Title: Lecture - Mathematical Physics, PHYS 777

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Subject: Mathematical physics

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LAST TIME

ADHM complex for a charge 1 instanton

$$\begin{tikzcd}
 & \mathcal{O} \arrow[dr, "v_1"] & \\
\mathcal{O}(-1) \arrow[ur, "-v_2"] \arrow[r, "v_1"] \arrow[dr, "z"] \arrow[ddr, "-1"'] & \mathcal{O} \arrow[r, "v_2"] & \mathcal{O}(1) \\
 & \mathcal{O} \arrow[ur, "1"] & \\
 & \mathcal{O} \arrow[uur, "z"'] &
\end{tikzcd}$$

$$\mathcal{O}(-1) \xrightarrow{g} \mathcal{O}^4 \xrightarrow{f} \mathcal{O}(1)$$

Bundle on $\mathbb{P}\mathbb{T}$

$$E = \frac{\operatorname{Ker} f}{\operatorname{Im} g}$$

- E is a rank 2 bundle on $\mathbb{P}\mathbb{T}$
- Instantunton charge 1 (charge from $\mathcal{O}(-1), \mathcal{O}(1)$)

a charge 1 instanton

$$\begin{tikzcd} & \mathcal{O} \arrow[dr, "1\rho"] & \\ \mathcal{O}(-1) \arrow[ur, "2\rho"] \arrow[dr, "-1\rho"'] & & \mathcal{O}(1) \\ & \mathcal{O} \arrow[ur, "2\rho"'] & \end{tikzcd}$$

$$\mathcal{O}(-1) \xrightarrow{g} \mathcal{O}^4 \xrightarrow{f} \mathcal{O}(1)$$

Bundle on $\mathbb{PT}$

$$E = \frac{\ker f}{\operatorname{Im} g}$$

- E is a rank 2 bundle on $\mathbb{PT}$
- Instantanton charge 1 (charge from $\mathcal{O}(-1), \mathcal{O}(1)$)
- Gives a sol^n to SDYM for $SL(2,\mathbb{C})$ on $\mathbb{R}^4$

What happens at ∞?

Consider rescaling $V_{\dot{\alpha}} \to c V_a$

Send $c \to \infty$

f, g are

$$\begin{pmatrix} c v_1 \\ c v_2 \\ 1 \\ z \end{pmatrix}, \quad \begin{pmatrix} -c v_2 \\ c v_1 \\ -z \\ 1 \end{pmatrix}$$

Rescaling the basis of $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ by $c^{\pm 1}$

sends f, g to

$$\begin{pmatrix} v_1 \\ v_2 \\ c^{-1} \\ c^{-1} z \end{pmatrix}, \quad \begin{pmatrix} -v_2 \\ v_1 \\ -c^{-1} z \\ c^{-1} \end{pmatrix}$$

as $c \to \infty$ we get

Have

$$\begin{tikzcd}[column sep=small]
 & \mathcal{O} \arrow[dr, "v_1"] & \\
\mathcal{O}(-1) \arrow[ur, "-v_2"] \arrow[dr, "v_1"'] & & \mathcal{O}(1) \\
 & \mathcal{O} \arrow[ur, "v_2"'] &
\end{tikzcd}$$

$$\mathcal{O}^2$$

Claim if v_α not both zero,
the coho is $\mathcal{O}^2$.
I.e get trivial bundle of rank 2.

$$A \to 0 \text{ as } x_{\alpha a} \to \infty$$

If $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathcal{O}^2$, is in kernel of f if

$$v_1 a = -v_2 b$$

If $\underline{v_1} \neq 0$,

$$a = -\frac{v_2 b}{\underline{v_1}}$$

which is true

$\begin{pmatrix} a \\ b \end{pmatrix}$ is in the image of g

If $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -v_2 e \\ \underline{v_1} e \end{pmatrix}$ some e, which is true with $e = \frac{b}{V_1}$.

If $V_1 \neq 0$, no cohomology

Similarly if $V_2 \neq 0$.

age of g

e) some e,

e)

$1) \xrightarrow{g} \mathcal{O}^4 \xrightarrow{f} \mathcal{O}(1)$

...e with $e = \frac{b}{v_{\dot 1}}$.

0, no cohomology

$v_{\dot 2} \neq 0$.

What happens if $\rho \to 0$?

$$f = \begin{pmatrix} v_{\dot 2} \\ v_{\dot 2} \\ 1\rho \\ z\rho \end{pmatrix} \quad g = \begin{pmatrix} -v_{\dot 2} \\ v_{\dot 1} \\ -z\rho \\ \rho \end{pmatrix}$$

if $\rho \to 0$ we get the same phenomenon

At $\rho = 0$, the gauge field on $\mathbb{R}^4$ is trivial except at $0 \in \mathbb{R}^4$ (corresponds to $v_{\dot a} = 0$)

In this limit, $\frac{1}{8\pi^2}\mathrm{tr}\left(F(A)\wedge F(A)\right)$

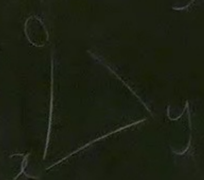
$$= \delta_{x=0}$$

$$b_0 = \frac{11}{3}N_c - \frac{2}{3}N_f$$

measure $\sim \rho^{b_0-5}\, d\rho$

Consider null triangle given by 3 points

- 0
- $x_{\alpha\dot\alpha}$, where $x_{\alpha\dot 2}=0$, $x_{2\dot 1}=1$, $x_{1\dot 1}=0$
- $y_{\alpha\dot\alpha}$, $y_{\alpha\dot 2}=0$, $y_{1\dot 1}=1$, $y_{2\dot 1}=-1$



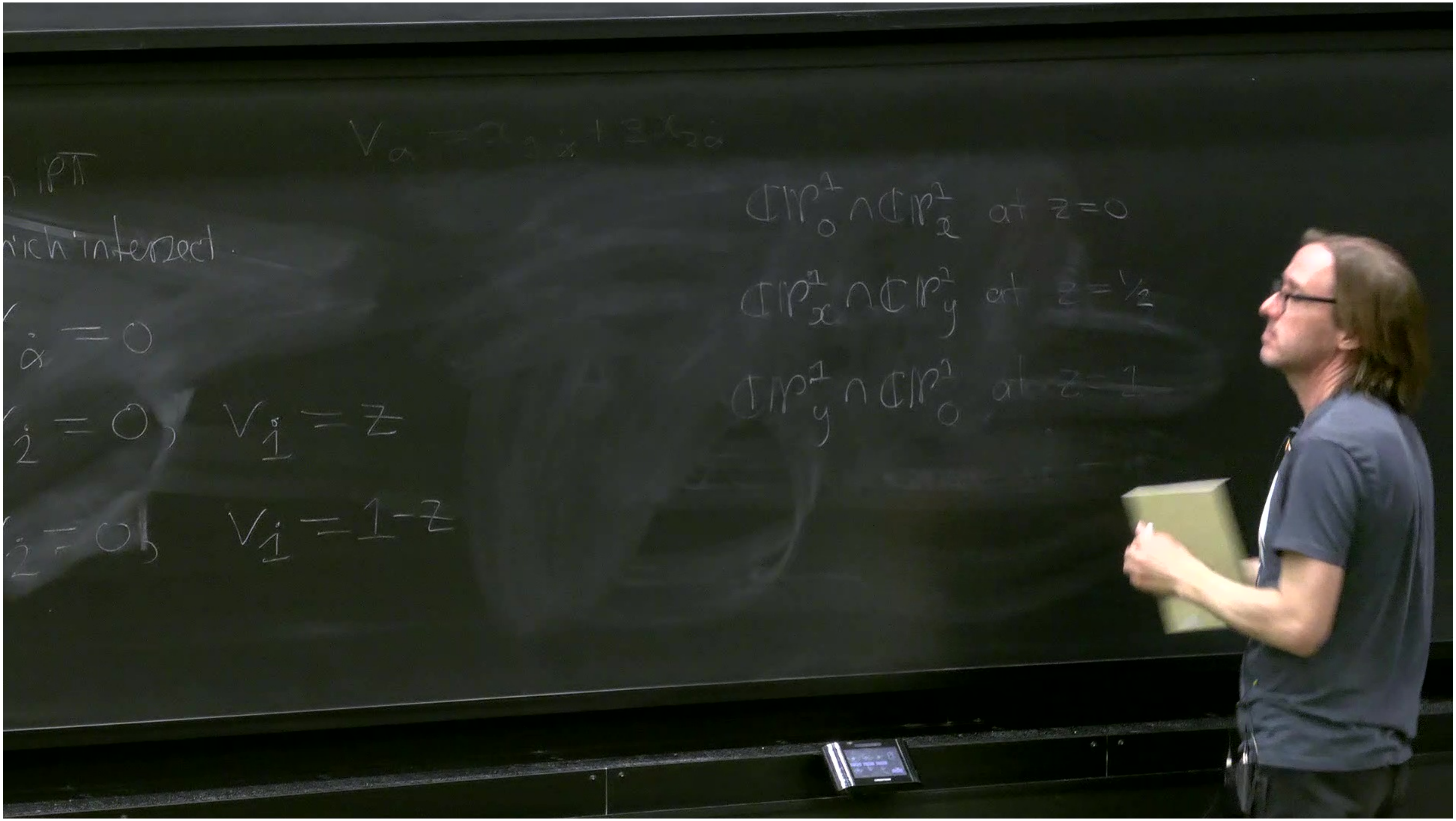
We'll compute parallel transport of charge 1 ADHM instanton.

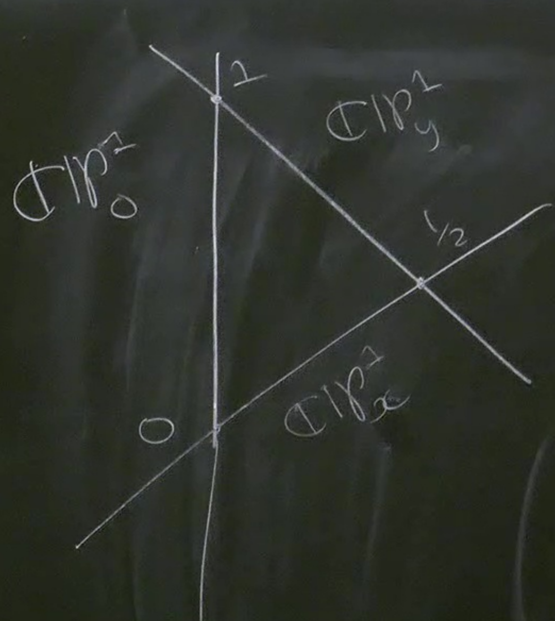
$V_{\alpha} = \omega_{1\dot\alpha} + z\,\lambda_{2\dot\alpha}$

3 $\mathbb{CP}^1$'s in $\mathbb{PT}$

each of which intersect.

- 0 – $V_{\dot\alpha} = 0$
- x : $V_{\dot 2} = 0$, $V_{\dot 1} = z$
- y : $V_{\dot 2} = 0$, $V_{\dot 1} = 1 - z$


hich intersect
$\mathbb{CP}^1_0 \cap \mathbb{CP}^1_2$ at $z=0$
$\mathbb{CP}^1_x \cap \mathbb{CP}^1_y$ at $z=\frac{1}{2}$
$\mathbb{CP}^1_y \cap \mathbb{CP}^1_0$
$V_{\dot 1} = z$
$V_{\dot 1} = 1-z$



On $\mathbb{CP}^1_0$, ADHM complex is

$$\begin{tikzcd}
 & \mathcal{O} \arrow[dr, "0"] & \\
\mathcal{O}(-1) \arrow[ur, "0"] \arrow[r, "0"] \arrow[dr, "-z"] \arrow[ddr, "1"'] & \mathcal{O} \arrow[r, "0"] & \mathcal{O}(1) \\
 & \mathcal{O} \arrow[ur, "1"] & \\
 & \mathcal{O} \arrow[uur, "-z"'] &
\end{tikzcd}$$

On $\mathbb{CP}^1_{0,}$ ADHM complex is

$$\begin{tikzcd}
 & \mathcal{O} \arrow[dr, "0"] & \\
\mathcal{O}(-1) \arrow[ur, "0"] \arrow[r, "0"] \arrow[dr, "-z"] \arrow[ddr, "1"'] & \mathcal{O} \arrow[r, "0"] & \mathcal{O}(1) \\
 & \mathcal{O} \arrow[ur, "1"] & \\
 & \mathcal{O} \arrow[uur, "z"'] &
\end{tikzcd}$$

A basis of $H^0(\mathbb{CP}^1, E)$ is

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$$

(Note, $H^0(\mathbb{CP}^1, \mathcal{O}^4) = \mathbb{C}^4$

$H^0(\mathbb{CP}^1, \mathcal{O}(-1)) = 0$

$H^0(\mathbb{CP}^1, \mathcal{O}(1)) = \mathbb{C}^2$

Cohomology is th

of the map

$H^0(\mathbb{CP}^1, \mathcal{O}^4) \to H$

On $\mathbb{CP}^1_x$ we have

$$f = \begin{pmatrix} z \\ 0 \\ \frac{1}{z} \end{pmatrix}$$

Kernel is

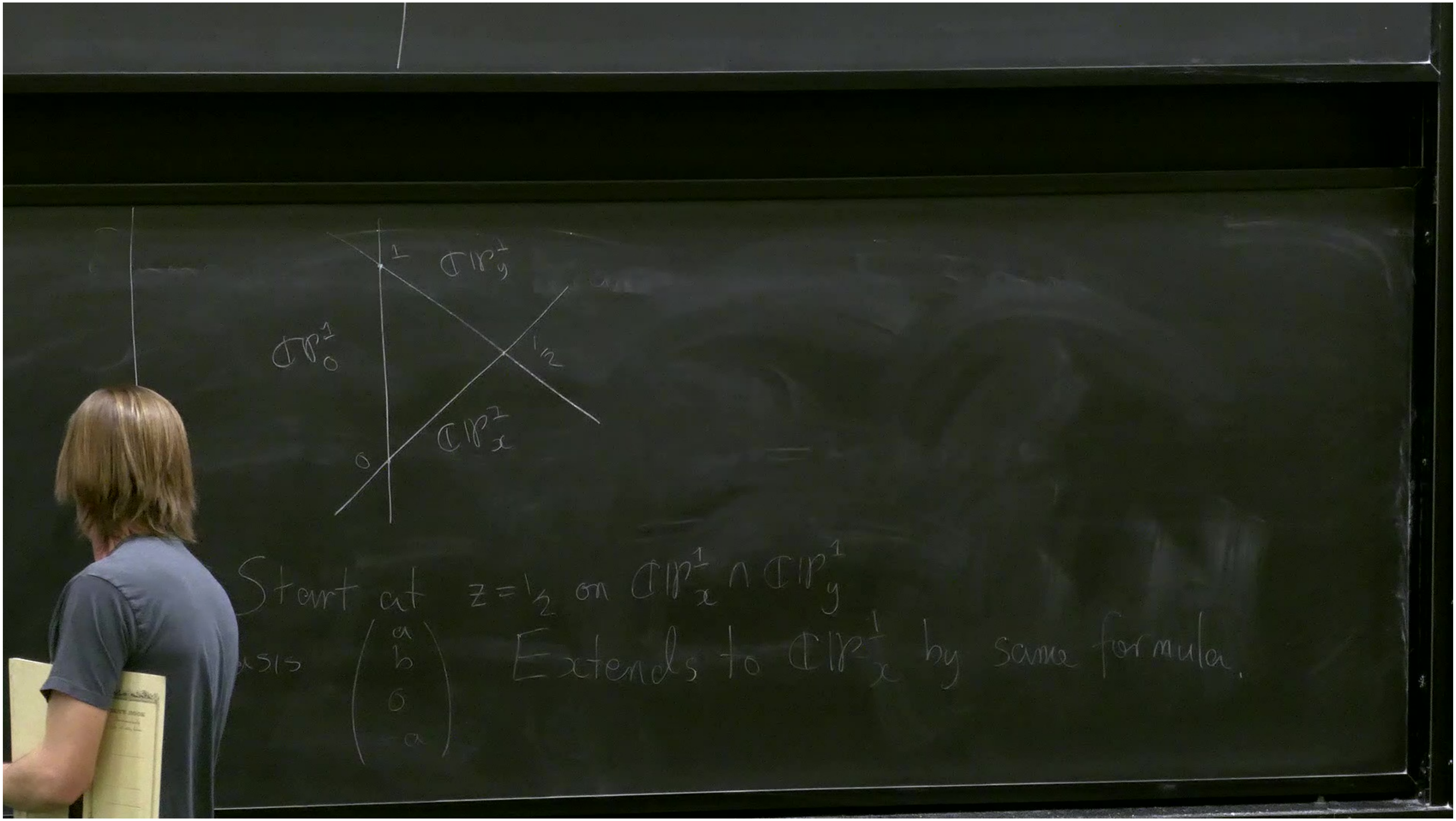
$$\begin{pmatrix} a \\ b \\ 0 \\ -a \end{pmatrix}$$

$\mathbb{CP}^1_y$

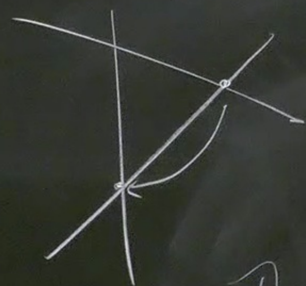
$$f = \begin{pmatrix} 1-z \\ 0 \\ \frac{1}{z} \end{pmatrix}$$

Kernel is

$$\begin{pmatrix} a \\ b \\ -a \\ a \end{pmatrix}$$


1
$\mathbb{CP}^1_y$
$\mathbb{CP}^1_0$
1/2
$\mathbb{CP}^1_x$
0
Start at $z = 1/2$ on $\mathbb{CP}^1_x \cap \mathbb{CP}^1_y$
asis
$\begin{pmatrix} a \\ b \\ 0 \\ -a \end{pmatrix}$
Extends to $\mathbb{CP}^1_x$ by same formula.

A $z=0, v_a=0$ have



$$\begin{pmatrix} a \\ b \\ 0 \\ -a \end{pmatrix}$$

This does not extend to $\mathbb{CP}^1_0$.

$v_a = 0$ have

$$\begin{pmatrix} a \\ b \\ 0 \\ -a \end{pmatrix}$$

oes not extend to

But: at $z = 0$,
we have to mod out by
image of g

$$g = \begin{pmatrix} -v_2 \\ -v_1 \\ -z \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ at } v_a = 0,\ z = 0$$

: at $z=0$,

have to mod out by

ge of g

$$\begin{pmatrix} -v_2 \\ v_1 \\ -z \\ 1 \end{pmatrix}$$

at $V_a=0, z=0$

In cohomology, at $z=0$, $V_a=0$

$$\begin{pmatrix} a \\ b \\ 0 \\ -a \end{pmatrix}$$ is equivalent to

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$$

to

At $z=1$, have

$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$. Want to extend this along $\mathbb{CP}^1_y$.

g at $z=1$ is $\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ so $\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$ is equivalent

A basis of $H^0(\mathbb{CP}^1, E)$ is

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$$

(Note, $H^0(\mathbb{CP}^1, \mathcal{O}^4) = \mathbb{C}^4$

$H^0(\mathbb{CP}^1, \mathcal{O}(-1)) = 0$

$H^0(\mathbb{CP}^1, \mathcal{O}(1)) = \mathbb{C}^2$

Cohomology is the kernel of the map

$$H^0(\mathbb{CP}^1, \mathcal{O}^4) \to H^0(\mathbb{CP}^1, \mathcal{O}(1)^2)$$

$$g = \begin{pmatrix} -v_2 \\ v_1 \\ -z \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix} \text{ at this point.}$$

Need to add something in $\operatorname{Im} g$ to $\begin{pmatrix} a \\ b \\ -a \\ a \end{pmatrix}$ to remove 3^{rd} entry.

$$g = \begin{pmatrix} -v_2 \\ v_1 \\ -z \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix} \text{ at this point.}$$

Need to add something in $\operatorname{Im} g$ to $\begin{pmatrix} a \\ b \\ -a \\ a \end{pmatrix}$ to remove 3^{rd} entry.

Add $-2a\begin{pmatrix} 0 \\ 1/2 \\ -1/2 \\ 1 \end{pmatrix}$

t this point.

n Img

Add $-2a\begin{pmatrix}0\\ 1/2\\ -1/2\\ 1\end{pmatrix}$

So $\begin{pmatrix}a\\ b\\ -a\\ a\end{pmatrix}$ is equivalent to

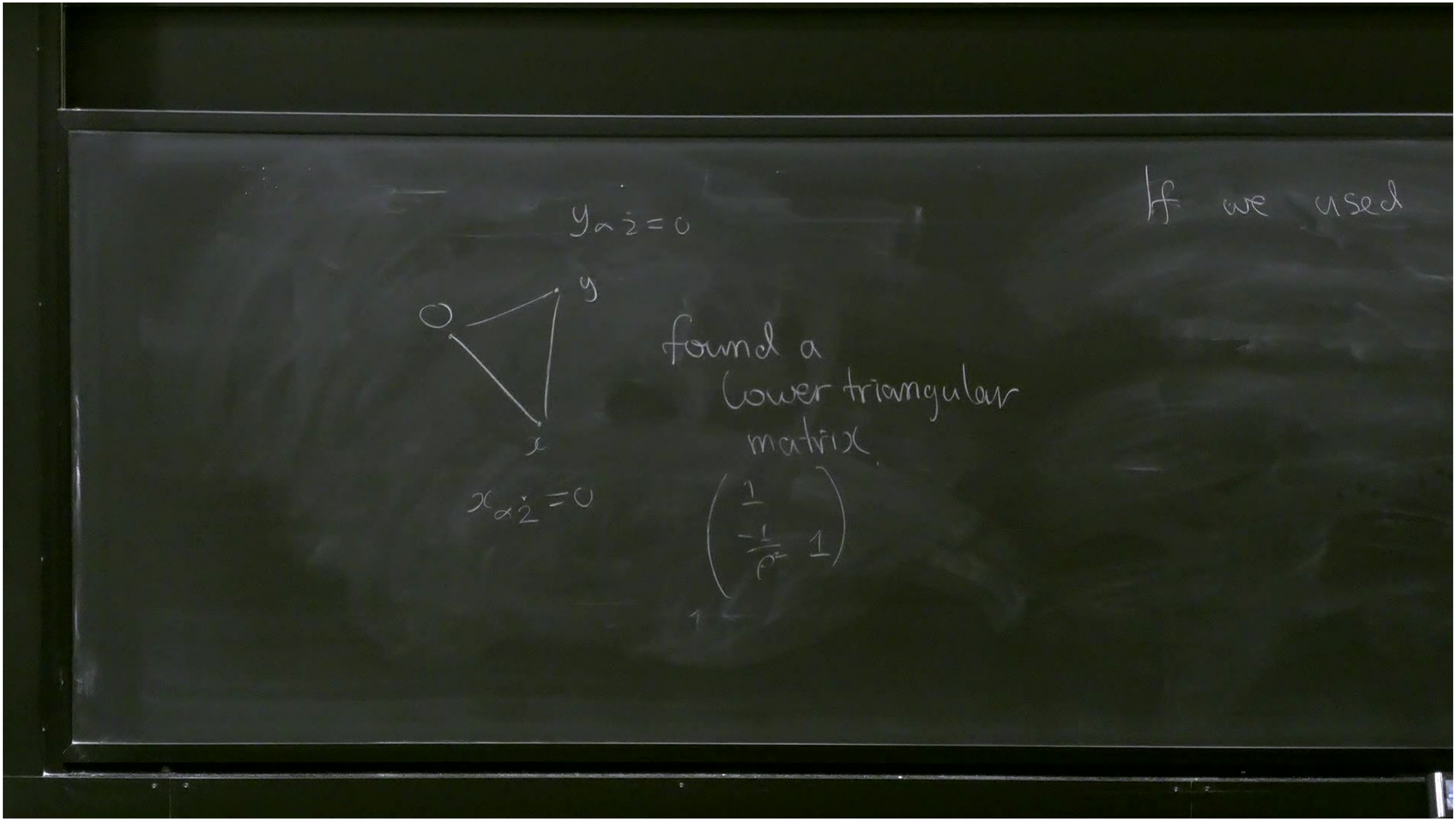
$$\begin{pmatrix}a\\ b-a\\ 0\\ -a\end{pmatrix}$$

Transformation sends

$$\begin{pmatrix}a\\ b\end{pmatrix}\to\begin{pmatrix}a\\ b-a\end{pmatrix}$$

i.e. it's the matrix

$$\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$$


$y_{\alpha\dot{2}}=0$
y
0
x
$x_{\alpha\dot{2}}=0$
found a lower triangular matrix
$\begin{pmatrix} 1 & \\ -\frac{1}{\rho^2} & 1 \end{pmatrix}$
If we used

$y_{\alpha\dot{2}} = 0$

y

x

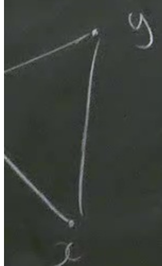
$x_{\alpha\dot{2}} = 0$

found a
lower triangular
matrix

$$\begin{pmatrix} 1 & \\ -\frac{1}{\rho^2} & 1 \end{pmatrix}$$

If we used x, y where
$y_{\alpha\dot{1}} = 0,\ x_{\alpha\dot{1}} = 0$
$\Rightarrow$ upper triangular matrix.

$y_{\alpha\dot{2}} = 0$



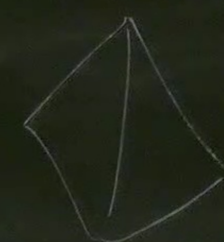
$\alpha\dot{2} = 0$

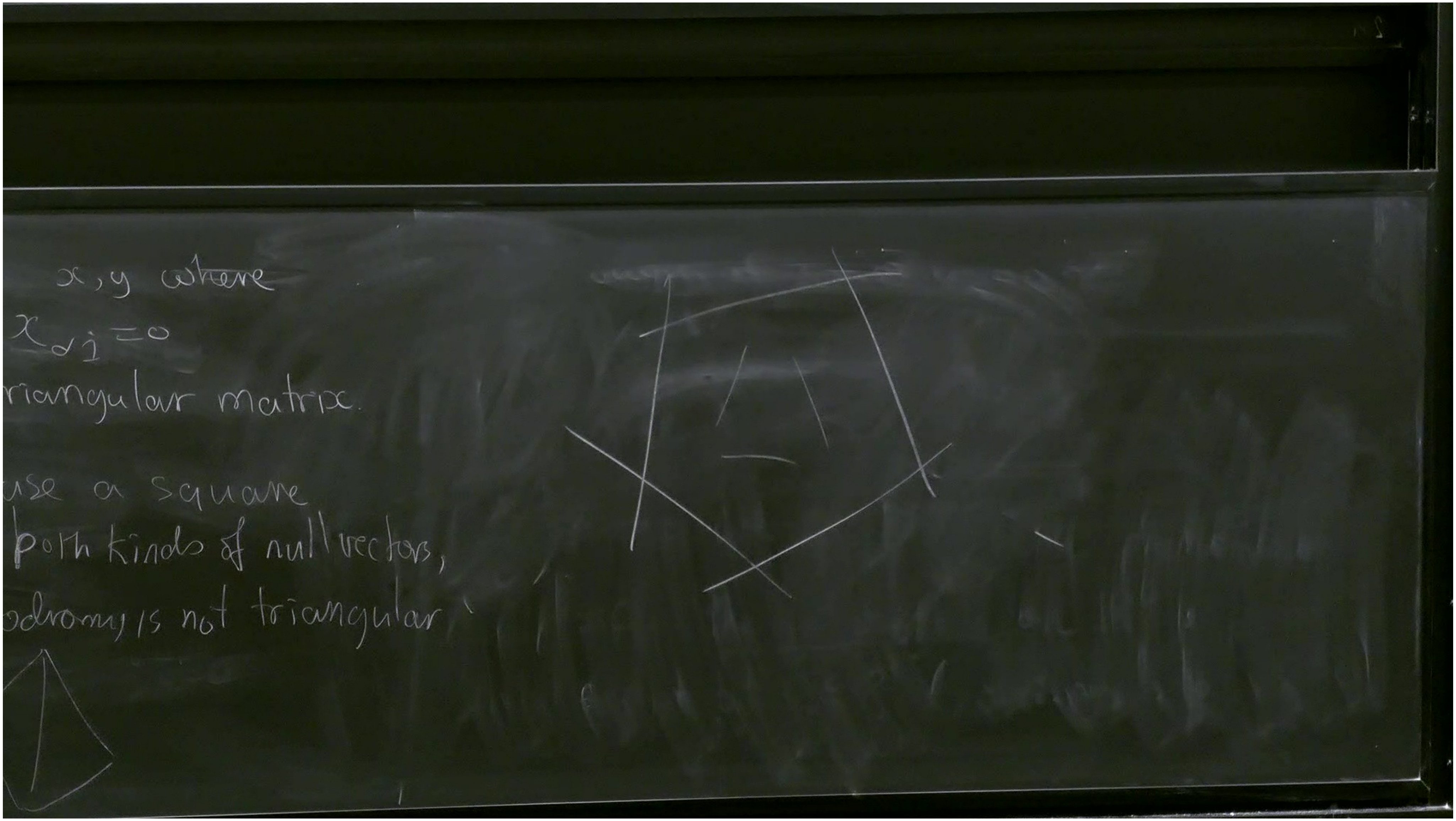
found a
lower triangular
matrix

$$\begin{pmatrix} 1 & \\ -\frac{1}{e^2} & 1 \end{pmatrix}$$

If we used x, y where
$y_{\alpha\dot{1}} = 0, \; x_{\alpha\dot{1}} = 0$
$\Rightarrow$ upper triangular matrix.

If we use a square
with both kinds of null vectors,
Monodromy is not triangular.




x, y where
$x_{\alpha i} = 0$
riangular matrix.
use a square
both kinds of null vectors,
odromy is not triangular.