}, K \geq 2$

- Defined by Nicolas Guay in 2007.
- $\mathbf{Y}^{(K)}_{\hbar_1,\hbar_2}$ deform certain universal central extension of $\mathbf{U}(\mathfrak{sl}_K[u^{\pm 1}, v])$.
- They have a triangular decomposition and the positive half is defined as a $\mathbb{C}[\hbar_1, \hbar_2]$ algebra generated by elements $X^+_{i,r}$ where $i \in \mathbb{Z}/K\mathbb{Z}; r \geq 0$ with specific relations.
- Let $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ be a torus action, where first copy acts on arrows a by weight $(1,0)$, arrow a^* by weight $(0,1)$ and loops ω_i with weight $(-1,-1)$. There exist an algebra morphism

$$\Phi : \mathbf{Y}^{(K+1),+}_{\hbar_1,\hbar_2} \to \mathcal{A}^{\mathbb{T}}_{\widetilde{Q^K},\widetilde{W}}$$

  defined by $X^+_{i,r} \mapsto \alpha^r_i$, where $\alpha^r_i = u^r[\mathrm{pt}_i] \in \mathrm{H}^T(\mathfrak{M}_{\delta_i}(\widetilde{Q^K}), \varphi_{\mathrm{Tr}(\widetilde{W})}) \simeq \mathrm{H}_T(\mathrm{pt})[u], \delta_i = (\underbrace{0, \cdots, 0}_{i\ 0s}, 1, 0, \cdots, 0)$.
- This morphism is not a surjection, since there are elements of negative degrees in $\mathcal{A}^{\mathbb{T}}_{\widetilde{Q^K},\widetilde{W}}$, $\gamma_\delta := [S_K] \in \mathrm{H}^{\mathrm{BM}}_T(\mathfrak{M}_{(1,1,\cdots,1)}(\Pi_{Q^K}), \mathbb{Q}^{\mathrm{vir}})$ of cohomological degree -2.

Embedding to Shuffle Algebra

- Then usual properties of vanishing cycles gives a morphism of graded vector spaces $\mathrm{H}^{(T)}(\mathfrak{M}_{\mathbf{d}}(\widetilde{Q^K}), \varphi_{\mathrm{Tr}(\widetilde{W})}) \to \mathrm{H}^{(T)}(\mathfrak{M}_{\mathbf{d}}(\widetilde{Q^K}), \mathbb{Q}^{\mathrm{vir}})$.
- This morphism upgrades to a morphism of graded algebras

$$i^{(T)} : \mathcal{A}^{(T)}_{\widetilde{Q^K},\widetilde{W}} \to \mathcal{A}^{(T)}_{\widetilde{Q^K}}.$$

- i is not always an injection however i is an embedding when $T = \mathbb{T}$.

$$\begin{array}{ccc} \mathcal{A}^{(\mathbb{T})}_{\widetilde{Q^K},\widetilde{W}} & \xhookrightarrow{\ i\ } & \mathcal{A}^{(\mathbb{T})}_{\widetilde{Q^K}} \\ \Phi \uparrow & \nearrow & \\ \mathbf{Y}^{(K+1),+}_{\hbar_1,\hbar_2} & & \end{array}$$

Localized Algebras

- Thus it is enough to understand the image of $i^{\mathbb{T}}$.
- After localization, we have that

Theorem (Schiffmann-Vasserot 2017, Negut 2023)

$\mathcal{A}^{\mathbb{T}}_{\widetilde{Q},\widetilde{W}} \otimes_{\mathbb{C}[\hbar_1,\hbar_2]} \mathbb{C}(\hbar_1,\hbar_2)$ *is spherically generated. i.e generated by dimension vectors* $\delta_i = (\underbrace{0,\cdots,0}_{i\ 0s},1,0,\cdots,0) \in \mathbb{N}^{Q_0}$.

- Yang-Zhao 2018 for ADE, Rapcak-Soibelman-Yang-Zhao 2018 for Jordan quiver study localized algebras and relate them to Yangians. One can similarly show that $\Phi \otimes \mathbb{C}(\hbar_1,\hbar_2)$ is an isomorphism.
- However, without localizing, these algebras are typically (Except ADE and Jordan quiver) NOT spherically generated. These moduli spaces are singular, localizing looses lot of geometry.

Integral form $\mathbf{Y}_{\hbar_1,\hbar_2}^{\mathrm{IntCoHA}(n),+}$

Consider the $\mathbb{C}[\hbar_1, \hbar_2]$ algebra generated by elements $X^+_{i,r}$ where $i = 0, \cdots, K; r \geq 0$ and K_r where $r \geq 0$ such that

$$
\begin{aligned}
[X^+_{i,r+1}, X^+_{i+1,s}] - [X^+_{i,r}, X^+_{i+1,s+1}] &= (\hbar_1 + \hbar_2/2)[X^+_{i,r}, X^+_{i+1,s}] - \hbar_2/2\{X^+_{i,r}, X^+_{i+1,s}\} \\
[X^+_{i,r+1}, X^+_{i-1,s}] - [X^+_{i,r}, X^+_{i-1,s+1}] &= -(\hbar_1 + \hbar_2/2)[X^+_{i,r}, X^+_{i-1,s}] - \hbar_2/2\{X^+_{i,r}, X^+_{i-1,s}\} \\
[X^+_{i,r+1}, X^+_{i,s}] - [X^+_{i,r}, X^+_{i,s+1}] &= \hbar_2(X^+_{i,r}X^+_{i,s} + X^+_{i,s}X^+_{i,r}) \\
[X^+_{i,r}, X^+_{j,s}] &= 0 \ \forall |i-j| > \\
\mathrm{Sym}_{r_1,r_2}[X^+_{i,r_1}, [X^+_{i,r_2}, X^+_{i+1,s}]] &= 0 \\
\hbar_1\hbar_2 K_r &= T^r((\hbar_1 + \hbar_2)Y_K - Z_K) \\
Y_K &= \sum X_{i,0}[X_{i+1,0}, [X_{i+2,0}, \cdots, X_{i-1,0}]] \\
Z_K &= \sum [X_{i,0}, [X_{i+1,1}, [X_{i+2,0}, [\cdots, X_{i-1,0}]]]]
\end{aligned}
$$

where $T(X^+_{i,r}) = X^+_{i,r+1}$, $T(X^+_{i,r}X^+_{j,s}) = X^+_{i,r+1}X^+_{j,s} + X^+_{i,r}X^+_{j,s+1}$

Theorem

For $K \geq 2$, we have an isomorphism of algebras

$$\mathbf{Y}^{\mathrm{IntCoHA},(K+1),+}_{\hbar_1,\hbar_2} \to \mathcal{A}^{\mathbb{T}}_{\widetilde{Q^K},\widetilde{W}}$$

given by $X^{+}_{i,r} \to \alpha^r_i, K_r \mapsto u^r \gamma_\delta$.

- How? By a purity result, $\mathcal{A}^{\mathbb{T}}_{\widetilde{Q^K},\widetilde{W}}$ is flat deformation of $\mathcal{A}_{\widetilde{Q^K},\widetilde{W}}$. We first study $\mathcal{A}_{\widetilde{Q^K},\widetilde{W}}$ which allows to give minimal generators of $\mathcal{A}^{\mathbb{T}}_{\widetilde{Q^K},\widetilde{W}}$ and compute the image of $i^{\mathbb{T}}$.

Cohomological Integrality and PBW theorem[Davison-Meinhardt 2016]

- For any symmetric Q and potential W, there exist a (perverse) filtration $\mathfrak{P}$ on $\mathcal{A}_{Q,W}^{(T)}$ such that we have cohomologically graded PBW isomorphism

$$\mathrm{Sym}\left(\mathfrak{g}_{Q,W}^{\mathrm{BPS}} \otimes \mathbb{C}[u]\right) \simeq \mathrm{Gr}_{\mathfrak{P}}(\mathcal{A}_{Q,W}^{(T)})$$

  - The BPS Lie algebra $\mathfrak{g}_{Q,W}^{\mathrm{BPS},(T)} \hookrightarrow \mathcal{A}_{Q,W}^{(T)}$ is a sub-lie algebra closed under commutator Lie bracket.
  - The $\mathbb{C}[u]$ action is given by the determinant line bundle: for each vertex i, we have a tautological bundle $\mathcal{R}_i$ on each vertex

$$\mathrm{Det} : \mathfrak{M}_{\mathbf{d}}^{(T)}(\mathbb{C}Q) \to \mathrm{pt}/\mathbb{C}^*, \mathcal{R}_i \mapsto \det(\mathcal{R}_i)$$

  u acts by $\sum c_1(\mathcal{R}_i)$.
- $\mathfrak{g}_{Q,W}^{\mathrm{BPS},(T)} \otimes \mathbb{C}[u] \hookrightarrow \mathcal{A}_{\tilde{Q},\widetilde{W}}^{T}$ is not always closed under the Lie bracket.

Affinized BPS Lie Algebra [Davison-Kinjo]

- Note that $\mathrm{Jac}(\widetilde{Q}, \widetilde{W}) \simeq \Pi_Q[\omega]$, we can analyze generalized eigenvalues of ω.
- For U an open disc of $\mathbb{A}^1$, let $\mathfrak{M}_{\mathbf{d}}^U(\widetilde{Q})$ be the open substack where generalized eigenvalues of ω_i are in $U \subset \mathbb{A}^1$. Thus we have

$$\mathrm{H}(\mathfrak{M}_{\mathbf{d}}(\widetilde{Q}), \varphi_{\mathrm{Tr}(W)}) \simeq \mathrm{H}(\mathfrak{M}_{\mathbf{d}}^U(\widetilde{Q}), \varphi_{\mathrm{Tr}(W)_{|U}})$$

- If U_1 and U_2 are two disjoint open sets, then we have a version of Thom-Sebastiani implying

$$\mathrm{H}(\mathfrak{M}^{U_1 \amalg U_2}(\widetilde{Q}), \varphi_{\mathrm{Tr}(W)}) \simeq \mathrm{H}(\mathfrak{M}^{U_1}(\widetilde{Q}), \varphi_{\mathrm{Tr}(W)}) \otimes \mathrm{H}(\mathfrak{M}^{U_2}(\widetilde{Q}), \varphi_{\mathrm{Tr}(W)})$$

- This gives a cocommutative coproduct on $\mathcal{A}_{\widetilde{Q},\widetilde{W}}$ which means $\mathcal{A}_{\widetilde{Q},\widetilde{W}} \simeq \mathbf{U}(\hat{\mathfrak{g}}^{\mathrm{BPS}}_{\widetilde{Q},\widetilde{W}})$ for some lie algebra $\hat{\mathfrak{g}}^{\mathrm{BPS}}_{\widetilde{Q},\widetilde{W}}$.
- As a vector space $\mathfrak{g}^{\mathrm{BPS}}_{\widetilde{Q},\widetilde{W}}[u] \simeq \hat{\mathfrak{g}}^{\mathrm{BPS}}_{\widetilde{Q},\widetilde{W}}$ but not as a Lie algebra.

Cyclic quivers

- We have a relation with Kac plynomial[Davison]

$$\chi_{q^{1/2}}(\mathfrak{g}^{\mathrm{BPS}}_{\widetilde{Q},\widetilde{W}}) := \sum_i (-1)^i (q^{1/2})^i \dim \mathrm{H}^i(\mathfrak{g}^{\mathrm{BPS}}_{\widetilde{Q},\widetilde{W},\mathbf{d}}) = a_{Q,\mathbf{d}}(q^{-1})$$

  where $a_{Q,\mathbf{d}}(q)$ is the Kac polynomial.
  This gives generators $\alpha_i^{(r)} := u^r \cdot \alpha_{\delta_i}$ of cohomological degrees $2r, r \geq 0$ and $\gamma^{(r)}_{k\cdot\delta} := u^r \cdot \gamma_{k\cdot\delta}, k \geq 1$ of cohomology degrees $2r-2, r \geq 0$, where $\delta = (1, 1, \cdots, 1)$ is the imaginary root.
- There is an action of $\mathcal{A}_{\widetilde{Q^K},\widetilde{W}}$ on the equivariant Hilbert scheme $\oplus_{V\in R(\mathbb{Z}_{K+1})}\mathrm{H}(\mathrm{Hilb}^V(\mathbb{C}^2), \mathbb{Q})$, and there is an action of $\mathcal{A}^{\mathrm{Im}}_{\widetilde{Q^K},\widetilde{W}} := \oplus_{k\geq 0}\mathrm{H}(\mathfrak{M}_{k\cdot\delta}(\widetilde{Q^K}), \varphi_{\mathrm{Tr}(\widetilde{W})})$ on the Hilbert scheme $\bigoplus_{k\geq 0}\mathrm{H}(\mathrm{Hilb}^k(S_K), \mathbb{Q})$.
- Geometric McKay corrrepondence and Lehn relations allows us to show that $\mathcal{A}_{\Pi_{Q^K}}$ is generated by $\alpha_i^{(r)}$ and $\gamma_\delta^{(r)}$ where $r \geq 0$ with explicit relations.

- Let $\widetilde{W}_K$ be the Lie algebra generated by $T_{k,a}(X)$ where $X \in \mathfrak{sl}_K$ and $t_{k,a}$ where $k \in \mathbb{Z}$ and $a \geq 0$ with the relations

$$[t_{m,a}, T_{n,b}(X)] = (na - mb)T_{m+n,a+b-1}(X)$$
$$[t_{m,a}, t_{n,b}] = (na - mb)t_{m+n,a+b-1}$$
$$[T_{m,a}(X), T_{n,b}(Y)] = T_{m+n,a+b}([X, Y])$$

Theorem

Let Q^K be the cyclic quiver of length $K+1$. Then there is an isomorphism of algebras

$$\mathcal{A}_{\widetilde{Q^K},\widetilde{W}} \simeq \mathbf{U}(\widetilde{W}^+_{K+1})$$

- This provides enough relations to compute the image of $i : \mathcal{A}^T_{\widetilde{Q^K},\widetilde{W}} \to \mathcal{A}^T_{\widetilde{Q^K}}$, allowing us to prove previous theorem. We can also consider a $\mathbb{C}^*$ deformation.

- Let $\hbar$ be a formal variable and let $D_\hbar(\mathbb{C}^*)$ be the algebra of $\hbar-$differential operators on $\mathbb{C}^*$. It is defined as a unital associative $\mathbb{C}[\hbar]$ linear algebra generated by $z^{\pm 1}, D$ subject to the relations:

$$Dz = z(D+\hbar),\ z^{+1}z^{-1} = z^{-1}z^{+1} = 1. \tag{1}$$

- Let $D_\hbar(\mathbb{C}^*) \otimes \mathfrak{gl}_K$ be the algebra of polynomial differential operators on $\mathbb{C}^*$ with coefficients in $\mathfrak{gl}_K$. This forms a $\mathbb{C}[\hbar]$ Lie algebra with basis $z^n D^a X, n \in \mathbb{Z}, a \geq 0, X \in \mathfrak{gl}_K$.
- Let

$$\mathcal{W}_K \subset (D_\hbar(\mathbb{C}^*) \otimes \mathfrak{gl}_K) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{\pm 1}]$$

be the $\mathbb{C}[\hbar]$ linear subspace spanned by $T_{k,a}(X), X \in \mathfrak{gl}_K$ and

$$t_{k,a} := T_{k,a}(1)/\hbar$$

where $k \in \mathbb{Z}, a \geq 0, X \in \mathfrak{gl}_K$ and by 1, we mean the identity matrix $\mathrm{Id} \in \mathfrak{gl}_K$.
- $\mathcal{W}_K$ is closed under Lie bracket and $\lim_{\hbar \to 0} \mathcal{W}_K = W_K$.

Deformed Affinized

It is shown by Tsymabliuk [2016] that the classical limit of affine Yangian $\mathbf{Y}^{(K)}_{\hbar_1,\hbar_2}$ is $\mathbf{U}_{\mathbb{C}[\hbar]}(D_\hbar(\mathbb{C}^*) \otimes \mathfrak{gl}_K)$. Since equivariant CoHA $\mathcal{A}^{\mathbb{C}^*}_{\Pi_Q}$ is a flat deformation of $\mathcal{A}_{\Pi_Q}$, this gives

Theorem

Let Q^K be cyclic quiver of length $K+1$. Assume $K \geq 2$. Let $\mathbb{C}^*$ be a torus acting on the arrows a with weight 1 and arrows a^* with weight -1 and weight 0 on the arrows. Then there is an isomorphism of algebras

$$\mathcal{A}^{\mathbb{C}^*}_{\widetilde{Q^K},\widetilde{W}} \simeq \mathbf{U}_{\mathbb{C}[\hbar]}(\mathcal{W}^+_{K+1})$$

3d Quantum Groups

$$
\begin{array}{lccc}
2d: & Y_Q^{\mathrm{MO},+} & \xrightarrow{\simeq} & \mathcal{A}^{(T)}_{\Pi_Q} \\
 & & & \uparrow \text{dimension reduction when}(Q,W)=(\widetilde{Q},\widetilde{W}) \\
3d: & ?? & & \mathcal{A}^{(T)}_{Q,W}
\end{array}
$$

Ongoing work: Super Quantum Groups

- The Borel-Moore homology of the moduli space of representations of preprojective algebra is all even dimensional.
- Similarly the construction of the Maulik-Okounkov Yangian is done by considering cohomology of Nakajima quiver varieties, which are all even dimensional.
- If one wants to construct super quantum groups geometrically, one needs to fermionize these constructions.
- A way to do so is by considering "mass terms" or quadratic terms to the potentials. One can then aim to define 'critical stable envelopes' using BPS cohomology. The Jacobi algebra of resolved conifold can be obtained by doing fermionizing construction to Q^1 quiver.
- In an ongoing work of Ben Davison, we realize Lie superalgebras as example of BPS Lie algebras and we study the cohomological Hall algebra of resolved conifold which is expected (Kelvin Costello) to be isomorphic to the positive Half of the Affine Yangian of $\mathfrak{gl}(1,1)$.

Thanks :)