Title: Lecture - Mathematical Physics, PHYS 777

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TODAY

ADHM construction

in the general case:

Rank N instantons of charge K

Rank 2, charge 1 we have the complex

$$
\begin{tikzcd}
 & \mathcal{O} \arrow[dr, "v_1"] & \\
\mathcal{O}(-1) \arrow[ur, "-v_2"] \arrow[r, "v_1"] \arrow[dr, "-z"] \arrow[ddr, "1"] & \mathcal{O} \arrow[r, "v_2"] & \mathcal{O}(1) \\
 & \mathcal{O} \arrow[ur, "1"] & \\
 & \mathcal{O} \arrow[uur, "z"] &
\end{tikzcd}
$$

General ADHM complex:

$$\begin{tikzcd}
 & \mathcal{O}^K \arrow[dr, "V_1 + X_1(z)"] & \\
\mathcal{O}(-1)^K \arrow[ur, "-V_2 - X_2(z)"] \arrow[r, "V_1 + X_1(z)"'] \arrow[dr, "I(z)"'] & \mathcal{O}^K \arrow[r, "V_2 + X_2(z)"] & \mathcal{O}(1)^K \\
 & \mathcal{O}^N \arrow[ur, "J(z)"'] &
\end{tikzcd}$$

$\frac{1}{}$ $\mathcal{O}$

$\mathcal{O}$

z

$\mathcal{O}(1)^k$

$I(z)$ is a $K \times N$ matrix

$I(z) = I_1 + z I_2$ $\quad I_\alpha$ are $K \times N$ matrices

$J(z) = J_1 + z J_2$ $\quad J_\alpha$ are $N \times K$ matrices

$X_{\dot\alpha}(z) = X_{1\dot\alpha} + z X_{2\dot\alpha}$ $\quad X_{\alpha\dot\alpha}$ are $K \times K$ matrices

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$I(z)$ is a $K \times N$ matrix

$I(z) = I_1 + z I_2$ $\quad$ I_α are $K \times N$ matrices

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$X_{\dot\alpha}(z) = X_{1\dot\alpha} + z X_{2\dot\alpha}$ $\quad$ $X_{\alpha\dot\alpha}$ are $K \times K$ matrices

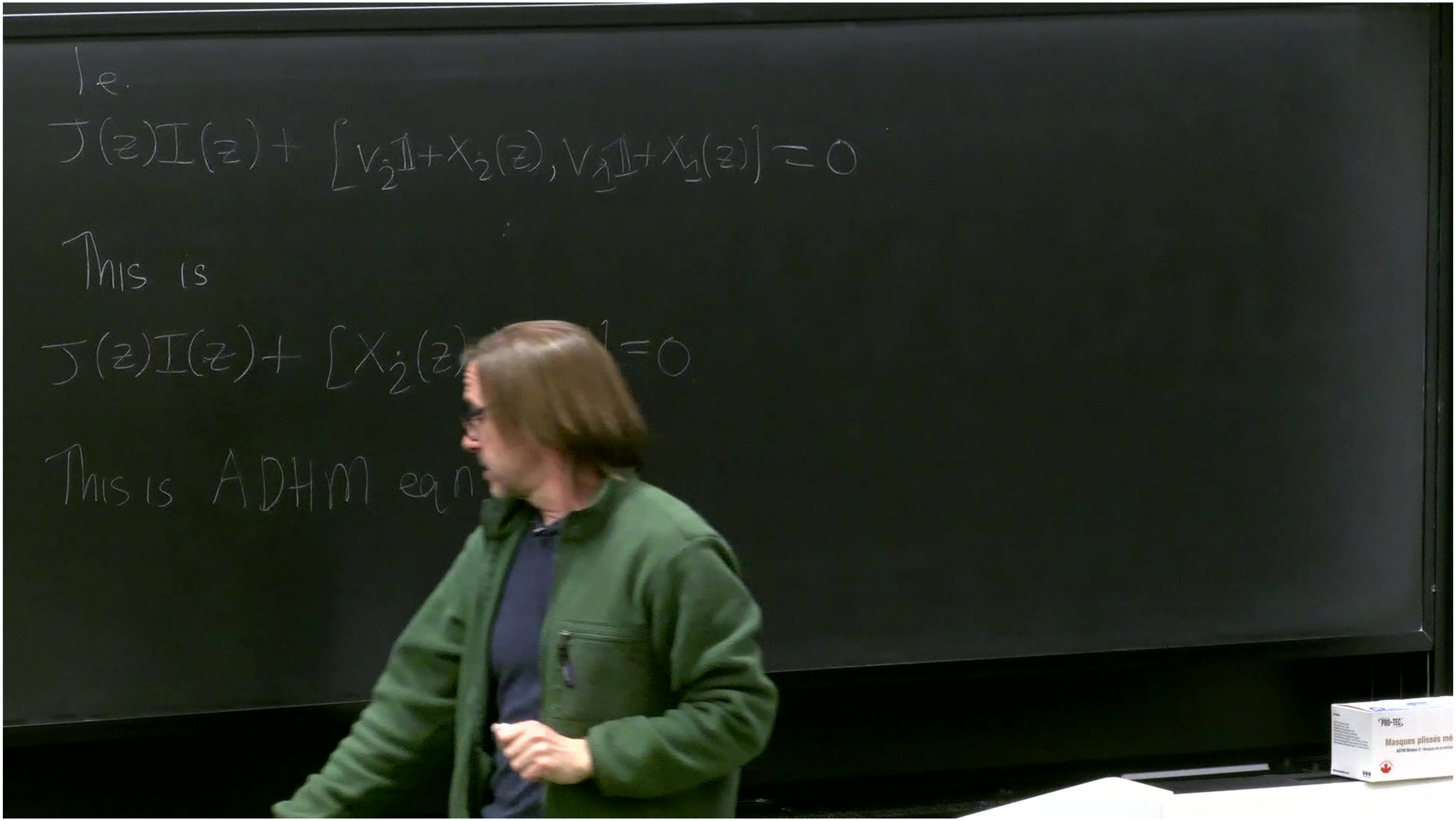
We want to take cohomology so we need the composition of the 2 maps to be zero.

This is the equation:

$$J(z)I(z) + (V_2 + X_2(z))(V_1 + X_1(z)) - (V_1 + X_1(z))(V_2 + X_2(z)) = 0$$

i.e.

$J(z)I(z$


i.e.
$J(z)I(z) + [v_2\mathbb{1} + X_2(z), v_1\mathbb{1} + X_1(z)] = 0$
This is
$J(z)I(z) + [X_2(z)$
$] = 0$
This is ADHM eqn

This is an eq^n of $K \times K$
matrices which depend on z,
but have constant, linear, quadratic
terms in z.

3 $K \times K$ matrix equations.

In index notation, we have

$$J_\alpha I_\beta - \frac{1}{2}[X_{\alpha\dot{\alpha}}, X_{\beta\dot{\beta}}]\,\varepsilon^{\dot{\alpha}\dot{\beta}} = 0$$

where α, β indices are symmetrized.

$$\underline{I}(z) \longrightarrow \mathcal{O}^N$$

These are taken up to gauge trans.

$u \in GL(k)$,

$$X(z) \longrightarrow u\,X(z)\,u^{-1}$$

$$\underline{I}(z) \longrightarrow \underline{I}(z)\,u^{-1}$$

$$J(z) \longrightarrow u\,J(z)$$

$J(z)$

$X_{\dot\alpha}(z) = X_{1\dot\alpha} + z X_{2\dot\alpha}$ $\quad X_{\alpha\dot\alpha}$ are $k \times k$ matrices

preserves the eqn and gives an equivalent

instanton

Rank 2
Charge 1 $\quad X_{\alpha\dot\alpha}$ is a 1×1 matrix

$\rightsquigarrow X_{\alpha\dot\alpha}$ is a point in space-time. center of instanton.

ADHM eqn is $\quad J_1 I_1 = 0 \quad J_2 I_2 = 0 \quad J_2 I_1 + I_1 J_2 = 0$

After a change of basis on $\mathcal{O}^2$, $I = \begin{pmatrix} \rho \\ \rho z \end{pmatrix}$ $J = \begin{pmatrix} -\rho z \\ \rho \end{pmatrix}$ is only soln e.g. We can build a charge 2 rank 4 instanton by taking 2 different charge 1 rank 1 instantons, and adding

2

arge 1

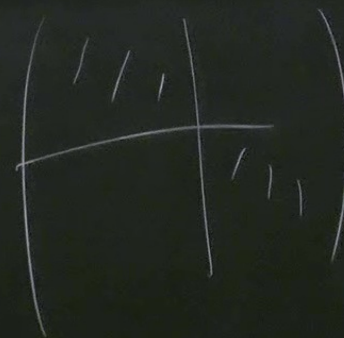
lding

1st charge 1 has $X=0$

2nd has some $x \in \mathbb{C}^4$

Rank 4, charge 2

$$X = \begin{pmatrix} 0 & \\ & x \end{pmatrix}$$



I = Sum of I's of charge 1, rk 2 case

Same for J

As before, if $V_{\dot{\alpha}} \longrightarrow \infty$

i.e. we are at ∞ on space-time,

then after a rescaling of $\mathcal{O}(-1), \mathcal{O}(1)$

bundles, complex becomes

...ge of basis on $\mathcal{O}^2$, $I = \begin{pmatrix}\rho \\ \rho z\end{pmatrix}$ $J = \begin{pmatrix}-\rho z \\ \rho\end{pmatrix}$ is only sol^{n}

$$\begin{tikzcd} & \mathcal{O}^k \arrow[dr, "V_1"] & \\ \mathcal{O}(-1)^k \arrow[ur, "-V_2"] \arrow[dr, "V_1"'] & & \mathcal{O}(1)^k \\ & \mathcal{O}^k \arrow[ur, "V_2"'] & \end{tikzcd}$$

$$\mathcal{O}^N$$

Same computation tells us coho is $\mathcal{O}^N$

At ∞, we have a gauge where $A \to 0$.

Also, charge of $\mathcal{O}(1)^k$ or $\mathcal{O}(-1)^k$ is k times the charge of $\mathcal{O}(1)$ or $\mathcal{O}(-1)$, so we get charge k.

Most degenerate instanton

all charge localizes at K points

This happens if $I, J = 0$.

Then, ADHM eq^{n}s tell us

$$[X_{\alpha\dot{\alpha}}, X_{\beta\dot{\beta}}]\,\varepsilon^{\dot{\alpha}\dot{\beta}} = 0$$

degenerate instanton

harge localizes at K points

appens if $I, J = 0$

ADHM eq^{n}s tell us

$$[X_{\alpha\dot{\alpha}}, X_{\beta\dot{\beta}}]\, \epsilon^{\dot{\alpha}\dot{\beta}} = 0$$

$$[X_{1\dot{\alpha}}, X_{1\dot{\beta}}] = 0$$

$$[X_{2\dot{\alpha}}, X_{2\dot{\beta}}] = 0$$

$$[X_{1\dot{\alpha}}, X_{2\dot{\beta}}] + [X_{2\dot{\alpha}}, X_{1\dot{\beta}}] = 0$$

We can diagonalize the X's

eigenvalues are locations of the charge.

Dim^n of the space of instantons is:

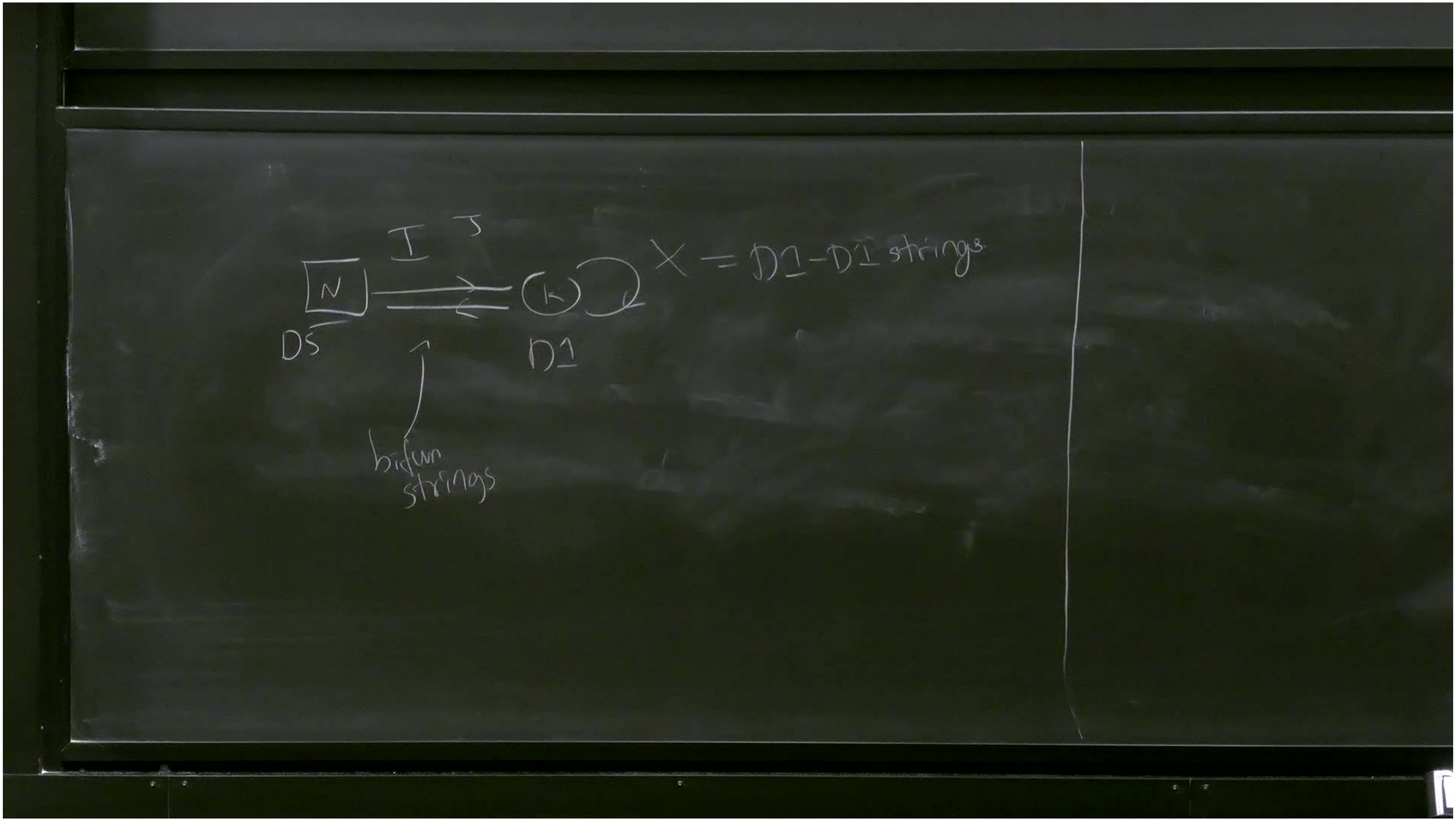
- $\underline{I}, J$ give $4Nk$
- $X_{\alpha\dot{\alpha}}$ gives $4k^2$
- 3 ADHM eqns are eq^{n}s for $k\times k$ matrices gives $-3k^2$
- $GL(k,\mathbb{C})$ gauge trans. gives $-k^2$

Dim^n is

$$4Nk + 4k^2 - 4k^2$$
$$= 4Nk$$

Theorem (ADHM, using work of Ward + others)

Every $GL(N, \mathbb{C})$ instanton on $\mathbb{R}^4$, of charge k, with a gauge at ∞ so $A = 0$ there, arises uniquely (up to gauge tr) from ADHM construction.


I J
N
D5
k
X = D1-D1 strings
D1
bifun
strings

After a change of basis on $\mathcal{O}^2$, $I = \begin{pmatrix} \rho \\ \rho z \end{pmatrix}$ $J = \begin{pmatrix} -\rho z \\ \rho \end{pmatrix}$ is only sol^n

Reality conditions

If we want a $U(N)$ gauge field

what do we do?

...ge of basis on Θ^2; $I = \begin{pmatrix} \rho \\ \rho z \end{pmatrix}$ $J = \begin{pmatrix} -\rho z \\ \rho \end{pmatrix}$ is only sol^n

conditions

...ant a $U(N)$ gauge field

...e do?