Title: Lecture - Mathematical Physics, PHYS 777

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Subject: Mathematical physics

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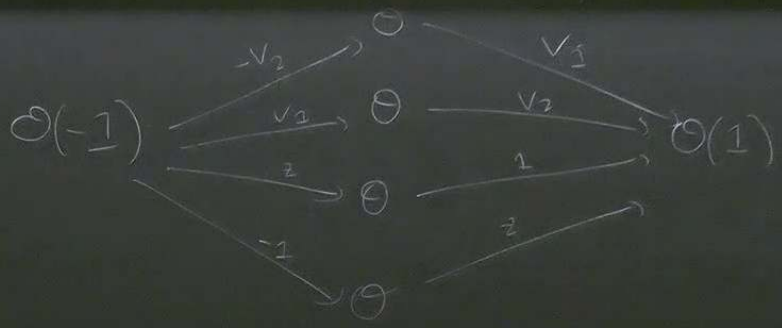
LAST TIME

ADHM complex for a charge 1 instanton



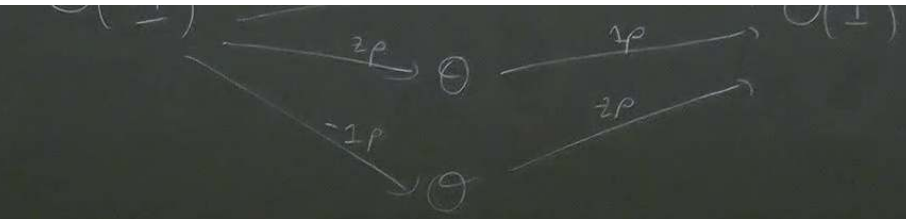
$$\mathcal{O}(-1) \xrightarrow{g} \mathcal{O}^4 \xrightarrow{f} \mathcal{O}(1)$$

Bundle on $\mathbb{PT}$

$$E = \frac{\ker f}{\mathrm{Im}\, g}$$

- E is a rank 2 bundle on $\mathbb{PT}$
- Instanton charge 1 (charge from $\mathcal{O}(-1), \mathcal{O}(1)$)

a charge 1 instanton



$$\mathcal{O}(-1) \xrightarrow{g} \mathcal{O}^4 \xrightarrow{f} \mathcal{O}(1)$$

Bundle on $\mathbb{PT}$

$$E = \frac{\ker f}{\mathrm{Im}\, g}$$

- E is a rank 2 bundle on $\mathbb{PT}$
- Instanton charge 1 (charge from $\mathcal{O}(-1), \mathcal{O}(1)$)
- Gives a sol^n to SDYM for $SL(2,\mathbb{C})$ on $\mathbb{R}^4$

What happens at ∞?

Consider rescaling $V_a \to cV_a$

Send $c \to \infty$

f, g are

$$\begin{pmatrix} cV_1 \\ cV_2 \\ 1 \\ z \end{pmatrix}, \quad \begin{pmatrix} -cV_2 \\ cV_1 \\ -z \\ 1 \end{pmatrix}$$

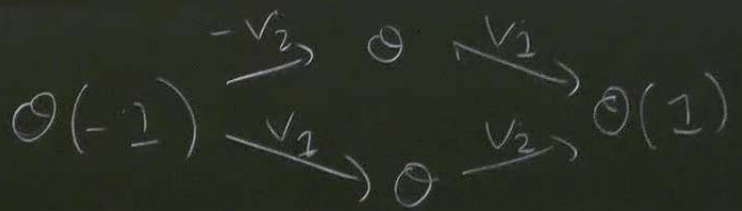
Rescaling the basis of $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ by $c^{\pm 1}$

sends f, g to

$$\begin{pmatrix} V_1 \\ V_2 \\ c^{-1} \\ c^{-1}z \end{pmatrix}, \quad \begin{pmatrix} -V_2 \\ V_1 \\ -c^{-1}z \\ c^{-1} \end{pmatrix}$$

as $c \to \infty$ we get

Have



$\mathcal{O}^2$

Claim if v_α not both zero, the coho is $\mathcal{O}^2$.

I.e get trivial bundle of rank 2.

$A \to 0$ as $x_{\alpha\alpha} \to \infty$

If $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathcal{O}^2$, is in kernel of f if

$v_1 a = -v_2 b$

If $v_1 \neq 0$,

$$a = \frac{-v_2 b}{v_1}$$

$\begin{pmatrix} a \\ b \end{pmatrix}$ is in the image of g

if $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -v_2 e \\ v_1 e \end{pmatrix}$ some e,

which is true

which is true with $e = \frac{b}{V_1}$

If $V_1 \neq 0$, no cohomology

Similarly if $V_2 \neq 0$.

age of g

e)

e) some e,

$1) \xrightarrow{g} \mathcal{O}^4 \xrightarrow{f} \mathcal{O}(1)$

e with $e = \frac{b}{v_1}$.

$\mathcal{O}$, no cohomology

$v_2 \neq 0$.

What happens if $\rho \to 0$?

$$f = \begin{pmatrix} v_2 \\ v_2 \\ 1\rho \\ 2\rho \end{pmatrix} \quad g = \begin{pmatrix} -v_2 \\ v_1 \\ -2\rho \\ \rho \end{pmatrix}$$

if $\rho \to 0$ we get the same phenomenon

At $\rho = 0$, the gauge field on $\mathbb{R}^4$ is trivial except at $0 \in \mathbb{R}^4$ (corresponds to $v_i = 0$)

In this limit, $\frac{1}{8\pi^2} \mathrm{Tr}\left(F(A) \wedge F(A)\right)$

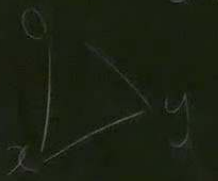
$= \delta_{x=0}$

$b_0 = \frac{11}{3} N_c - \frac{2}{3} N_f$

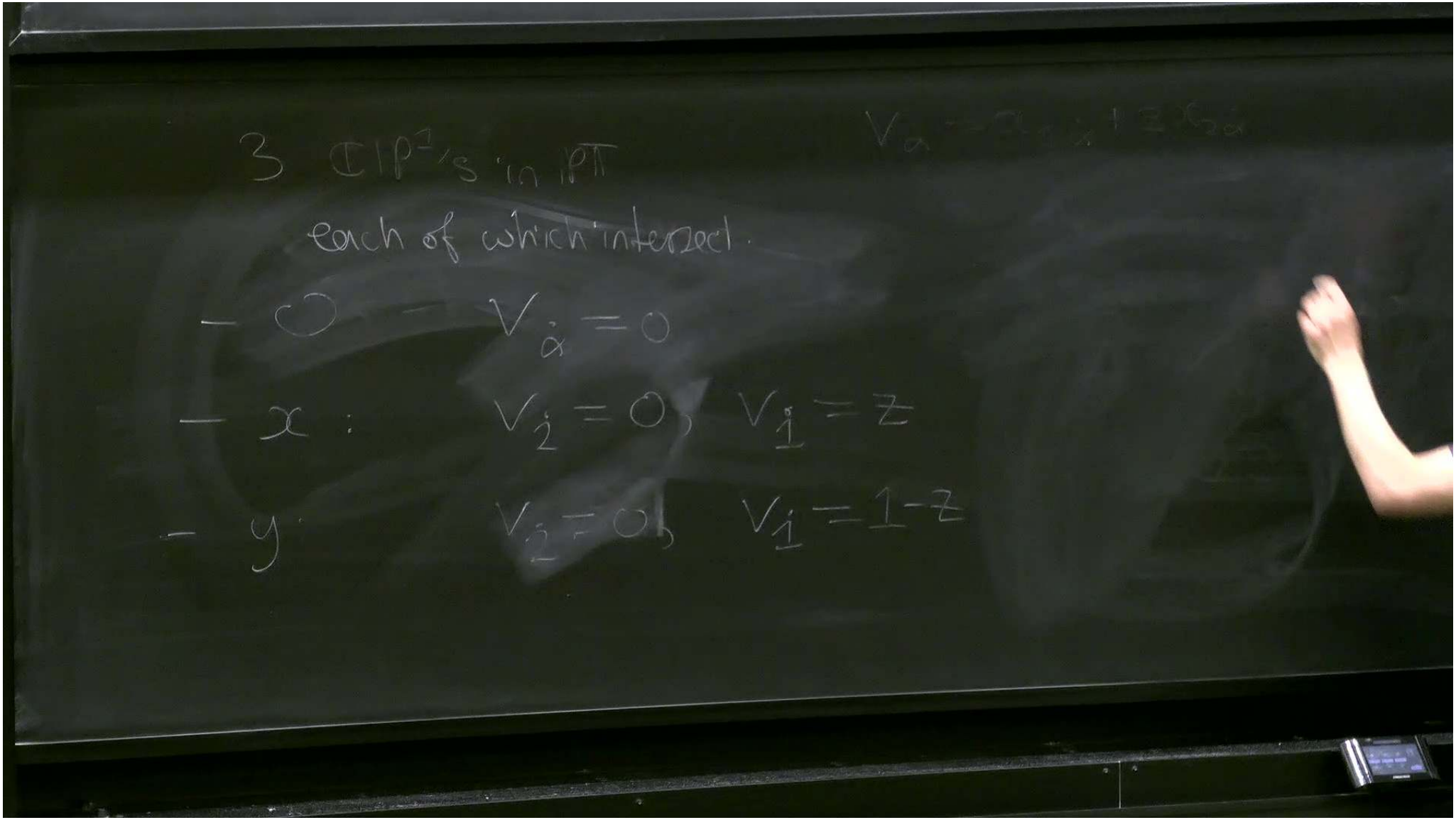
measure $\sim \rho^{b_0 - 5} d\rho$

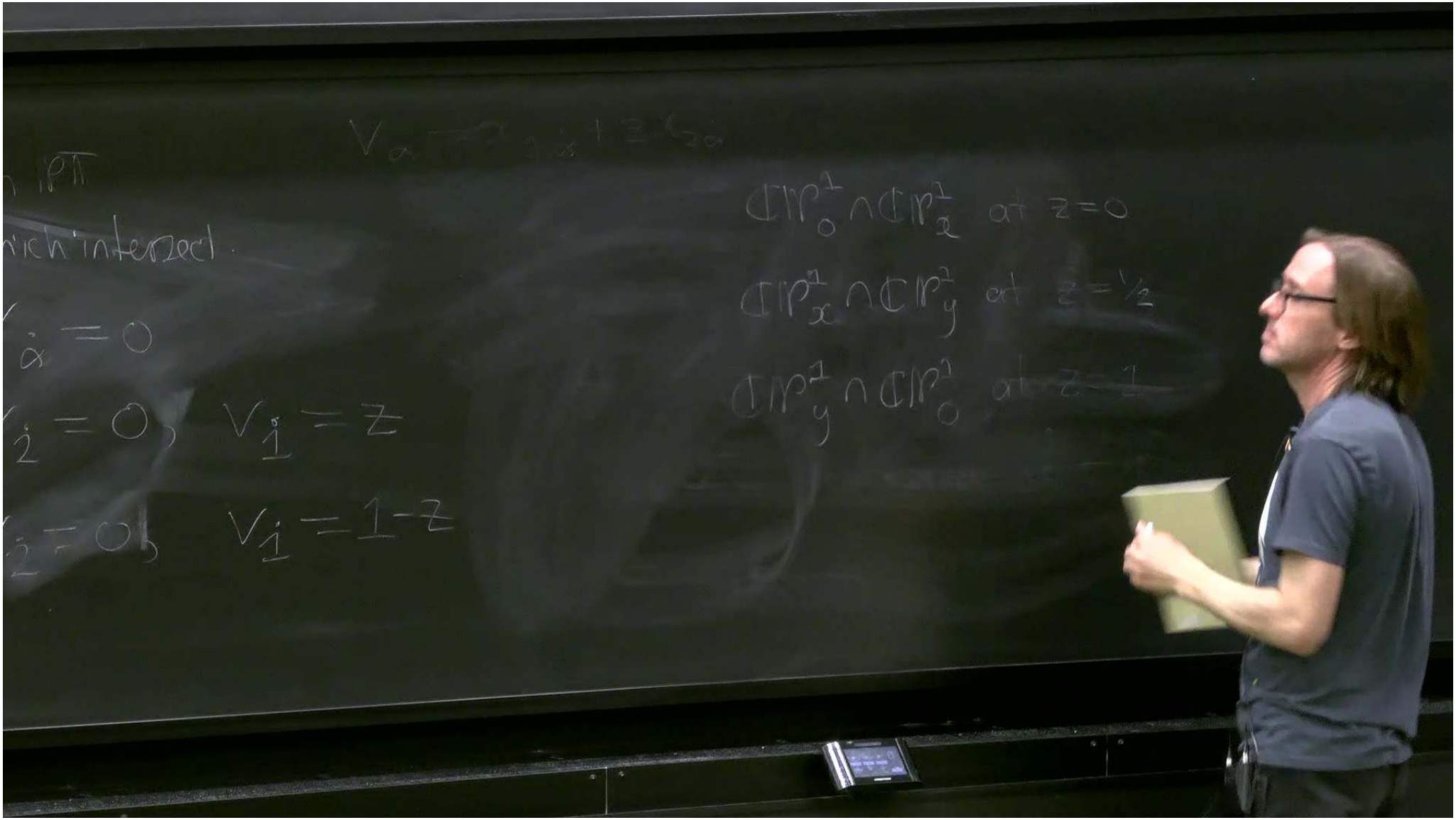
Consider null triangle given by 3 points

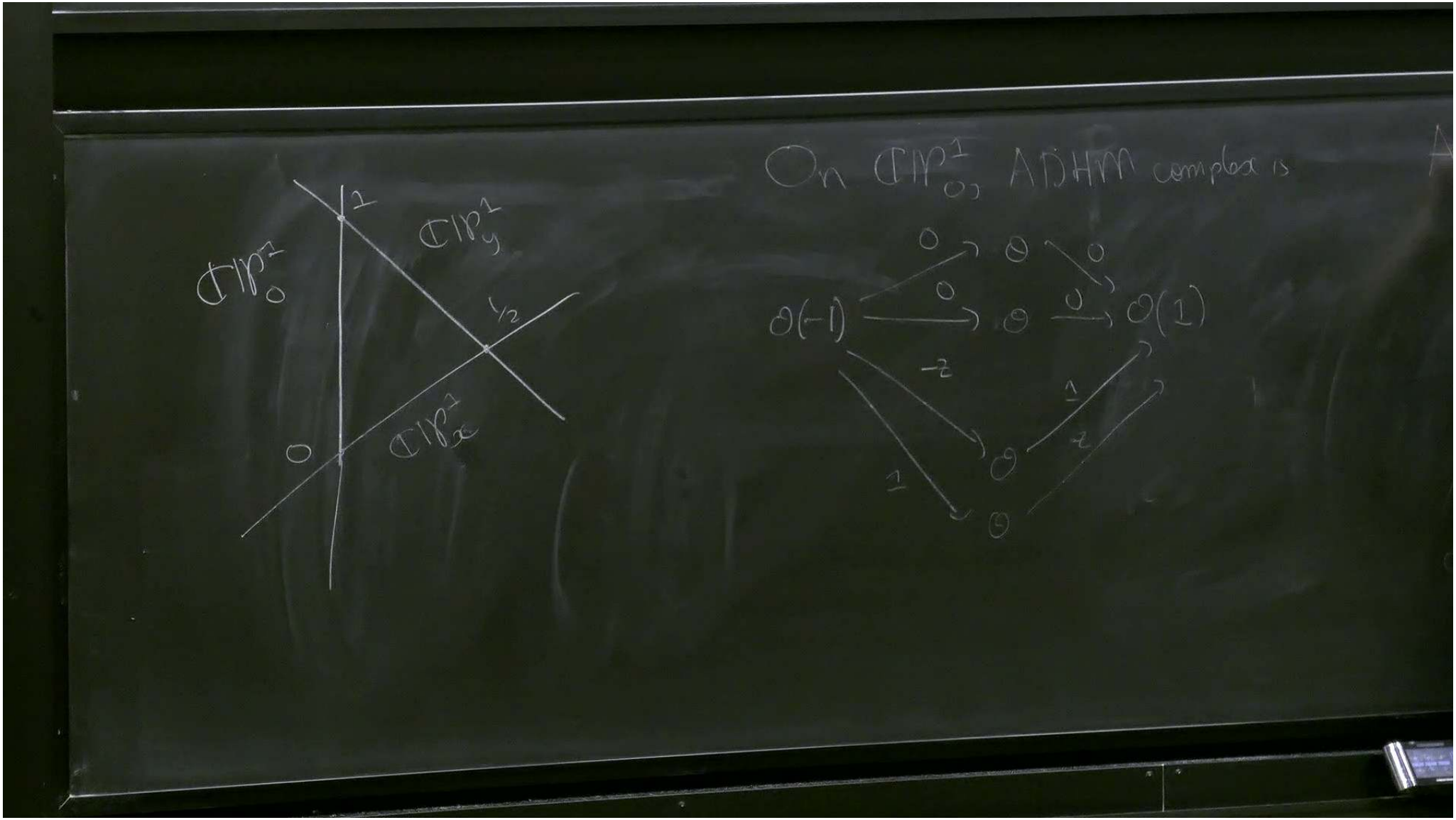
- O
- $x_{\alpha\dot{\alpha}}$, where $x_{\alpha\dot{2}} = 0$, $x_{2\dot{1}} = 1$, $x_{1\dot{1}} = 0$
- $y_{\alpha\dot{\alpha}}$, $y_{\alpha\dot{2}} = 0$, $y_{1\dot{1}} = 1$, $y_{2\dot{1}} = -1$



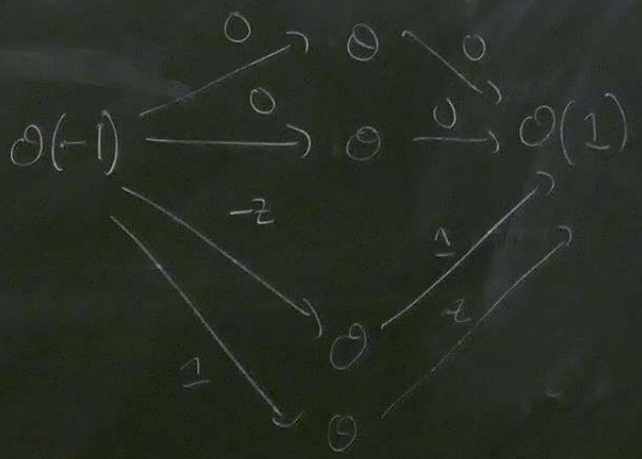
We'll compute parallel transport of charge 1 ADHM instanton.


3 $\mathbb{CP}^2$'s in IPT
each of which intersect.
V_a
$V_{\dot{\alpha}} = 0$
$V_2 = 0, \quad V_1 = z$
$V_2 = 0, \quad V_1 = 1 - z$


IPT
which intersect
$\alpha = 0$
$V_1 = z$
$V_1 = 1 - z$
$\mathbb{CP}^1_0 \cap \mathbb{CP}^1_z$ at $z = 0$
$\mathbb{CP}^1_x \cap \mathbb{CP}^1_y$ at $z = \frac{1}{2}$
$\mathbb{CP}^1_y \cap \mathbb{CP}^1_0$ at $z = 1$


On $\mathbb{CP}^1_0$, ADHM complex is
$\mathbb{CP}^1_0$
$\mathbb{CP}^1_y$
$\mathbb{CP}^1_x$
1
1/2
0
$\mathcal{O}(-1)$
$\mathcal{O}(1)$
0
0
0
0
0
0
$\mathcal{O}$
$\mathcal{O}$
$\mathcal{O}$
$-z$
1
1
$-z$

On $\mathbb{CP}^1_0$, ADHM complex is



A basis of $H^0(\mathbb{CP}^1, E)$ is

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$$

(Note, $H^0(\mathbb{CP}^1, \mathcal{O}^4) = \mathbb{C}^4$

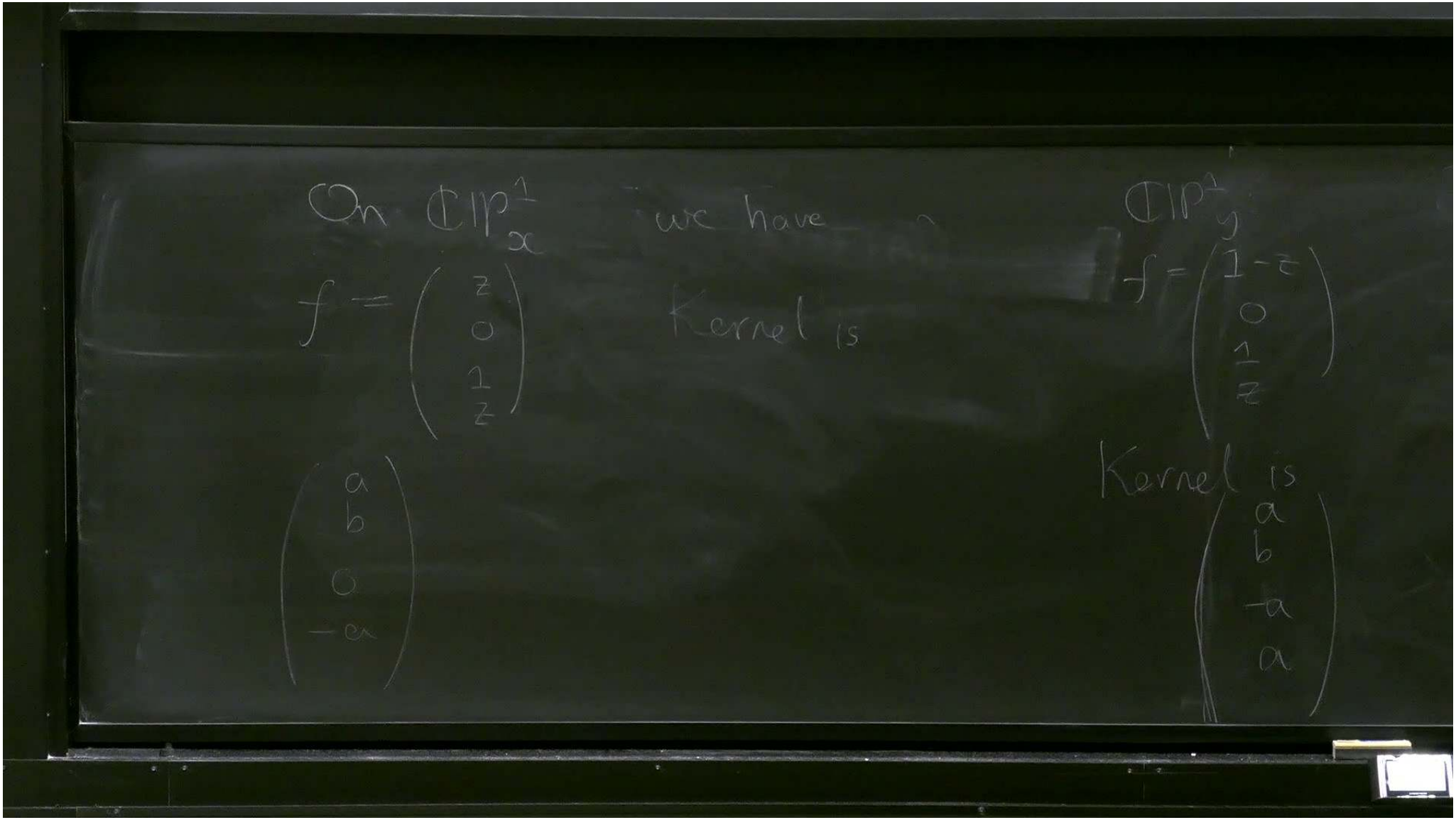
$H^0(\mathbb{CP}^1, \mathcal{O}(-1)) = 0$

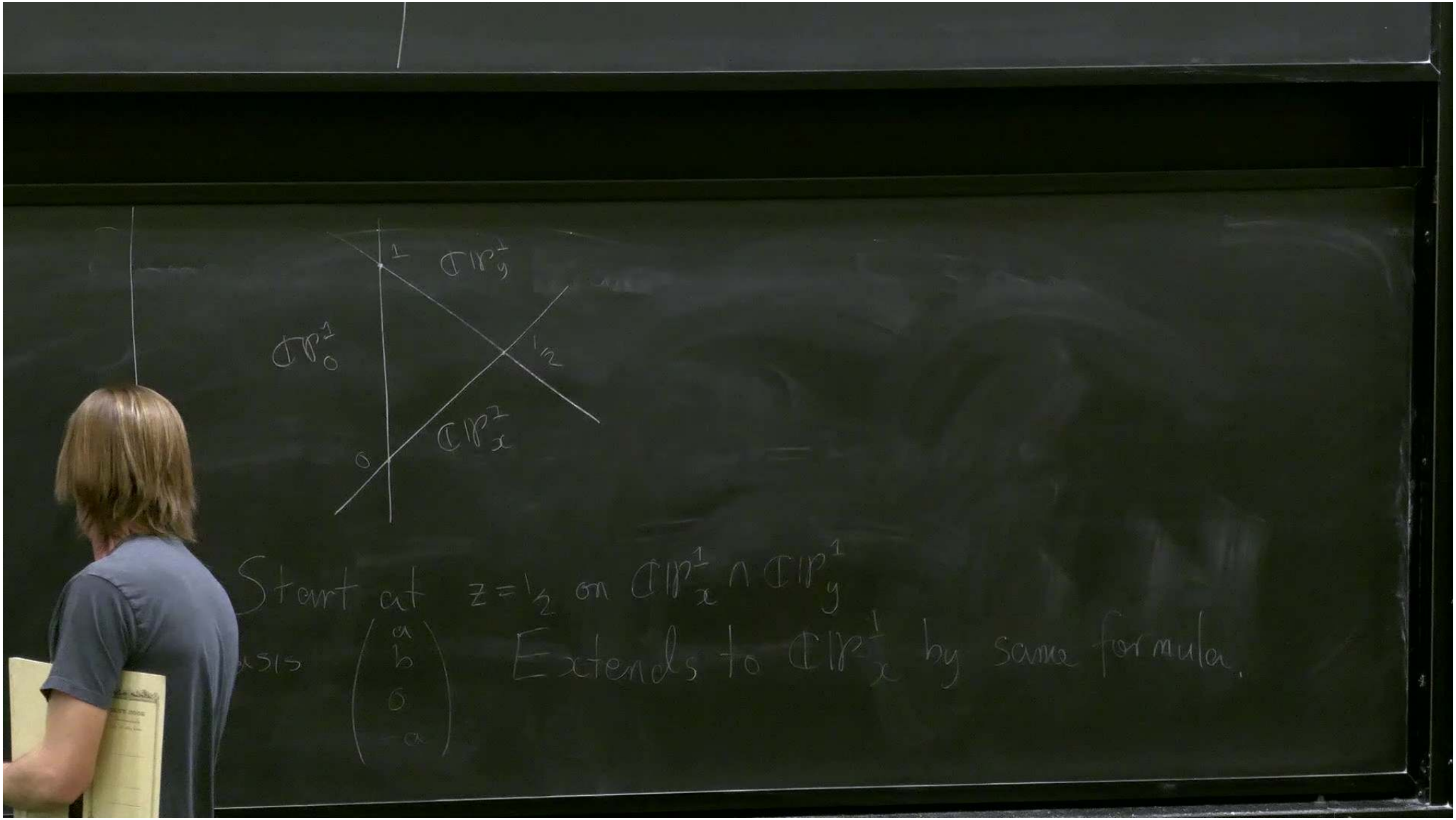
$H^0(\mathbb{CP}^1, \mathcal{O}(1)) = \mathbb{C}^2$

Cohomology is H

of the map

$H^0(\mathbb{CP}^1, \mathcal{O}^4) \to H$


On $\mathbb{CP}^1_x$ we have
$f = \begin{pmatrix} z \\ 0 \\ 1 \\ z \end{pmatrix}$
Kernel is
$\begin{pmatrix} a \\ b \\ 0 \\ -a \end{pmatrix}$
$\mathbb{CP}^1_y$
$f = \begin{pmatrix} 1 \\ 0 \\ 1 \\ z \end{pmatrix}$
Kernel is
$\begin{pmatrix} a \\ b \\ -a \\ a \end{pmatrix}$


1
$\mathbb{CP}^1_y$
$\mathbb{CP}^1_0$
∞
0
$\mathbb{CP}^2_x$
Start at $z=\frac{1}{2}$ on $\mathbb{CP}^1_x \cap \mathbb{CP}^1_y$
asis $\begin{pmatrix} a \\ b \\ 0 \\ a \end{pmatrix}$
Extends to $\mathbb{CP}^1_x$ by same formula.

At $z=0, v_0=0$ have

$$\begin{pmatrix} a \\ b \\ 0 \\ -a \end{pmatrix}$$

This does not extend to $\mathbb{CP}^1_0$.

$V_a = 0$ have

$$\begin{pmatrix} a \\ b \\ 0 \\ -a \end{pmatrix}$$

does not extend to

But: at $z = 0$,
we have to mod out by
image of g

$$g = \begin{pmatrix} -V_2 \\ V_1 \\ -z \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ at } V_a = 0, z = 0$$

: at $z=0$,
have to mod out by
ge of g

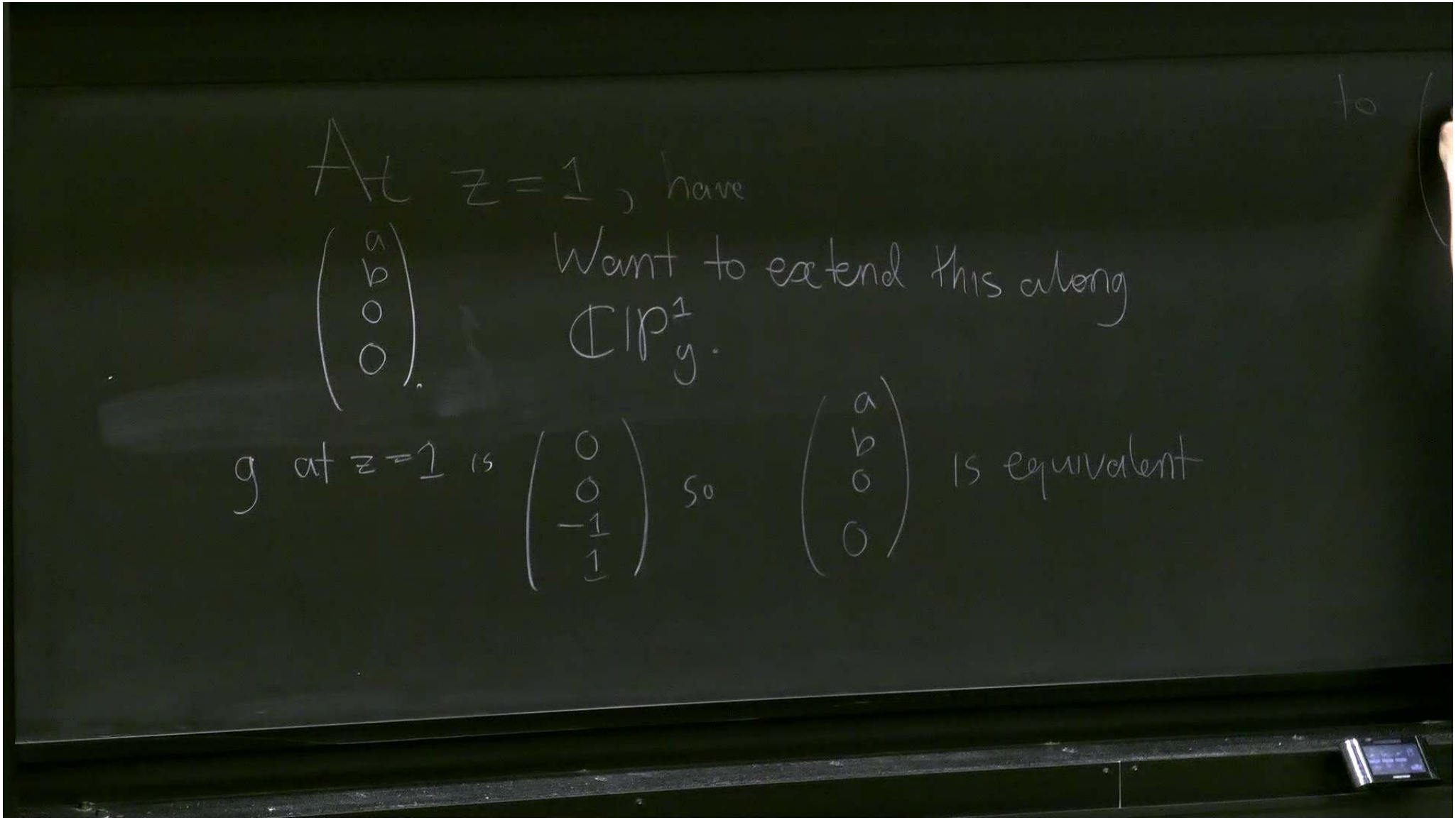
$$\begin{pmatrix} -v_2 \\ v_1 \\ -z \\ 1 \end{pmatrix}$$

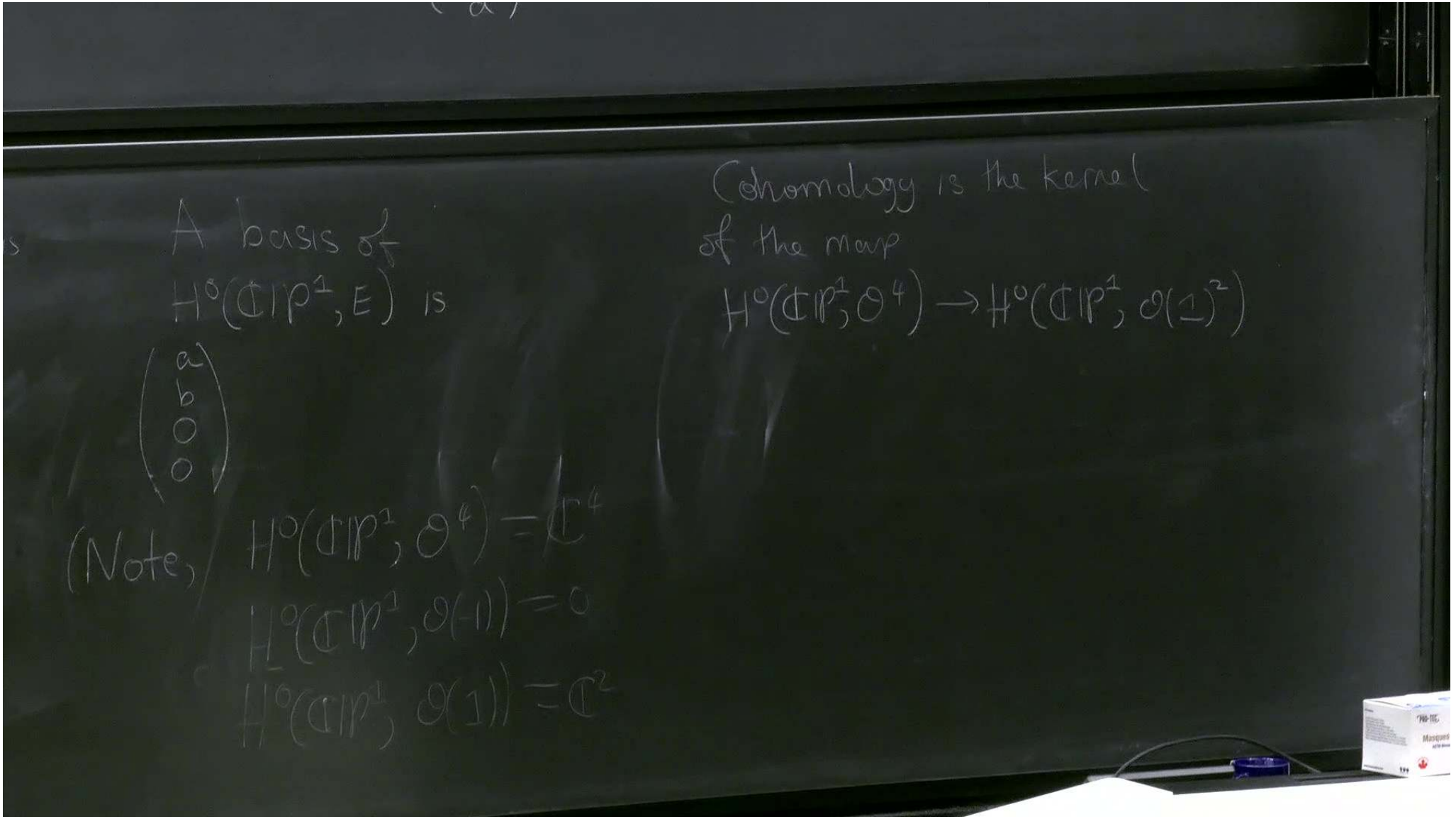
at $v_2=0, z=0$

In cohomology, at $z=0, v_2=0$

$$\begin{pmatrix} a \\ b \\ 0 \\ -a \end{pmatrix}$$ is equivalent to

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$$


At $z=1$, have
$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$.
Want to extend this along $\mathbb{CP}^1_y$.
g at $z=1$ is $\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ so $\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$ is equivalent


A basis of
$H^0(\mathbb{CP}^1, E)$ is
$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$
(Note, $H^0(\mathbb{CP}^1, \mathcal{O}^4) = \mathbb{C}^4$
$H^0(\mathbb{CP}^1, \mathcal{O}(-1)) = 0$
$H^0(\mathbb{CP}^1, \mathcal{O}(1)) = \mathbb{C}^2$
Cohomology is the kernel
of the map
$H^0(\mathbb{CP}^1; \mathcal{O}^4) \to H^0(\mathbb{CP}^1; \mathcal{O}(1)^2)$

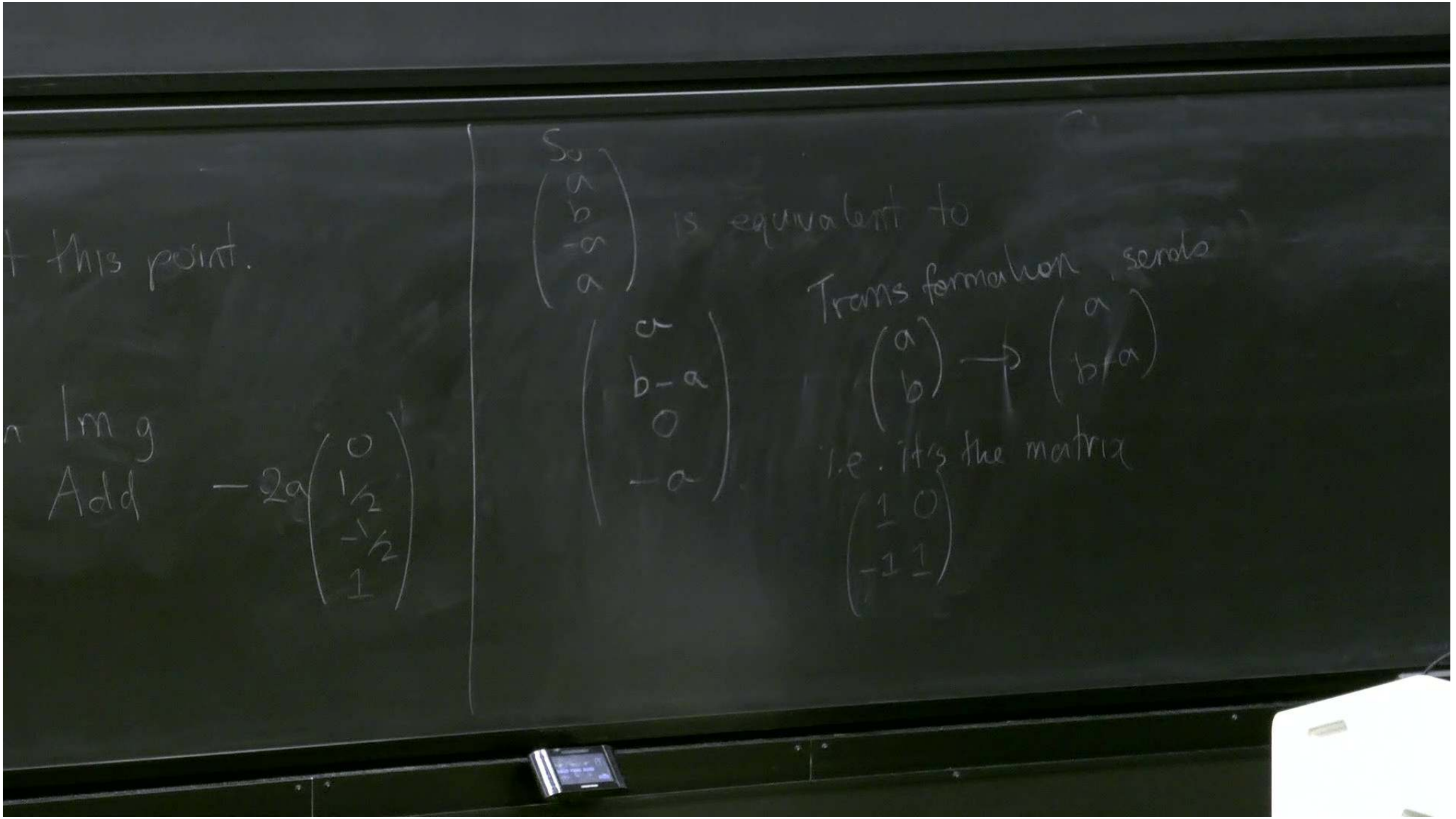
$$g = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{1} \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$ at this point.

Need to add something in $\operatorname{Im} g$

to $\begin{pmatrix} a \\ b \\ -a \\ a \end{pmatrix}$ to remove 3rd entry.

$g = \begin{pmatrix} -1/2 \\ 1/1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}$ at this point.

Need to add something in $\operatorname{Im} g$ to $\begin{pmatrix} a \\ b \\ -a \\ a \end{pmatrix}$ to remove 3rd entry. Add $-2a \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}$


t this point.
Im g
Add $-2a\begin{pmatrix}0\\1/2\\-1/2\\1\end{pmatrix}$
So $\begin{pmatrix}a\\b\\-a\\a\end{pmatrix}$ is equivalent to $\begin{pmatrix}a\\b-a\\0\\-a\end{pmatrix}$
Transformation, sends $\begin{pmatrix}a\\b\end{pmatrix} \to \begin{pmatrix}a\\b-a\end{pmatrix}$
i.e. it's the matrix $\begin{pmatrix}1&0\\-1&1\end{pmatrix}$

$y_{\wedge 2} = 0$

$x_{\wedge 2} = 0$

If we used

found a lower triangular matrix.

$$\begin{pmatrix} 1 & \\ -\frac{1}{\rho^2} & 1 \end{pmatrix}$$

$y_{\alpha i} = 0$

$x_{\alpha i} = 0$

found a lower triangular matrix.

$$\begin{pmatrix} 1 & & \\ \frac{-1}{\rho^2} & 1 & \\ & & \end{pmatrix}$$

If we used x, y where

$y_{\alpha i} = 0, x_{\alpha i} = 0$

$\Rightarrow$ upper triangular matrix.

$y \curvearrowright \dot{z} = 0$

$\alpha \dot{z} = 0$

found a lower triangular matrix.

$$\begin{pmatrix} 1 & \\ -\frac{1}{\rho^2} & 1 \end{pmatrix}$$

If we used x, y where

$y \alpha \dot{1} = 0, \; x \alpha \dot{1} = 0$

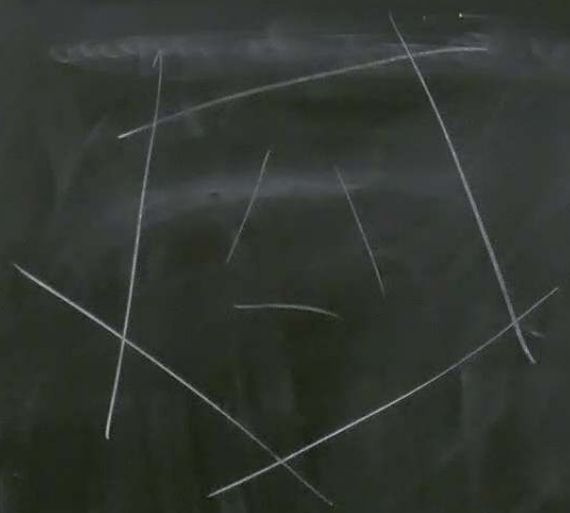
$\Rightarrow$ upper triangular matrix.

If we use a square with both kinds of null vectors, Monodromy is not triangular.

x, y where

$x_{ij} = 0$

riangular matrix.



use a square

both kinds of null vectors,

odromy is not triangular