Title: Asymptotic symmetries and algebras: a review with emphasis on gravity

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Abstract:

A review of asymptotic (more generally, boundary) symmetries will be given in the context of the Hamiltonian formulation. General features (such as the form of the symmetry generators and the structure of the algebra) as well as specific examples will be covered. A particular attention will be paid to asymptotically flat spaces and the asymptotic BMS algebra, where nonlinear redefinitions will be shown to yield a supertranslation-invariant angular momentum.

Asymptotic symmetries and algebras: a review with emphasis on gravity

Marc Henneaux
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Perimeter Institute, 30 April 2025

Asymptotic symmetries

What is an asymptotic symmetry?

And why are they important?

Perhaps the best example of an asymptotic symmetry arises in the context of defining a meaningful energy in general relativity,

The problem of the energy in general relativity

Providing a physically meaningful definition of the energy in general relativity is a subtle question.

The problem is the following :

the energy of a system is the value of the generator of time translations.

But which time should we use ?

General relativity is invariant under general changes of coordinates ("diffeomorphisms"), which can involve arbitrary time reparametrizations.

The time coordinate has in general no direct physical meaning.

The problem of the energy in general relativity

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This problem has been investigated and understood by many authors,

leading to the following conclusion :

"The spacetime must be asymptotically flat (i.e., approach Minkowski space at large spatial distances) if there is to be any possibility of defining energy and angular momentum."

(C.W. Misner, K.S. Thorne, J.A. Wheeler, in "Gravitation" (1973)).

The problem of the energy in general relativity

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Why can we define a meaningful energy when spacetime is asymptotically flat, i.e., becomes identical with Minkowski space asymptotically at spatial infinity?

A particularly lucid account of the question has been given by L. Faddeev in Sov. Phys. Usp. 25(3), March 1982.

With the asymptotic Minkowskian structure, one can define meaningful "time translations" :

"The asymptotic condition that spacetime should be flat at large distances makes it possible to define a dynamic displacement in time as a displacement with respect to an observer far from the gravitating matter, and to associate energy with the displacement."

In contrast, for gravity in spaces with no boundary or asymptotic region, "there is no natural time displacement and accordingly no meaningful energy concept."

The problem of the energy in general relativity

We thus come to the conclusion that in general relativity, there are diffeomorphisms that do not correspond to a mere redundancy in the description of the system,

but have a physical significance.

In particular, spacetime diffeomorphisms that go asymptotically to a translation in the asymptotic Minkowskian time

should be regarded as physical transformations.

These are examples of “asymptotic symmetries".

Asymptotic symmetries

More generally, the gauge symmetries of an arbitrary gauge system

can be "proper", corresponding to a redundancy in the description of the system,

or "improper", in which case they do have a non-trivial physical action.

An example of improper gauge transformations is, as we just saw, given by diffeomorphisms that go to an asymptotic time translation in the asymptotically flat context,

with non trivial corresponding conserved quantity given by the (ADM) energy.

Asymptotic symmetries

Both proper and improper gauge symmetries take the same form (e.g., $\delta_\epsilon A_i = \partial_i \epsilon$ in electromagnetism).

So, what makes improper gauge symmetries different from proper ones?

The distinction between proper and improper gauge symmetries can only be made if there is a boundary

which can be at infinity (as we shall assume here for definiteness).

In that case, one speaks about "asymptotic symmetries".

It is the behaviour of the gauge parameters at infinity (at the boundary) that makes the difference.

Proper versus improper gauge symmetries

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The distinction between proper and improper gauge symmetries is particularly clear in the Hamiltonian formalism.

Gauge systems are characterized by (first class) constraints (Dirac).

Proper gauge symmetries have generators that weakly vanish (proportional to the constraints),

while improper ones have non-vanishing generators, weakly equal to a non-zero boundary term.

One sometimes call improper gauge transformations “large gauge transformations" because the corresponding gauge parameters usually do not vanish at the boundary,

while the gauge parameters usually vanish for proper gauge transformations.

Proper versus improper gauge symmetries

Because they act non-trivially on the physical states, improper gauge transformations provide a lot of information on the system.

This is particularly so when the improper gauge symmetries form an infinite-dimensional algebra, of which the physical states provide (non-trivial) representations.

One example is AdS gravity in $2+1$ dimensions, where the asymptotic symmetry algebra is (twice) the Virasoro algebra, with central charge that provides insightful information on the number of physical states.

Another example is $3+1$ gravity in the asymptotically flat context,

where the group of asymptotic symmetries is infinite-dimensional

(Bondi-Metzner-Sachs or"BMS" group).

Purpose of talk

The purpose of this lecture is to review general properties of asymptotic symmetries.

Our strategy will be to treat explicitly gravity in the asymptotic flat case,

phrasing the concepts in a way that the method holds in fact for all gauge systems.

We shall rely on Hamiltonian methods.

ADM action

A central role in the analysis is played by the gravitational action which reads, in Hamiltonian form,

$$S[g_{ij},\pi^{ij},N,N^i] = \int dt \left\{ \int d^3x \left(\pi^{ij}\partial_t g_{ij} - N^i \mathscr{H}_i^{grav} - N\mathscr{H}^{grav} \right) - B_\infty \right\}$$

where B_∞ is a boundary term at infinity and where

$$\mathscr{H}^{grav} = -\sqrt{g}R + \frac{1}{\sqrt{g}}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2) \approx 0, \quad \mathscr{H}_i^{grav} = -2\nabla_j \pi_i^j \approx 0.$$

(Dirac, Arnowitt-Deser-Misner, Regge-Teitelboim)

Boundary conditions

The definition of the phase space of general relativity must be completed by boundary conditions on the canonical variables, which are g_{ij} and π^{ij}.

We shall insist that the boundary conditions make the action :

- finite
- and invariant under (at least) all (asymptotic) Poincaré symmetries, which are thus canonical transformations.

Boundary conditions

The usually assumed fall-off is (in cartesian coordinates)

$$g_{ij} = \delta_{ij} + O(r^{-1}), \qquad \pi^{ij} = O(r^{-2}).$$

Without additional requirement, this fall-off is too slow because it generically leads to a logarithmic divergence in the symplectic structure (kinetic term)

$$\int d^3x \, \pi^{ij} \dot{g}_{ij} \sim \ln r.$$

Parity conditions

One way to cure this problem would be to impose that the leading terms of the metric and its conjugate momentum have opposite parity properties under the antipodal map,

$$h_{ij} \equiv g_{ij} - \delta_{ij} = \frac{\overline{h}_{ij}(\mathbf{n}^k)}{r} + O(\frac{1}{r^2}), \quad \overline{h}_{ij}(-\mathbf{n}^k) = \overline{h}_{ij}(\mathbf{n}^k)$$

and

$$\pi^{ij} = \frac{\overline{\pi}^{ij}(\mathbf{n}^k)}{r^2} + O(\frac{1}{r^3}), \quad \overline{\pi}^{ij}(-\mathbf{n}^k) = -\overline{\pi}^{ij}(\mathbf{n}^k).$$

.

These strict parity conditions are compatible with Poincaré invariance and are obeyed by all known asymptotically flat solutions.

But these strict parity conditions leave no room for the BMS group.

Symmetries

Asymptotic symmetries and algebras: a review with emphasis on gravity

Marc Henneaux (Université Libre de Bruxelles and Collège de France)

To understand this point, we recall that a symmetry transformation is a phase space transformation

(i.e., a transformation $(g_{ij}, \pi^{ij}) \rightarrow (g'_{ij}, \pi'^{ij})$ which preserves the boundary conditions)

that leaves the action invariant up to surface integrals at the time boundaries.

A symmetry transformation preserves therefore in particular the symplectic form ("canonical transformation")

$$\Omega = \int d^d x d_V \pi^{ij} \wedge d_V g_{ij}$$

exactly - and not up to surface terms.

Its canonical generator defines furthermore a constant of the motion.

Asymptotic symmetries and charges

An asymptotic symmetry is a gauge transformation that fulfills the above conditions.

Hence it preserves not only the boundary conditions,

but it is also canonically generated.

Its canonical generator is given by a bulk term proportional to the (first class) constraints plus a surface integral at infinity.

When the constraints hold, it reduces to the surface term at infinity.

If the surface term vanishes for all configurations obeying the boundary conditions, the symmetry is a proper gauge symmetry and corresponds to a mere redundancy in the description of the system.

The asymptotic symmetries with non-vanishing generator are the improper gauge symmetries and should not be factored out.

Asymptotic symmetries and gauge conditions

The proper gauge symmetries form an ideal.

The physical asymptotic symmetry algebra is the quotient of all the gauge transformations preserving the boundary conditions by the ideal of the proper gauge symmetries.

A physical asymptotic symmetry is determined by its asymptotic behaviour, hence the name "asymptotic symmetry".

Because the improper gauge symmetries define physical transformations, it would be incorrect to gauge-fix them.

In particular, the asymptotic conditions, which are expressed in terms of gauge-variant variables, can only gauge-fix the proper gauge symmetries.

Where are the supertranslations?

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The strict parity conditions are invariant under an infinite-dimensional group given by :

$$\xi = b_i x^i + T(\mathbf{n}) + O\left(r^{-1}\right)$$
$$\xi^i = b^i{}_j x^j + W^i(\mathbf{n}) + O\left(r^{-1}\right)$$

where $T(\mathbf{n}) = a^0$ + "odd" and $W^i(\mathbf{n}) = a^i$ + "odd" (the linear growing terms describe boosts and spatial rotations).

However, a direct computation shows that the only symmetry generators with non-vanishing charges are the Poincaré generators.

Therefore, with the strict parity conditions, the asymptotic symmetry algebra is finite-dimensional and given by the Poincaré algebra.

Parity-twisted boundary conditions

But the BMS group found at null infinity is infinite-dimensional and contains much more (supertranslations, or angle-dependent translations) ! What is the origin of this tension?

The strict parity conditions are too strong and kill the pure supertranslations.

To see the full BMS group, one must "unfreeze" the non-trivial BMS supertranslations, which are improper gauge transformations which have been incorrrectly gauge-fixed by the strict parity conditions.

Thus, one must relax these parity conditions... but not completely if one wants to maintain finiteness of the action.

The idea is to allow a "parity-twisted component" of a specific form in the leading orders in the asymptotic expansions of the metric and its conjugate momentum.

Parity conditions twisted by a diffeomorphism

This parity-twisted component takes the form of a transformation of the canonical variables generated by $\mathcal{O}(1)$ diffeomorphisms (rewritten in Hamiltonian form).
One takes thus for the metric and its conjugate momentum

$$h_{ij} \equiv g_{ij} - \delta_{ij} = \frac{(\overline{h}_{ij})^{even}(\mathbf{n}^k)}{r} + U_{ij} + + O(\frac{1}{r^2}),$$

$$\pi^{ij} = \frac{(\overline{\pi}^{ij})^{odd}(\mathbf{n}^k)}{r^2} + V^{ij} + O(\frac{1}{r^3})$$

$U_{ij} = 2\partial_i\partial_j(rU)$ is the contribution that twists the strict parity condition on the metric by an $\mathcal{O}(1)$-diffeomorphism (at leading order), while $V^{ij} = \partial^i\partial^j V - \delta^{ij} V$ is the contribution that twists the strict parity of π^{ij} (at leading order).
One can assume that $U(\mathbf{n}^k)$ is odd and $V(\mathbf{n}^k)$ is even.
The twisting is thus characterized by two functions of the angles, one odd (U) and one even (V).

Parity-twisted boundary conditions

These relaxed parity conditions involving a twist still lead to a consistent dynamical description :

finite action, finite symplectic form, well-defined generators for all Poincaré generators including boosts...

and more!

BMS group at spatial infinity

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With these relaxed boundary conditions, one finds in fact that the asymptotic symmetries are given by hypersurface deformations that behave asymptotically as

$$\xi = b_i x^i + T(\mathbf{n}) + O\left(r^{-1}\right)$$
$$\xi^i = b^i{}_j x^j + W_i(\mathbf{n}) + O\left(r^{-1}\right), \quad b_{ij} = -b_{ji}, \quad W_i(\mathbf{n}) = \partial_i(rW(\mathbf{n})).$$

where T is even and W is odd.

The terms $b_i x^i$ and $b^i{}_j x^j$ describe respectively boosts and spatial rotations.

The zero mode of T and the first spherical harmonic component of W describe translations.

BMS group at spatial infinity

The higher spherical harmonics describe general supertranslations.

In fact, the even function T and the odd function W combine to form a single arbitrary function of the angles, as in the null infinity description of the supertranslations.

The symmetries are canonical transformations with generators

$$P_\xi[g_{ij}, \pi^{ij}] = \int d^3x \left(\xi \mathscr{H} + \xi^i \mathscr{H}_i\right) + \mathscr{B}_\xi[g_{ij}, \pi^{ij}]$$

where $\mathscr{B}_\xi[g_{ij}, \pi^{ij}]$ is a surface term.

The algebra of the generators (which is guaranted to form a Poisson algebra by the general theorems) can be easily verified to be the BMS algebra, but in a different basis than the one used at null infinity.

Spatial infinity versus null infinity

There is complete agreement with the null infinity results.

One can match explicitly the symmetries and the charges (in the limit $u \to -\infty$ or $v \to +\infty$; Bondi-like charges with ADM-like charges).

One can in fact do more :

- One can derive Strominger's matching conditions from the initial data.
- There is an improvement over the standard null infinity analysis, since the Hamiltonian analysis at spatial infinity **guarantees that the polylogarithmic terms that appear inevitably at null infinity do not spoil the BMS symmetry** (not an issue at spatial infinity).

The BMS algebra

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The BMS algebra is the semi-direct sum of the Lorentz algebra $\mathscr{L}$ and the infinite-dimensional abelian algebra $\mathscr{A}$ spanned by the supertranslations,

$$\text{BMS} = \mathscr{L} \oplus_\sigma \mathscr{A}$$

Schematically, if we denote the generators of the homogeneous Lorentz group by M_a, one finds

$$\{M_a, M_b\} = f^c_{ab} M_c \,,$$
$$\{M_a, T_i\} = R_{ai}{}^j T_j \,,$$
$$\{M_a, S_\alpha\} = G_{a\alpha}{}^i T_i + G_{a\alpha}{}^\beta S_\beta \,,$$
$$\{T_i, T_j\} = 0 = \{T_i, S_\alpha\} = \{S_\alpha, S_\beta\}$$

where T_i and S_α are respectively the generators of the standard translations and of the pure supertranslations.

The BMS algebra

The structure constants ${R_{ai}}^{j}$, ${G_{a\alpha}}^{i}$ and ${G_{a\alpha}}^{\beta}$ are non zero.

The ordinary translations transform in the 4-dimensional vector representation of the Lorentz group (${R_{ai}}^{j}$).

Modulo the ordinary translations, the pure supertranslations transform in an infinite-dimensional representation of the Lorentz group (${G_{a\alpha}}^{\beta}$).

There are intriguing physical features resulting from the structure of the BMS algebra

which are somewhat uncomfortable.

Angular momentum ambiguity

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All the puzzles originate from the fact that the Poincaré subalgebra is not an ideal. The most discussed one is the so-called " angular momentum ambiguity".

It follows from the non-vanishing of the bracket of the pure supertranslations with the homogeneous Lorentz transformations that the angular momentum transforms under pure supertranslations.

This non-invariance comes on top of the familiar non-invariance of the angular momentum under ordinary translations, but there one knows how to define an intrinsic angular momentum, which amounts to computing the angular momentum with respect to the center of mass worldline.

Can one provide a similar construction for the supertranslation ambiguity?

Not within the BMS algebra (Sachs).

Relaxing further the boundary conditions

Another question, apparently unconnected is :

Given boundary conditions leading to a consistent dynamical description,

how do we know that one cannot relax them further,

i.e., that we have not frozen improper gauge transformations that we therefore do not see?

There is no systematic way to answer this question. It is a bit of an art.

It turns out that in the present case, one can relax even more the boundary conditions in a useful way,

which enlarges the asymptotic symmetry group to include “logarithmic supertranslations” conjugate to the (pure) supertranslations.

This enables one (somewhat unexpectedly) to overcome the angular momentum puzzle raised above.

Logarithmic supertranslations – Idea

To exhibit the logarithmic supertranslations, one must allow terms in the canonical variables that result from diffeomorphisms that grow like $\ln r$ at infinity.

The logarithmic supertranslations are then diffeomorphisms that grow like $\ln r$ at infinity and which preserve by construction the new boundary conditions.

They define true symmetries of the action provided some conditions on the logarithmic diffeomorphisms are imposed.

The (allowed) logarithmic supertranslations are then canonical transformations with a well-defined, finite charge.

They depend on a single function of the angles, but the $\ell = 0$ and $\ell = 1$ harmonics define proper gauge transformations : they have zero charge.

Structure of the logarithmic BMS algebra

One can compute the Poisson brackets of the logarithmic supertranslation generators with themselves and with the generators of the BMS algebra.

The computation is direct and follows standard canonical methods.

One finds that the generators L^α of the logarithmic supertranslations commute among themselves,

$$\{L^\alpha, L^\beta\} = 0,$$

Logarithmic BMS algebra

Putting everything together, one thus gets as non-zero brackets

$$\{M_a, M_b\} = f^c_{ab} M_c\,,$$
$$\{M_a, T_i\} = R_{ai}{}^{j} T_j\,,$$
$$\{M_a, S_\alpha\} = G_{a\alpha}{}^{i} T_i + G_{a\alpha}{}^{\beta} S_\beta\,,$$
$$\{M_a, L^\alpha\} = -G_{a\beta}{}^{\alpha} L^\beta\,,$$
$$\{L^\alpha, S_\beta\} = \delta^\alpha_\beta\,,$$

i.e.,

$$\text{LogBMS} = \mathscr{L} \oplus_\sigma (\mathscr{A} \oplus_c \mathscr{B})$$

with $\mathscr{B}$ being the abelian algebra of the logarithmic supertranslations.

Poisson manifolds

The generators of the asymptotic symmetries form a Poisson algebra,

which can a priori be nonlinear (any function of a conserved quantity is a conserved quantity; the PB of two conserved quantity is a conserved quantity).

[A true Hamiltonian description of the asymptotic symmetries is crucial here.]

There are many examples where the Poisson bracket of two asymptotic symmetry charges involves central charges or is non-linear (extended supergravity models in 3D, higher spins in 3D, asymptotically flat spacetimes in higher dimensions).

Even when the Poisson brackets are linear, one can consider non-linear redefinitions of the charges, which enables one to use Darboux-like constructions.

Logarithmic BMS algebra

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In the case of the logarithmic BMS algebra, which has the structure

$$\{M_a, M_b\} = f^c_{ab} M_c\,,$$
$$\{M_a, T_i\} = R_{ai}{}^j T_j\,,$$
$$\{M_a, S_\alpha\} = G_{a\alpha}{}^i T_i + G_{a\alpha}{}^\beta S_\beta\,,$$
$$\{M_a, L^\alpha\} = -G_{a\beta}{}^\alpha L^\beta\,,$$
$$\{L^\alpha, S_\beta\} = \delta^\alpha_\beta\,,$$

the fact that the central charge in $\{L^\alpha, S_\beta\}$ is invertible enables one to rewrite the algebra as the direct sum $\mathscr{P} \oplus (\mathscr{A}' \oplus_c \mathscr{B})$, where $\mathscr{A}'$ is the abelian algebra of pure supertranslations.

The logarithmic supertranslations are canonically conjugate to the pure supertranslations, and a Darboux-like procedure enables one to "decouple" them from the other generators by making appropriate redefinitions.

Logarithmic BMS_4 algebra

The only generators that need to be redefined are actually the Lorentz generators, as follows,

$$\tilde{M}_a = M_a - G_{a\beta}{}^{i} L^\beta T_i - G_{a\beta}{}^{\gamma} L^\beta S_\gamma \quad (5.1)$$

$$= M_a - L^\beta \{M_a, S_\beta\}\,. \quad (5.2)$$

One easily verifies

$$\{\tilde{M}_a, S_\alpha\} = \{\tilde{M}_a, L^\alpha\} = 0\,, \quad (5.3)$$

while the bracket $\{\tilde{M}_a, T_i\}$ does not suffer any modification.

Logarithmic BMS algebra

Thus we have achieved :

$$\{\tilde{M}_a, \tilde{M}_b\} = f^c_{ab}\tilde{M}_c\,,$$
$$\{\tilde{M}_a, T_i\} = R_{ai}{}^{j}\, T_j\,,$$
$$\{\tilde{M}_a, S_\alpha\} = 0\,,$$
$$\{\tilde{M}_a, L^\alpha\} = 0\,,$$
$$\{L^\alpha, S_\beta\} = \delta^\alpha_\beta\,,$$

i.e., LogBMS $= \mathscr{P} \oplus (\mathscr{A}' \oplus_c \mathscr{B})$

The fact that the Lorentz generators commute with the supertranslation generators (in the PB) implies that the Lorentz charges are supertranslation invariant.

Logarithmic BMS_4 algebra

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The redefinitions of the Lorentz generators

involve quadratic terms of the forme TL and SL.

This means that the new Lorentz transformations will differ from the old ones by field dependent supertranslations and logarithmic supertranslations.

Conclusions and comments

Marc Henneaux (Université Libre de Bruxelles and Collège de France)

In gauge theories formulated on spaces with boundaries (which can be at infinity), improper (or large) gauge transformations play an important role.

They provide important information on the system (physical states must transform in representations of the asymptotic symmetry group, which can be non-trivial).

Gravity in the asymptotically flat context is a superb example, with the infinite-diimensional BMS group emerging in that case, but many other examples are known.

Because we have a well-defined symplectic structure and a well-defined Hamiltonian action, standard Hamiltonian theorems apply to asymptotic symmetries (Poisson algebra, nonlinear redefinitions possible...).

Enlarging the algebra to include logarithmic supertranslations is possible and useful (matches null-infinity works by Porrati et al, Yau et al, Compère et al).

Conclusions and comments

Many open questions remain on the quantum side ("flat space holography").

(Talk based on work mostly carried out in collaboration with O. Fuentealba and C. Troessaert)

THANK YOU!