Title: The extremal black hole threshold

Speakers: Ryan Unger

Collection/Series: Strong Gravity

Subject: Strong Gravity

Date: April 03, 2025 - 1:00 PM

URL: https://pirsa.org/25040083

Abstract:

In this talk, I will present a proof that extremal Reissner-Nordström arises on the black hole formation threshold for the Einstein equations coupled to a self-gravitating plasma. This constitutes the first rigorous result on critical collapse. I will then discuss a formulation of the stability problem for extremal black holes, and present a positive resolution in the case of extremal Reissner-Nordström perturbed by a self-gravitating neutral scalar field in spherical symmetry, despite the presence of the celebrated Aretakis instability. This is based on joint work with Yannis Angelopoulos (Caltech) and Christoph Kehle (MIT) (2402.10190, 2410.16234).

The extremal black hole threshold

Ryan Unger

Department of Mathematics, Stanford University

Strong gravity seminar, Perimeter Institute, April 2025

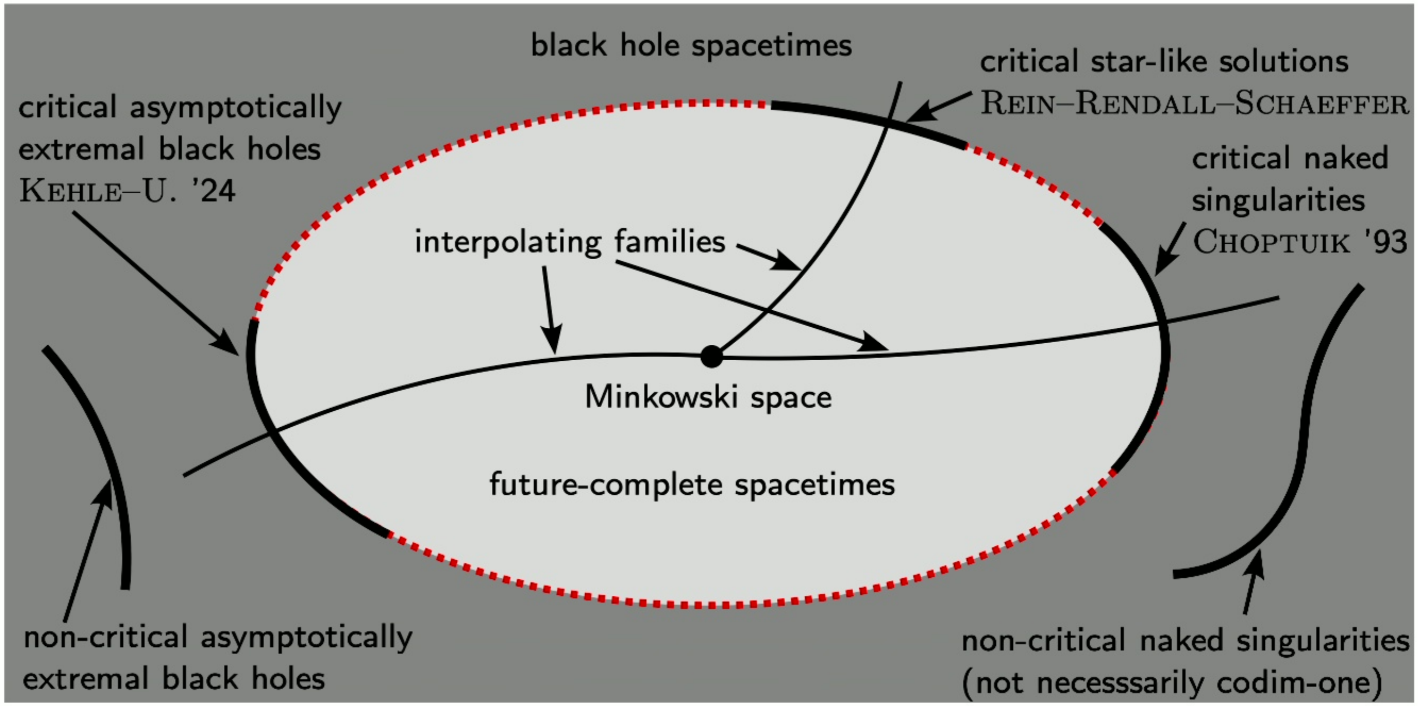
joint work with Yannis Angelopoulos (Caltech) and Christoph Kehle (MIT)
2402.10190, 2410.16234

Outline

1. Critical collapse and extremal Reissner–Nordström
2. The stability problem for extremal Reissner–Nordström
3. What's next

"Moduli space" of classical GR

The set of asymptotically flat solutions of the Einstein equations coupled to reasonable matter, parametrized by initial data



KEHLE–U. '22

CHRISTODOULOU '94

Extremal Reissner–Nordström

The **Reissner–Nordström** family of solutions

$$g_{M,e} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- $M > 0$ **mass**, $e \in \mathbb{R}$ **charge** ($A = -er^{-1}dt$)
- Spherically symmetric, static, solves the **Einstein–Maxwell** equations
- Describes a black hole when $|e| \leq M$
- $|e| < M$ **subextremal** (incl. Schwarzschild)
- $|e| = M$ **extremal**, characterized by **vanishing surface gravity**
- $|e| > M$ **superextremal** contains a "naked singularity"
- *Charge is a poor man's angular momentum!*

Extremal critical collapse

The role of extremality:

- In subextremal RN/Kerr/KN, the black hole interior is **foliated by trapped surfaces**
- The property of having a trapped surface is stable, so an asymptotically subextremal black hole cannot be critical
- Extremal RN/Kerr/KN **do not** have trapped surfaces!

Definition.

An interpolating family $\{\Psi_\lambda\}_{\lambda\in[0,1]}$ exhibits **extremal critical collapse** if the critical data set Ψ_{λ_*} forms an (asymptotically) extremal black hole.

Theorem (Kehle–U. '24).

Extremal critical collapse occurs in the Einstein–Maxwell–charged Vlasov model:

There exist smooth interpolating families for which the critical solution forms an exact extremal Reissner–Nordström black hole after finite time.

We expect the phenomenon of ECC to be more general, but this is the simplest setting with a well-posed matter model in which one can prove its occurance.

The Einstein–Maxwell–charged Vlasov system

- General-relativistic version of the special-relativistic Vlasov–Maxwell model
- Charged spacetime $(\mathcal{M}^4, g, F)$
- Distribution function $f(x,p) \geq 0$ defined on

$$P^{\mathfrak{m}} \doteq \{(x,p) \in T\mathcal{M} : g_x(p,p) = -\mathfrak{m}^2, p^0 > 0\},$$

  models a collisionless gas of massive or massless particles with charge $\mathfrak{e}$
- Einstein field equations:

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 2 \underbrace{\left(F_\mu{}^\alpha F_{\nu\alpha} - \tfrac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)}_{T^{\mathrm{EM}}_{\mu\nu}} + 2 \underbrace{\int_{P^{\mathfrak{m}}_x} p_\mu p_\nu f \, d\mu}_{T^{\mathrm{Vlasov}}_{\mu\nu}}$$

- Maxwell's equations:

$$\nabla^\alpha F_{\mu\alpha} = \mathfrak{e} \underbrace{\int_{P^{\mathfrak{m}}_x} p_\mu f \, d\mu}_{N_\mu}, \quad J^{\mathrm{EM}} = \mathfrak{e} N$$

- Vlasov equation:

$$\left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} + \mathfrak{e} F^\mu{}_\alpha p^\alpha \frac{\partial}{\partial p^\mu}\right) f = 0$$

- $f =$ constant along trajectories $\gamma : I \to P^{\mathfrak{m}}$ of the *Lorentz force*

$$\dot\gamma^\nu \nabla_\nu \dot\gamma^\mu = \mathfrak{e} F^\mu{}_\nu \dot\gamma^\nu,$$

  also known as **electromagnetic geodesics**

Precise statement of the theorem

Theorem (Kehle–U. '24).
Extremal critical collapse occurs in EMV for both massless and massive particles:

There exist smooth 1-parameter families of spherically symmetric Cauchy data $\{\Psi_\lambda\}_{\lambda\in[0,1]}$ for the EMV system on $\mathbb{R}^3$ such that the resulting maximally globally hyperbolic developments $\{\mathcal{D}_\lambda\}_{\lambda\in[0,1]}$ have the following properties:

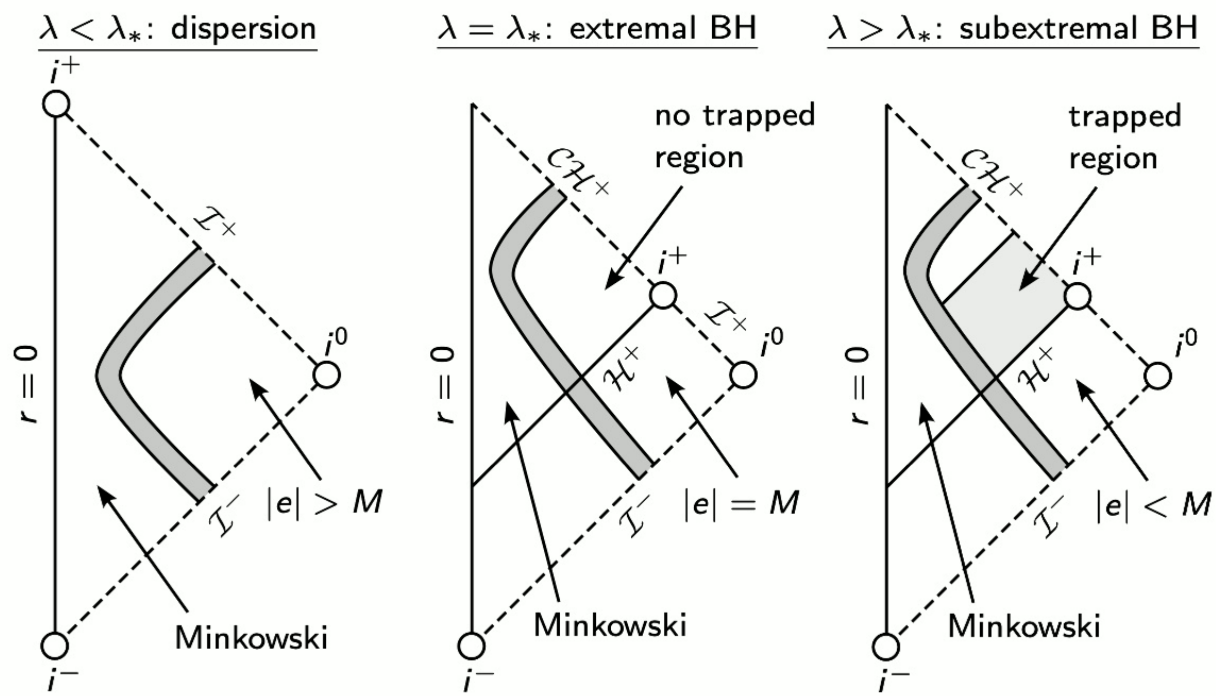
1. $\mathcal{D}_0$ *is Minkowski space and there exists* $\lambda_* \in (0,1)$ *such that for* $\lambda < \lambda_*$, $\mathcal{D}_\lambda$ *is future causally geodesically complete and disperses towards Minkowski space. No black hole or naked singularity forms.*

2. *If* $\lambda = \lambda_*$, *an* **extremal** *Reissner–Nordström black hole forms in finite time. The spacetime contains no trapped surfaces.*

3. *If* $\lambda > \lambda_*$, *a* **subextremal** *Reissner–Nordström black hole forms in finite time. The spacetime contains an open set of trapped surfaces.*

In addition, for every $\lambda \in [0,1]$, $\mathcal{D}_\lambda$ *is past causally geodesically complete and is isometric to Minkowski space near the center* $r = 0$ *for all time.*

Penrose diagrams of extremal critical collapse (massless particles)



Corollary.

The very "black hole-ness" of an extremal black hole arising in gravitational collapse can be unstable. The alternative is dispersion, not a naked singularity.

Sketch of the proof: bouncing charged Vlasov beams

The problem is to construct charged Vlasov beams that:

- form a Reissner–Nordström exterior with specified parameters
- take advantage of EM repulsion and avoid getting too close to $r = 0$
- only form trapped surfaces where we want them to

Singular toy model: the charged Vaidya metric

$$ds^2 = -\left(1 - \frac{2M(v)}{r} + \frac{e(v)^2}{r^2}\right) dv^2 + 2dvdr + r^2 d\Omega^2$$

Want to think of this as describing radial charged null dust

Unfortunately, this "solution" has two fundamental issues:

1. Null dust severely breaks down when the dust reaches $r = 0$, so dispersion is not actually possible in a charged Vaidya spacetime.
2. Sullivan–Israel '80:

$$T^{\mathrm{tot}} = T^{\mathrm{EM}} + \frac{1}{r^2}\left(\dot{M} - \frac{e\dot{e}}{r}\right)(-\partial_r) \otimes (-\partial_r)$$

   If $r < e\dot{e}/\dot{M}$, then the dust sector violates the WEC! Dynamical violation of an energy condition!

Singular toy model: Ori's bouncing charged null dust

$$ds^2 = -\left(1 - \frac{2M(v)}{r} + \frac{e(v)^2}{r^2}\right) dv^2 + 2dvdr + r^2 d\Omega^2$$

Class. Quantum Grav. **8** (1991) 1559–1575.

Charged null fluid and the weak energy condition

Amos Ori
Theoretical Astrophysics, California Institute of Technology, Pasadena, CA 91125, USA

ORI proposed reinterpreting the charged Vaidya metric as a solution of actual equations of motion: **pressureless charged null fluid**

$$\begin{aligned}
g_{\mu\nu} k^\mu k^\nu &= 0 \\
R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + \rho k_\mu k_\nu\right) \\
\nabla^\alpha F_{\mu\alpha} &= \mathfrak{e}\rho k_\mu \\
k^\nu \nabla_\nu k^\mu &= \mathfrak{e} F^\mu{}_\nu k^\nu \\
\nabla_\mu (\rho k^\mu) &= 0
\end{aligned}$$

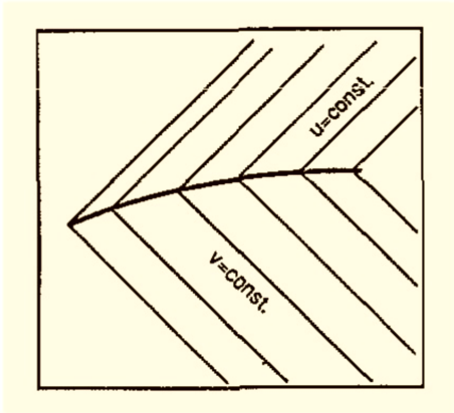
Singular toy model: Ori's bouncing charged null dust

The actual solution of the EoM:

$$k = \frac{\mathfrak{e}}{\dot{e}}\left(\dot{M} - \frac{e\dot{e}}{r}\right)(-\partial_r), \quad \rho = \frac{\dot{e}^2}{\mathfrak{e}^2 r^2}\left(\dot{M} - \frac{e\dot{e}}{r}\right)^{-1}$$

$$k^\mu = \frac{dx^\mu}{ds}, \quad \nabla_k k^\mu = \mathfrak{e} F^\mu{}_\nu k^\nu$$

If (g, F) is spherically symmetric and k is radial, then k is an eigenvector of F
$\implies$ $k(s)$ can decay exponentially, $x(s)$ has a limit point at $r = e\dot{e}/\dot{M}$ as $s \to +\infty$



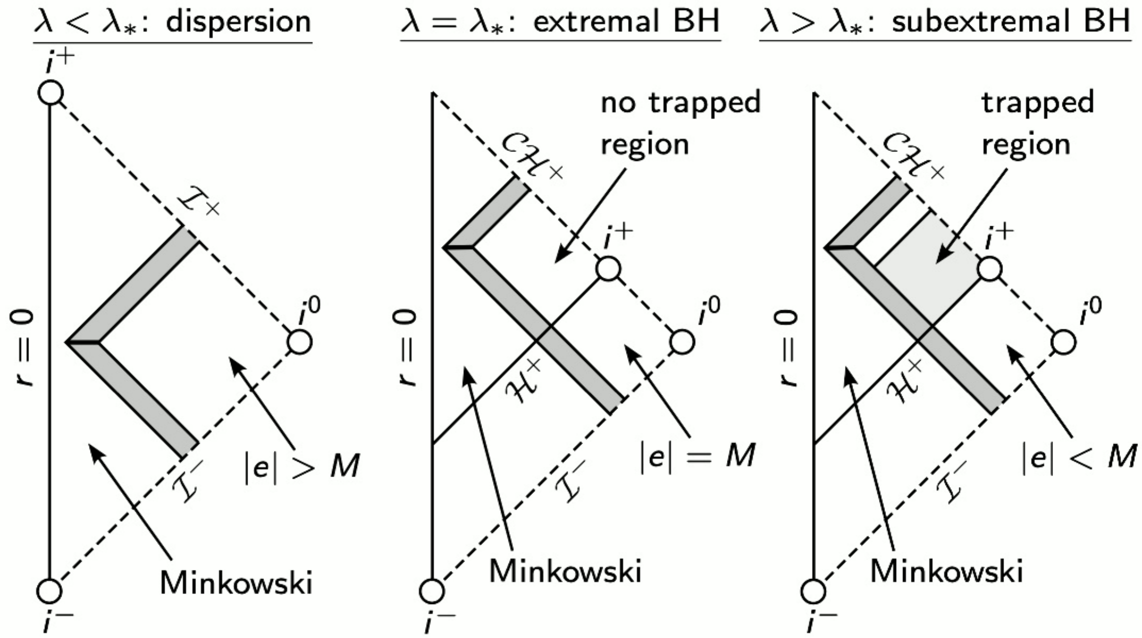
Proposition (Kehle–U. '24).

There exists a procedure for generating bouncing charged null dust spacetimes where the location of the bounce and initial and final RN parameters can be prescribed.

Singular toy model: Ori's bouncing charged null dust

Theorem (Kehle–U. '24).

Ori's charged null dust model exhibits extremal critical collapse.



Unfortunately, Ori's model still has several pathologies:

1. The model is not generally well-posed.
2. Null dust is ill-posed at $r = 0$.
3. ρ blows up at the bounce hypersurface.
4. ρ blows up where the Maxwell field vanishes.

Obtaining Vlasov spacetimes from dust spacetimes

Most of the work in [KU24] goes into proving the following:

Charged null dust approximation theorem.

Ori's bouncing charged null dust model arises as a limit of C^∞ solutions to the Einstein–Maxwell–charged Vlasov model:

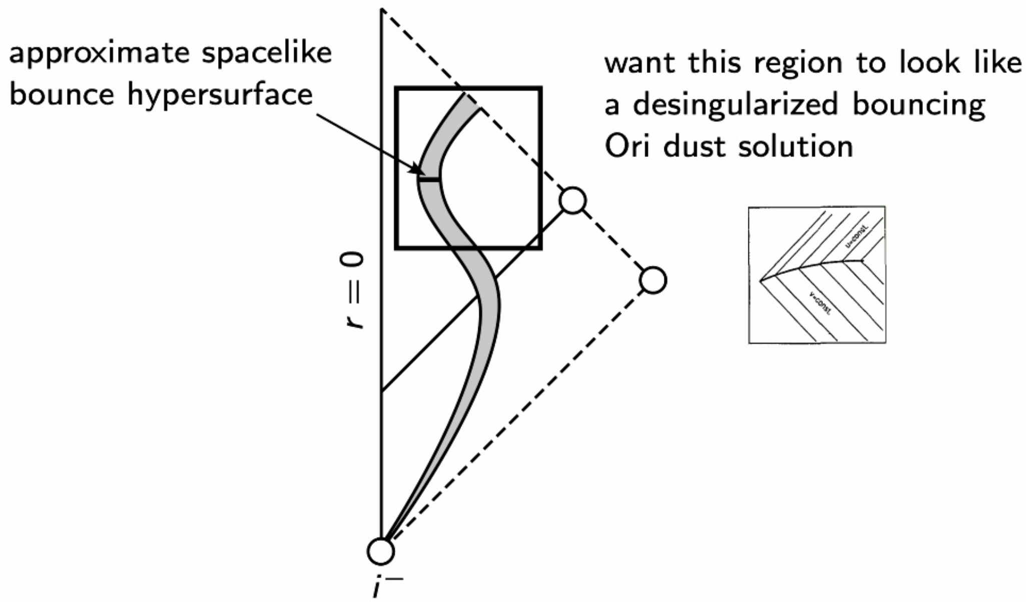
Given a compact subset of an Ori bouncing charged null dust spacetime $(K, g_{\mathrm{dust}}, F_{\mathrm{dust}}, k, \rho)$, there exist $\mathfrak{m}_i \searrow 0$ and $\ell_i \searrow 0$ and spherically symmetric C^∞ charged Vlasov solutions (K, g_i, F_i, f_i) such that

1. f_i *is supported on particles of mass* $\mathfrak{m}_i$ *and angular momentum* $\sim \ell_i$ *(caveat)*,
2. $g_i \to g_{\mathrm{dust}}$ *in* C^1,
3. $F_i \to F_{\mathrm{dust}}$ *in* C^0, *and*
4. $\int p_\mu f_i \, dp \to \rho k_\mu$, $\int p_\mu p_\nu f_i \, dp \to \rho k_\mu k_\nu$ *in the sense of distributions.*

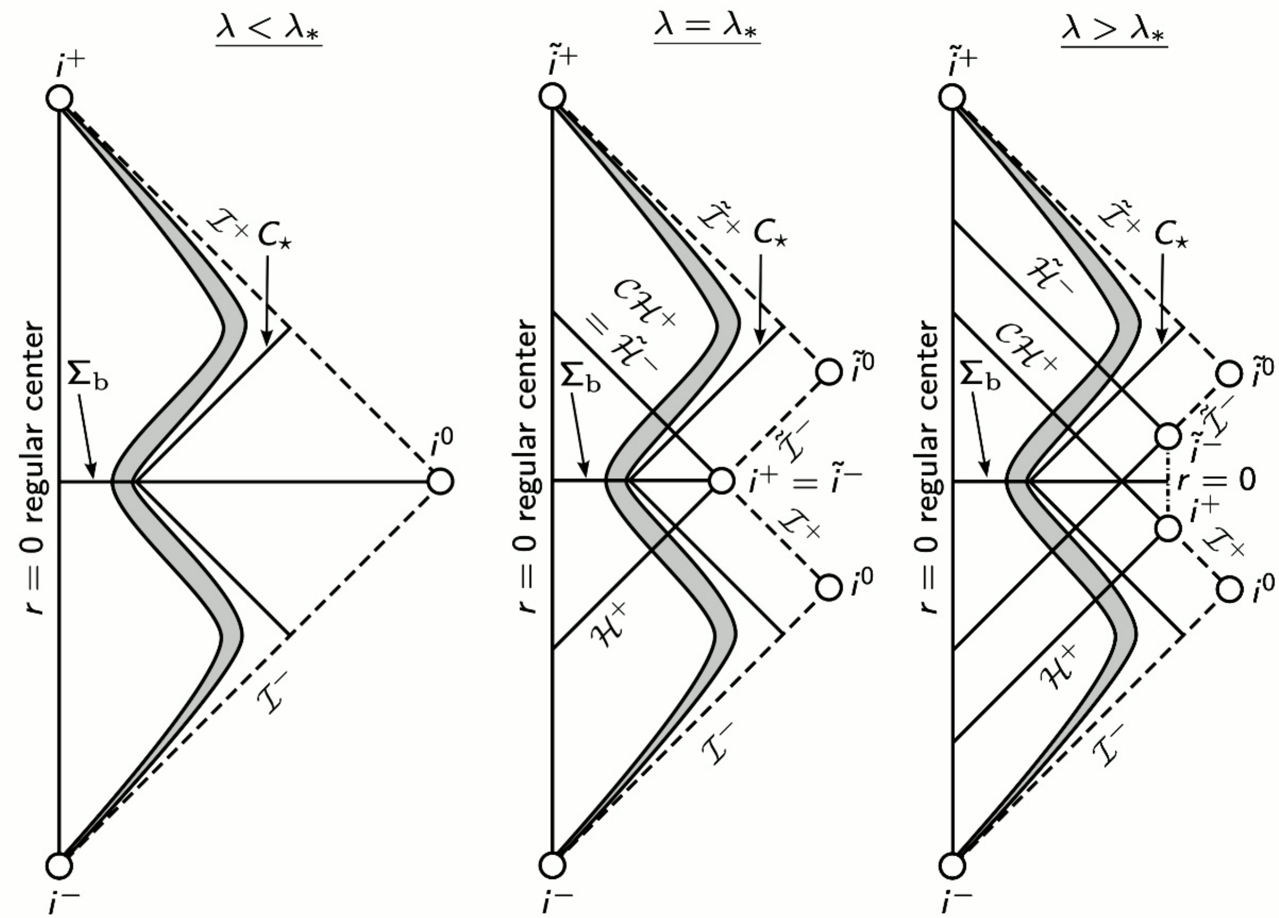
Outgoing Vlasov matter likes to remain outgoing.

Once the matter content of a spherically symmetric Einstein–Maxwell–Vlasov solution reaches "large enough r" and is "sufficiently outgoing," it will disperse.

Construction of bouncing charged Vlasov beams



Bouncing charged Vlasov beams—the time symmetric picture



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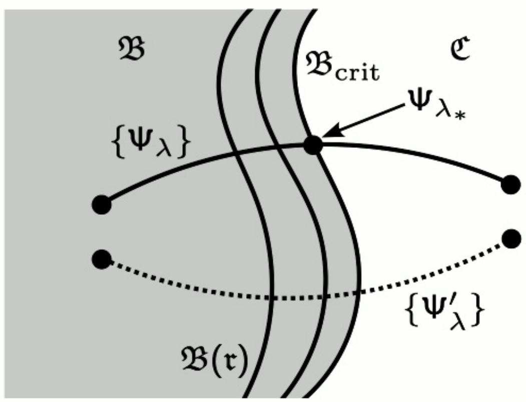
Stability of extremal critical collapse

Conjecture.

Extremal critical collapse is stable:

Let $\{\Psi_\lambda\}$ be one of the interpolating families given by [KU24]. Then there exists a "codimension-one submanifold" $\mathfrak{B}_{\mathrm{crit}}$ of the spherically symmetric moduli space $\mathfrak{M}$ such that $\Psi_{\lambda_*} \in \mathfrak{B}_{\mathrm{crit}} \subset \mathfrak{B}$, which has the following properties:

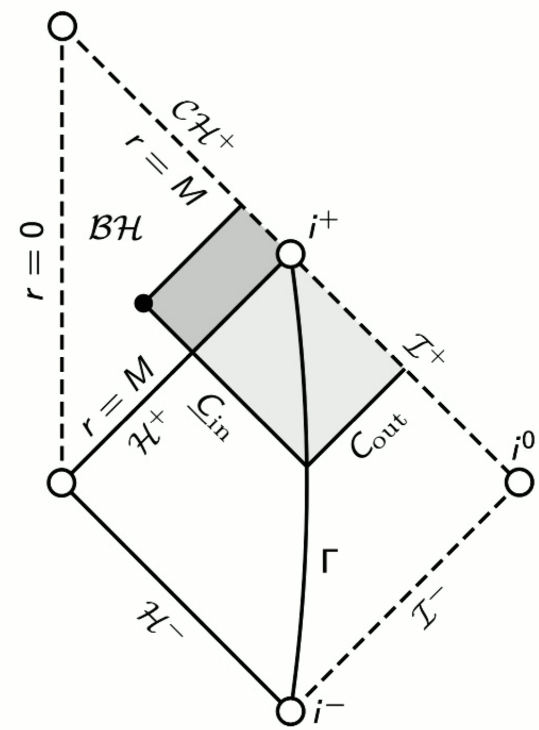
1. *$\mathfrak{B}_{\mathrm{crit}}$ is critical in the sense that $\mathfrak{B}$ and $\mathfrak{C}$ locally lie on opposite sides of $\mathfrak{B}_{\mathrm{crit}}$.*
2. *If $\Psi \in \mathfrak{B}_{\mathrm{crit}}$, the DOC of the maximal Cauchy development of Ψ asymptotically settles down to an extremal Reissner–Nordström black hole.*



- This is also a highly nontrivial statement about **interiors** of dynamical extremal black holes
- Fundamental difficulty: Aretakis instability associated to extremal horizons

The stability problem for extremal black holes

We first need to focus on dynamics near the event horizon of ERN.



The stability problem for extremal black holes

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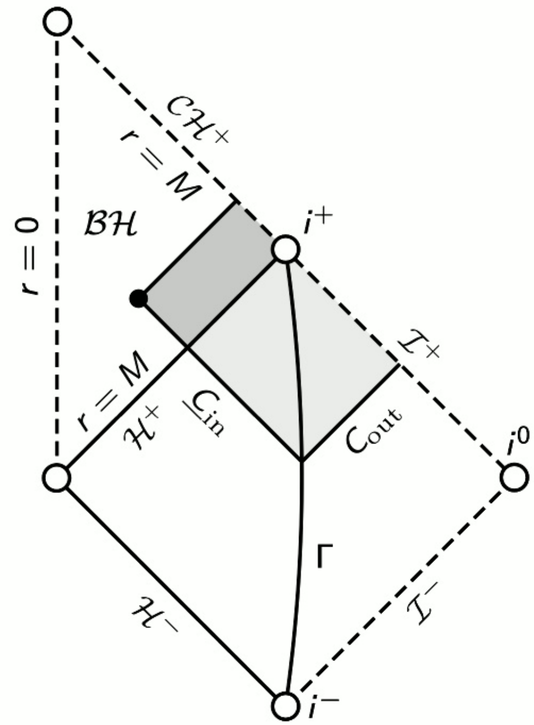


Question.

In what sense is the DOC of ERN stable as a solution to the Einstein equations? From considerations of the RN family, can expect *<u>at most codimension one</u>* *stable.*

Linear vs. nonlinear effects in the black hole stability problem

Linear

- Show sufficiently fast decay for gauge invariant quantities (e.g., Teukolsky), taking into account slow polynomial tails, trapped null geodesics, and redshift
- In fact need to estimate the full linearized system in a well-posed gauge, which might require "teleological normalization" of the gauge

Nonlinear:

- Need the nonlinearity to "cooperate" and allow the linear decay to persist

Example: JOHN's equation (1979)

$$\Box_\eta \phi = (\partial_t \phi)^2 \quad \text{on } \mathbb{R}^{3+1}_{t,x,y,z}$$

is *linearly stable* around $\phi = 0$, but experiences *finite time blowup* for nontrivial data.

A lesson taken from the **stability of Minkowski space** is that the Einstein equations have a cooperative structure at $\mathcal{I}^+$ (null structure).

The linear story for ERN: the Aretakis instability

Consider the scalar (massless) linear wave equation

$$\Box_{g_{\mathrm{ERN}}}\phi = \nabla^\mu\nabla_\mu\phi = 0, \qquad (*)$$

as an initial "poor man's linearization" of the Einstein equations around ERN.

Theorem (Aretakis '10).

Generic solutions to $(*)$ on ERN have the following behavior (where v and r are ingoing Eddington–Finkelstein coordinates):

1. $|\phi| \leq Cv^{-1}$ *along* $\mathcal{H}^+$,
2. $\partial_r\phi$ *does not decay along* $\mathcal{H}^+$,
3. $\partial_r^2\phi \sim v$ *along* $\mathcal{H}^+$, *and*
4. $\partial_r^k\phi \sim v^{k-1}$ *along* $\mathcal{H}^+$ *for* $k \geq 3$.

- The origin of 2. is that the (zeroth) Aretakis charge

$$H_0[\phi] := \int_{\mathcal{H}^+\cap\{v=\mathrm{const}\}} \partial_r(r\phi)\, d\omega$$

  is conserved along $\mathcal{H}^+$.
- Lucietti–Murata–Reall–Tanahashi and Apetroaie have established decay/growth hierarchies for Teukolsky on ERN.

Nonlinear ramifications of the Aretakis instability

The Aretakis instability implies a version of John's example for $\mathcal{H}^+$ of ERN:

Theorem (Aretakis '13).

The nonlinear wave equation

$$\Box_{g_{\mathrm{ERN}}}\phi = (\partial_r\phi)^2 \qquad (**)$$

experiences finite time blowup for certain initial data.

This "nonlinear instability" is driven by the linear instability in the following sense: For a solution to $(**)$, $H_0[\phi]$ is not conserved, but satisfies

$$\partial_v H_0[\phi] \geq c(H_0[\phi])^2$$

and hence $H_0[\phi] \to +\infty$ in finite time if $H_0[\phi] > 0$ initially.

Lesson: If ERN is to be stable, we need to identify appropriate null structure in the nonlinearity of the Einstein equations which is "compatible" with the Aretakis instability.

Weak stability of extremal Reissner–Nordström in spherical symmetry

Einstein–Maxwell-neutral/real scalar field model

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 2T^{\mathrm{EM}}_{\mu\nu} + 2\big(\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi\big),$$

$$\nabla^\nu F_{\mu\nu} = 0, \qquad \nabla^\mu\nabla_\mu\phi = 0$$

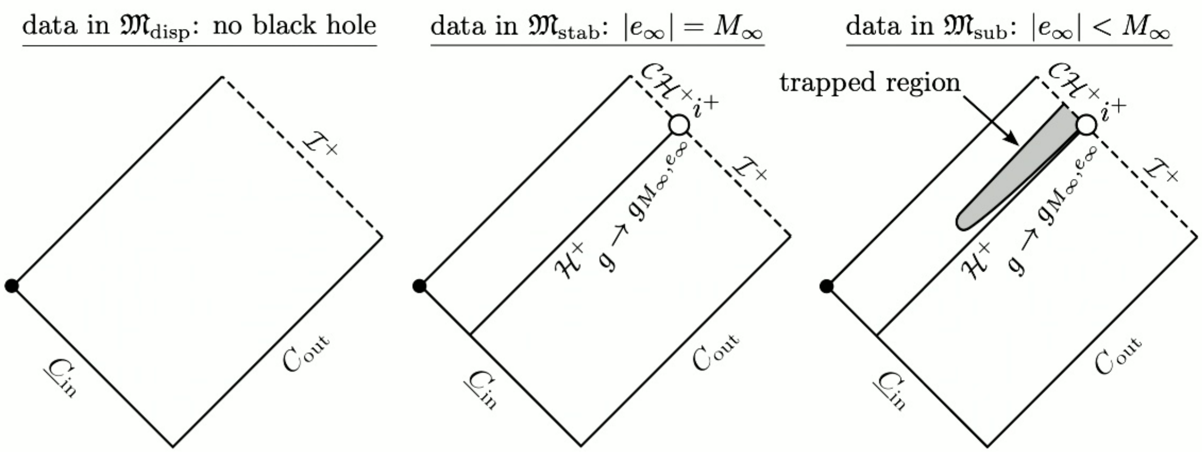
Theorem (Angelopoulos–Kehle–U. '24).
ERN is codimension-1 stable in the spherically symmetric EMSF model:

Given a scalar perturbation of ERN with $\|\phi_\circ\| \leq \varepsilon$, after making a (teleologically determined) $O(\varepsilon^2)$ adjustment to the mass, the following holds:

1. *A black hole forms,*
2. $\sup_{\Sigma_\tau} |g - g_{\mathrm{ERN}}| \to 0$ *uniformly up to $\mathcal{H}^+$,*
3. $\sup_{\Sigma_\tau} |\phi| \to 0$,
4. *$\partial_r\phi$ remains approximately constant on $\mathcal{H}^+$, and*
5. *for "generic" data, $\partial_r^2\phi \sim v$ along $\mathcal{H}^+$.*

- Due to the coupling of the scalar field to the geometry, suitable components and derivatives of $R_{\mu\nu}$ exhibit non-decay and growth phenomena along $\mathcal{H}^+$.
- We identify the "right" structure in the Einstein equations that allows the Aretakis instability for ϕ to be consistent with "weak stability" of g.

Local critical behavior in the neutral scalar field model



Conjecture.

Near ERN, the solution space of the spherically symmetric EMSF model can be decomposed as $\mathfrak{M}_{\rm disp} \cup \mathfrak{M}_{\rm stab} \cup \mathfrak{M}_{\rm sub}$, *where*

1. $\mathfrak{M}_{\rm disp}$ *consists of solutions that do not form a black hole in* $D^+(\underline{C}_{\rm in} \cup C_{\rm out})$,
2. $\mathfrak{M}_{\rm stab}$ *is a* C^1 *Banach hypersurface and consists of solutions asymptotic to ERN,*
3. $\mathfrak{M}_{\rm sub}$ *consists of solutions which are asymptotically subextremal.*

This has been observed numerically by MURATA–REALL–TANAHASHI '13

Stability of ERN for charged Vlasov matter

Conjecture.

An appropriate version of the Theorem of [AKU24] holds for charged, massless, self-gravitating Vlasov matter: ERN is codimension-1 stable in the Einstein–Maxwell–Vlasov model.

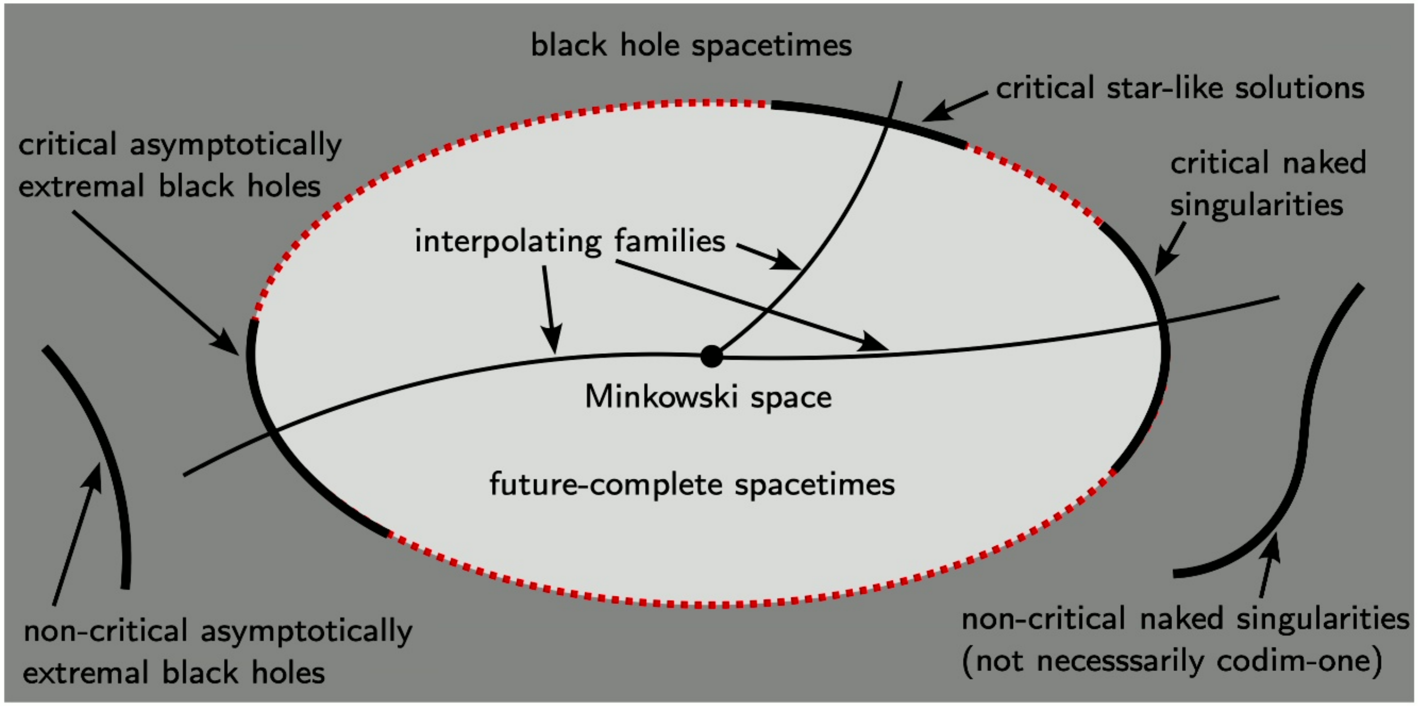
- This would be important evidence towards the codimension-1 property of extremal critical collapse in the charged Vlasov model.
- Complicated by several technical issues absent in the neutral scalar field model: trapping at the photon sphere, dynamical charge,

The vacuum case?

In principle, *extremal critical collapse* and its *stability* can be conjectured to also hold true in **vacuum** with extremal Reissner–Nordström replaced by **extremal Kerr**.

However, this is a very difficult problem which also relates to understanding

- the formation of extremal black holes in vacuum (the case $|a| \ll M$ has been resolved by KEHLE–U. '23)



black hole spacetimes
critical star-like solutions
critical asymptotically extremal black holes
critical naked singularities
interpolating families
Minkowski space
future-complete spacetimes
non-critical asymptotically extremal black holes
non-critical naked singularities (not necesssarily codim-one)

Thank you!