Title: Entanglement Hamiltonian and Heun operator

Speakers: Pierre-Antoine Bernard

Collection/Series: Mathematical Physics

Subject: Mathematical physics

Date: April 03, 2025 - 11:00 AM

URL: https://pirsa.org/25040107

Abstract:

The ground state of a bipartite quantum system resembles a thermal state from the perspective of an observer with access to only one of the two regions. This observation has led to the introduction of the entanglement Hamiltonian, which has numerous applications in quantum information and the study of quantum many-body systems.

This talk will explore two approaches to computing the entanglement Hamiltonian: one in the context of quantum field theories (QFTs), focusing on the Bisognano-Wichmann Hamiltonian, and the other for free fermion models on a lattice, where connections with algebraic Heun operators will be highlighted. Additionally, the relationship between the discrete and continuum cases will be examined.

Entanglement Hamiltonians and Heun operators

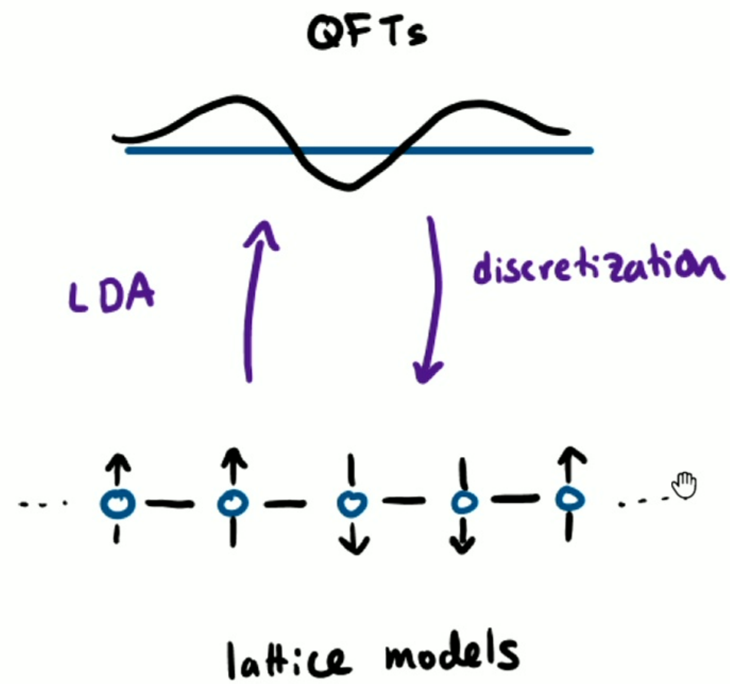
Pierre-Antoine Bernard
Centre de recherches mathématiques
Université de Montréal

Based on work realized with
Riccarda Bonsignori, Viktor Eisler, Gilles Parez, Luc Vinet

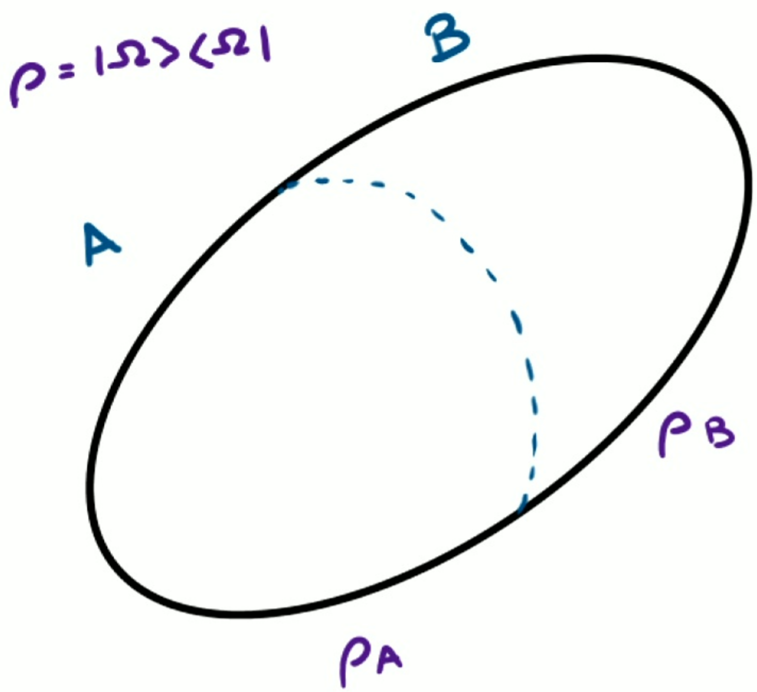
April 3, 2025

Overview

1. Motivation
2. QFT
3. Free Fermions on a lattice
4. Continuum limit



Motivation



Reduced density matrix:

$$\rho_A = \text{tr}_B\, \rho$$

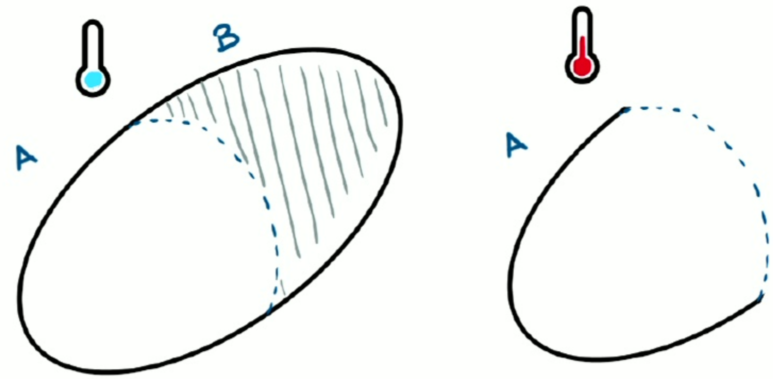
Since ρ_A is positive semi-definite, we can always write:

$$\rho_A \propto e^{-\mathcal{H}}$$

where $\mathcal{H}$ is defined as the **entanglement Hamiltonian** (or modular Hamiltonian).

Motivation

Idea: viewing ρ_A as a **thermal state** of a system governed by a Hamiltonian $\mathcal{H}$ defined solely on $\mathcal{A}$.



Applications:

- Computation of entanglement entropy:

$$S_A = -\mathrm{tr}_A \rho_A \ln \rho_A$$

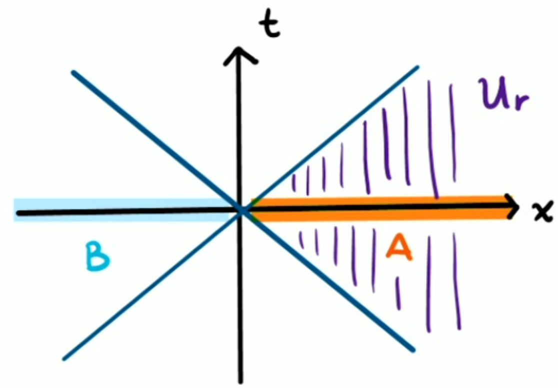
- Ansatz for quantum state tomography:

$$\rho_A(\vec{g}) \propto e^{-\mathcal{H}(\vec{g})}$$

- Connection with Unruh effect.

Computing $\mathcal{H}$: QFT case

Setting: a $1+1$d relativistic QFT. The region A is characterized by $x > 0$. $\mathcal{U}_r$ is the causal complement of B.



[Bisognano and Wichmann, 1975]

The modular operator Δ associated to the vacuum state Ω and the algebra of observable $\mathcal{A}_r$ of the right wedge $\mathcal{U}_r$ is

$$\Delta = e^{-2\pi K}$$

where K is the Lorentz boost generator.

Computing $\mathcal{H}$: QFT case

Corollary

Using Tomita-Takesaki theory, the modular operator Δ can be related to the density matrix as $\Delta = \rho_B^{-1} \otimes \rho_A$. Therefore, we can identify

$$\rho_A \propto e^{-2\pi K_A}, \quad \rho_B \propto e^{-2\pi K_B}$$

where

$$K_A = \int_{x \in A} dx\, x T_{00}(x), \quad K_B = -\int_{x \in B} dx\, x T_{00}(x)$$

$$K = K_A - K_B = \int dx\, x T_{00}(x)$$

Note: K_A is known as **Bisognano-Wichmann Hamiltonian**. We have the identification $\mathcal{H} = 2\pi K_A$.

Computing $\mathcal{H}$: QFT case

Alternative path integral approach [Witten, 2018]:

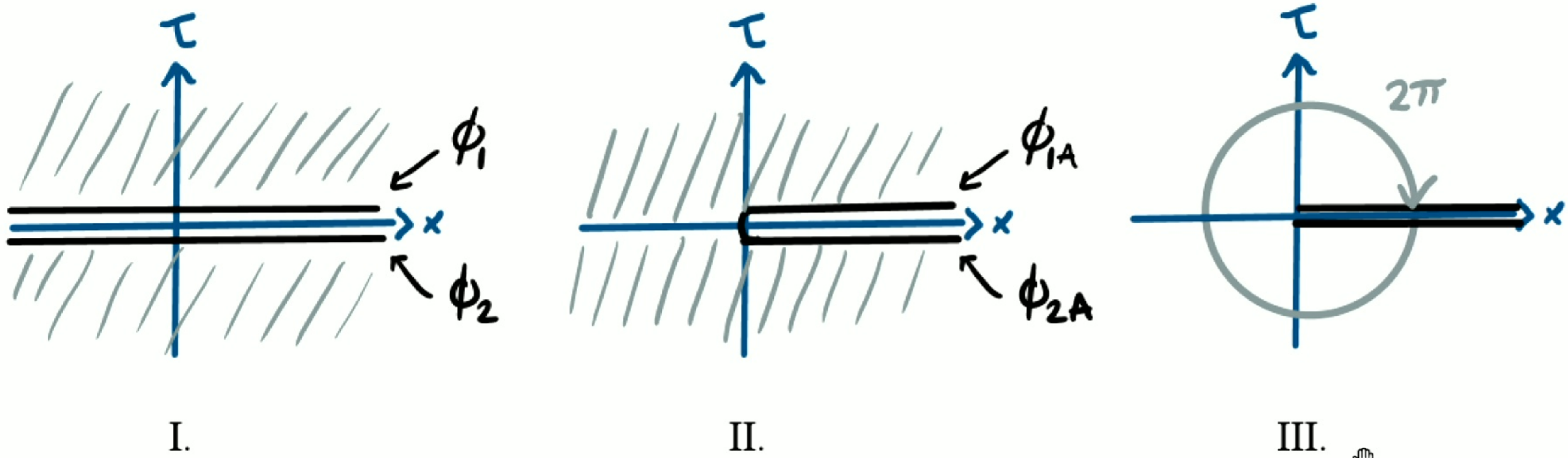
$$\rho[\phi_1, \phi_2] = \int_{\phi(x,0)=\phi_1(x)} D\phi e^{-S_E[\phi]} \int_{\phi'(x,0)=\phi_2(x)} D\phi' e^{-S_E[\phi']} = \int_{\substack{\phi(x,0^+)=\phi_1(x)\\ \phi(x,0^-)=\phi_2(x)}} D\phi e^{-S_E[\phi]}$$

We have $\phi_i = \phi_{i,B} \oplus \phi_{i,A}$. Taking the partial trace amounts to ask $\phi_{1,B} = \phi_{2,B}$.

$$\rho_A[\phi_{1,A}, \phi_{2,A}] = \int_{\substack{\phi_A(x,0^+)=\phi_{1,A}(x)\\ \phi(x,0^-)=\phi_{2,A}(x)}} D\phi e^{-S_E[\phi]}$$

The Hamiltonian interpretation of this path integral is a 2π rotation in the plane τx. In terms of the real time $t = -i\tau$, this is a Lorentz boots $e^{-2\pi K}$.

Computing $\mathcal{H}$: QFT case



Remark: This identification was made possible by the choice of region A and the presence of a continuous symmetry group.

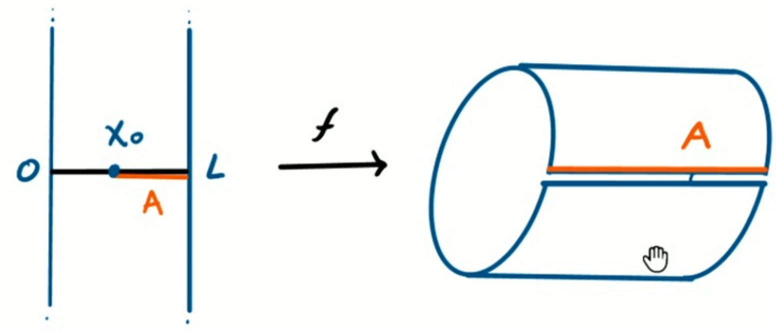
$$\mathcal{H} = 2\pi \int_{x \in A} dx\, x T_{00}(x)$$

Different geometries (CFT case)

Question: What happens for theories defined on a different geometry? What about a different choice of subsystem A?

Example: Consider a CFT, on a finite interval $(x, t) \in [0, L] \times \mathbb{R}$. The subsystem $A = [x_0, L]$. Let $z = x + it$ and consider the following map induced by

$$w = f(z) = \ln\left(\frac{\sin\frac{\pi(z-x_0)}{2L}}{\sin\frac{\pi(z+x_0)}{2L}}\right)$$



Reduced density matrix $\rho_A \to$ thermal state on annulus. Back to original coordinates:

$$\mathcal{H} = 2\pi \int_A \beta(x) T_{00}(x) dx, \quad \beta(x) = \frac{L}{\pi} \frac{\cos\frac{\pi x_0}{L} - \cos\frac{\pi x}{L}}{\sin\frac{\pi x_0}{L}}.$$

Computing $\mathcal{H}$: free fermions on a lattice

We consider the Heisenberg XX (or free fermions) model :

$$\boldsymbol{H} = -\frac{1}{2}\sum_n J_n(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y) - \frac{1}{2}\sum_n \mu_n(1+\sigma_n^z),$$

which using Jordan-Wigner transformation is mapped onto

$$\boldsymbol{H} = \sum_n J_n\left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right) - \sum_n \mu_n c_n^\dagger c_n,$$

where $\{c_m, c_n\} = \{c_m^\dagger, c_n^\dagger\} = 0$ and $\{c_m, c_n^\dagger\} = \delta_{mn}$. To solve this model, it is convenient to express the Hamiltonian in matrix form.

$$\boldsymbol{H} = \boldsymbol{c}^\dagger J \boldsymbol{c}, \quad \boldsymbol{c}^\dagger = (c_1^\dagger, c_2^\dagger, \dots, c_N^\dagger), \quad J = \begin{pmatrix} -\mu_1 & J_1 & 0 & \dots \\ J_1 & -\mu_2 & J_2 & \dots \\ 0 & J_2 & -\mu_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Computing $\mathcal{H}$: free fermions on a lattice

Idea: Diagonalize J

$$UJU^T = \text{diag}(\omega_1, \omega_2, \ldots, \omega_N), \quad U^T U = \mathbf{1}$$

Let $b_k = (U\boldsymbol{c})_k = \sum_i U_{ki} c_i$, then

$$\boldsymbol{H} = \boldsymbol{c}^\dagger J \boldsymbol{c} = \boldsymbol{c}^\dagger U^T UJU^T U\boldsymbol{c} = \sum_k \omega_k b_k^\dagger b_k$$

Observation: Favard's theorem states that three term recurrence are solved by orthogonal polynomials. Since J is a tridiagonal matrix, the entries of U can be expressed in terms of orthogonal polynomials:

$$U_{ki} \propto P_i(\omega_k).$$

Ground state: Let the vacuum state be denoted by $|0\rangle$ and $\omega_k < \omega_{k+1}$. Let K be the largest integer k such that $\omega_k < 0$, then the ground state $|\Omega\rangle$ is given by

$$|\Omega\rangle = b_1^\dagger \ldots b_K^\dagger |0\rangle .$$

Computing $\mathcal{H}$: free fermions on a lattice

[Peschel, 2003]

The entanglement Hamiltonian for the ground state of a lattice free fermion model is given by

$$\mathcal{H} = \sum_{ij\in A} h_{ij} c_i^\dagger c_j,$$

where h is an $|A| \times |A|$ matrix given in terms of an $|A| \times |A|$ truncated correlation matrix C

$$h = \ln\left(\frac{1-C}{C}\right), \quad C_{ij} = \langle\Omega| c_i^\dagger c_j |\Omega\rangle .$$

Sketch of proof: All higher correlation can be expressed in terms of two point correlation. Wick's theorem $\rightarrow$ true for gaussian fermionic states. We can identify h using

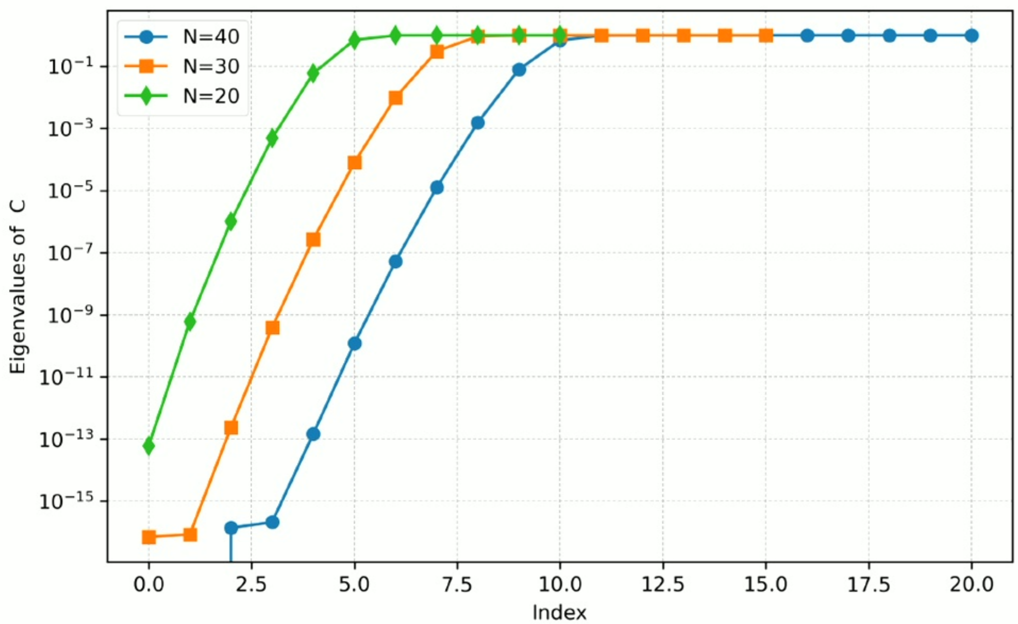
$$\frac{\mathrm{tr}(e^{-\mathcal{H}} c_i^\dagger c_j)}{\mathrm{tr}(e^{-\mathcal{H}})} = C_{ij}.$$

Computing $\mathcal{H}$: free fermions on a lattice

Computing C: we have

$$C_{ij} = \langle\Omega| c_i^\dagger c_j |\Omega\rangle = \sum_{k\leq K} U_{ki} U_{kj}$$

Observation: we have the issue that C is ill-conditioned.



Computing $\mathcal{H}$: free fermions on a lattice

[Eisler and Peschel, 2013] [Grünbaum et al., 2018] [Crampé et al., 2019]

If there exists a non-degenerate diagonal matrix X acting tridiagonally on the eigenbasis of the one-particle Hamiltonian J, then there exists a region A and coefficients μ and ν such that $[\mathcal{T}, C] = 0$, where

$$\mathcal{T} = JX + XJ + \mu J + \nu X.$$

Sketch of proof: The truncated correlation matrix can be factorized in terms of projectors π_K and π_A onto the eigenspaces of J and X respectively:

$$C = \sum_{k \leq K} U_{ki} U_{kj} = \pi_A \pi_K \pi_A$$

Two bases: one where J diagonal and X tridiagonal, one where J tridiagonal and X diagonal. In two bases, $\mathcal{T}$ tridiagonal.

$$[\mathcal{T}, \pi_K] = 0 \Rightarrow \text{constraint on } \nu, \quad [T, \pi_A] = 0 \Rightarrow \text{constraint on } \mu.$$

Computing $\mathcal{H}$: free fermions on a lattice

Remarks:

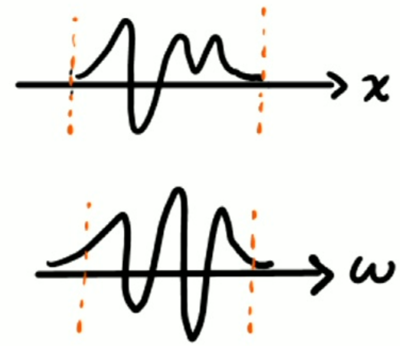
- In general, $\mathcal{T}$ is a tridiagonal matrix, well-conditioned.

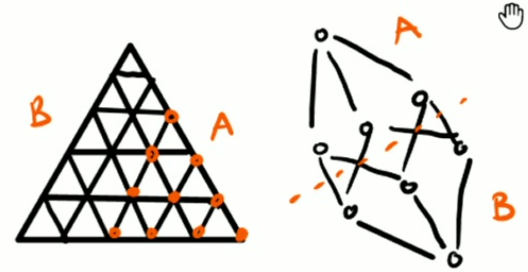
$$h = a_0 + a_1\mathcal{T} + a_2\mathcal{T}^2 + \dots$$

- Very good approximation:

$$h \approx a_0 + a_1\mathcal{T}$$

- Originates from the work [Slepian and Pollak, 1961] on time and band limiting.
- Generalizations: hyperplane lattices, Distance-regular graphs.





Computing $\mathcal{H}$: free fermions on a lattice

Remarks:

- In general, $\mathcal{T}$ is a tridiagonal matrix, well-conditioned.

$$h = a_0 + a_1\mathcal{T} + a_2\mathcal{T}^2 + \ldots$$

- Very good approximation:

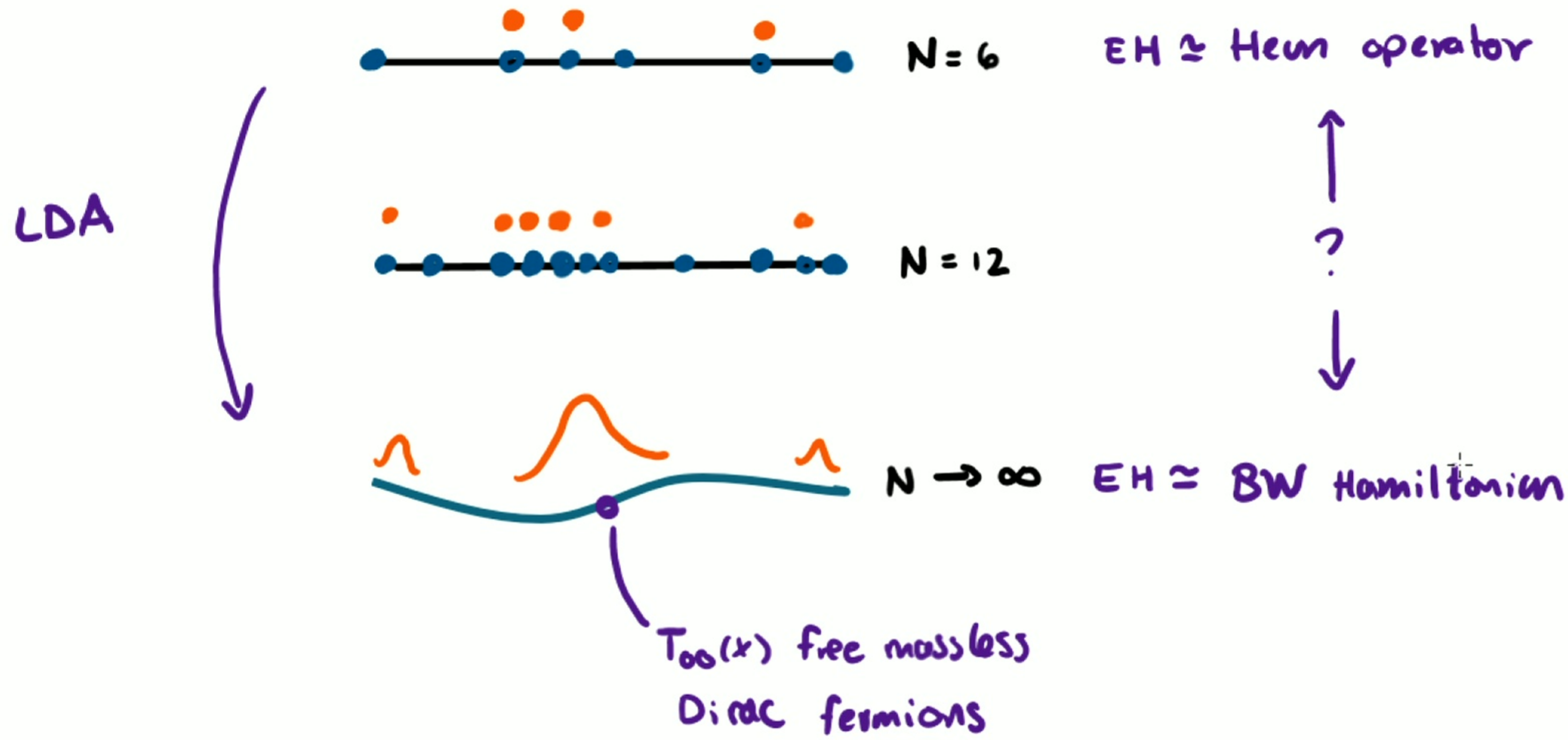
$$h \approx a_0 + a_1\mathcal{T}$$

- Originates from the work [Slepian and Pollak, 1961] on time and band limiting.
- Generalizations: hyperplane lattices, Distance-regular graphs.

- The pairs (J, X) are classified (Leonard pairs). One-to-one correspondence with q-Racah polynomials.
- Diagonalizing $\mathcal{T} \Rightarrow$ algebraic Bethe anstaz.
- In certain cases, $\mathcal{T}$ can be identified with differential Heun operator.

$$\frac{d^2w}{dz^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right]\frac{dw}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}w = 0.$$

Continuum limit



Continuum limit

Formal procedure for continuum limit: free fermions on a chain $\rightarrow$ **free massless Dirac fermions in a curved background**.

$$x = (n+1)N^{-1}, \quad J_n \rightarrow NJ(x), \quad \mu_n \rightarrow N\mu(x)$$

$$c_n \rightarrow \frac{1}{\sqrt{N}}\left(e^{i\varphi(x)}\psi_R(x) + e^{-i\varphi(x)}\psi_L(x)\right), \quad \varphi(x) = \int^x q_F(x)dx$$

Space-dependent Fermi velocity $v_F(x)$, Fermi momentum $q_F(x)$ and dispersion relation

$$q_F(x) = \pm \arccos\left(\mu(x)/2J(x)\right), \quad \omega_q = 2J(x)\cos q - \mu(x).$$

$$v_F = \left|\frac{\mathrm{d}\omega_q(x)}{\mathrm{d}q}\right|_{q_F(x)} = \sqrt{4J(x)^2 - \mu(x)}$$

Continuum limit

This leads to

$$\boldsymbol{H} = \sum_{i,j} [J]_{ij} c_i^\dagger c_j \xrightarrow[N\to\infty]{} \int_0^1 v_F(x) T_{00}(x) dx$$

where

$$T_{00}(x) = \frac{1}{2}\left[\psi_R^\dagger(x)(-i\partial_x)\psi_R(x) - \psi_L^\dagger(x)(-i\partial_x)\psi_L(x) + \text{h.c.}\right].$$

Similarly, we find for the entanglement Hamiltonian based on the Heun operator $\mathcal{T}$

$$\sum_{i,j\in A} [a_0 + a_1 \mathcal{T}]_{ij} c_i^\dagger c_j \xrightarrow[N\to\infty]{} \int_A \beta_{Heun}(x) v_F(x) T_{00}(x) dx$$

[Eisler et al., 2019] [Bonsignori and Eisler, 2024] [Bernard et al., 2024]

In all cases studied, the factor $\beta_{Heun}(x)$ from the **continuum limit of the Heun operator** matches the coefficient $\beta_{CFT}(x)$ predicted by conformal field theory (CFT) in curved spacetime.

Example: Krawtchouk chain

Setting: Pair $(J, X) = (S^x, S^z)$ where S^x and S^z are $(N+1) \times (N+1)$ matrices from an irreducible representation of $\mathfrak{su}(2)$ in the basis where S^z is diagonal.

$$\boldsymbol{H} = \sum_n J_n \left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n \right) - \sum_n \mu_n c_n^\dagger c_n,$$

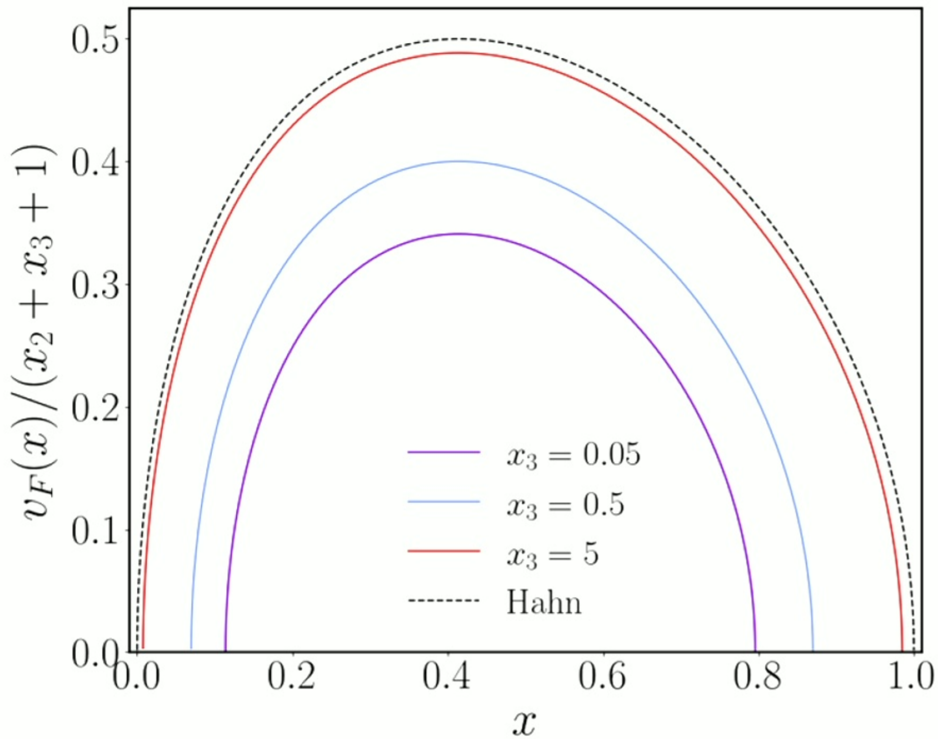
$$J_n = \frac{1}{2}\sqrt{(n+1)(N-n)}, \quad \mu_n = 0, \quad v_F(x) = \sqrt{x(1-x)}$$

Heun operator ($A =$ half-chain):

$$\mathcal{T} = \{S^x, S^z\}$$

$$\beta_{Heun}(x) = \beta_{CFT}(x) = \frac{\tilde{L}}{\pi} \frac{\cos\frac{\pi \tilde{x}_0}{\tilde{L}} - \cos\frac{\pi \tilde{x}(x)}{\tilde{L}}}{\sin\frac{\pi \tilde{x}_0}{\tilde{L}}} = \frac{x - x_0}{v_F(x_0)}$$

Example: Racah chain



$$\beta(x) = \frac{(x - x_0)(x + x_0 + x_1 - 1)}{(2x_0 + x_1 - 1)v_F(x_0)},$$

Outlook

Pratical applications

- Provides a scheme to fix a_0 and a_1 in the approximation $h \approx a_0 + a_1 \mathcal{T}$
- **(Bonsignori and Eisler, 2024)** Discretization of BW Hamiltonian for new (almost) commuting operator $\mathcal{T}$ for time and band limiting?

Open questions:

- Diagonalization of $\mathcal{T}$
- q-Racah case and other generalizations
- Proof for a_0 and a_1 without using CFT argument.
- XY model, MVOPs.

References I

Bernard, P.-A., Bonsignori, R., Eisler, V., Parez, G., and Vinet, L. (2024).
Entanglement hamiltonian and orthogonal polynomials.
arXiv preprint arXiv:2412.12021.

Bisognano, J. J. and Wichmann, E. H. (1975).
On the duality condition for a hermitian scalar field.
Journal of Mathematical Physics, 16(4):985–1007.

Bonsignori, R. and Eisler, V. (2024).
Entanglement hamiltonian for inhomogeneous free fermions.
Journal of Physics A: Mathematical and Theoretical, 57(27):275001.

Crampé, N., Nepomechie, R. I., and Vinet, L. (2019).
Free-fermion entanglement and orthogonal polynomials.
Journal of Statistical Mechanics: Theory and Experiment, 2019(9):093101.

Eisler, V. and Peschel, I. (2013).
Free-fermion entanglement and spheroidal functions.
Journal of Statistical Mechanics: Theory and Experiment, 2013(04):P04028.

Eisler, V., Tonni, E., and Peschel, I. (2019).
On the continuum limit of the entanglement hamiltonian.
Journal of Statistical Mechanics: Theory and Experiment, 2019(7):073101.

References II

Grünbaum, F. A., Vinet, L., and Zhedanov, A. (2018).
Algebraic heun operator and band-time limiting.
Communications in Mathematical Physics, 364:1041–1068.

Peschel, I. (2003).
Calculation of reduced density matrices from correlation functions.
Journal of Physics A: Mathematical and General, 36(14):L205.

Slepian, D. and Pollak, H. O. (1961).
Prolate spheroidal wave functions, fourier analysis and uncertainty—i.
Bell System Technical Journal, 40(1):43–63.

Witten, E. (2018).
Aps medal for exceptional achievement in research: Invited article on entanglement properties of quantum field theory.
Reviews of Modern Physics, 90(4):045003.

Thanks