**Title:** Lecture - Mathematical Physics, PHYS 777

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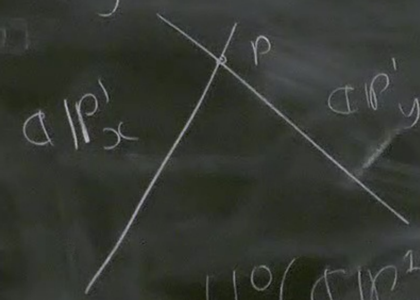
## Last Time:

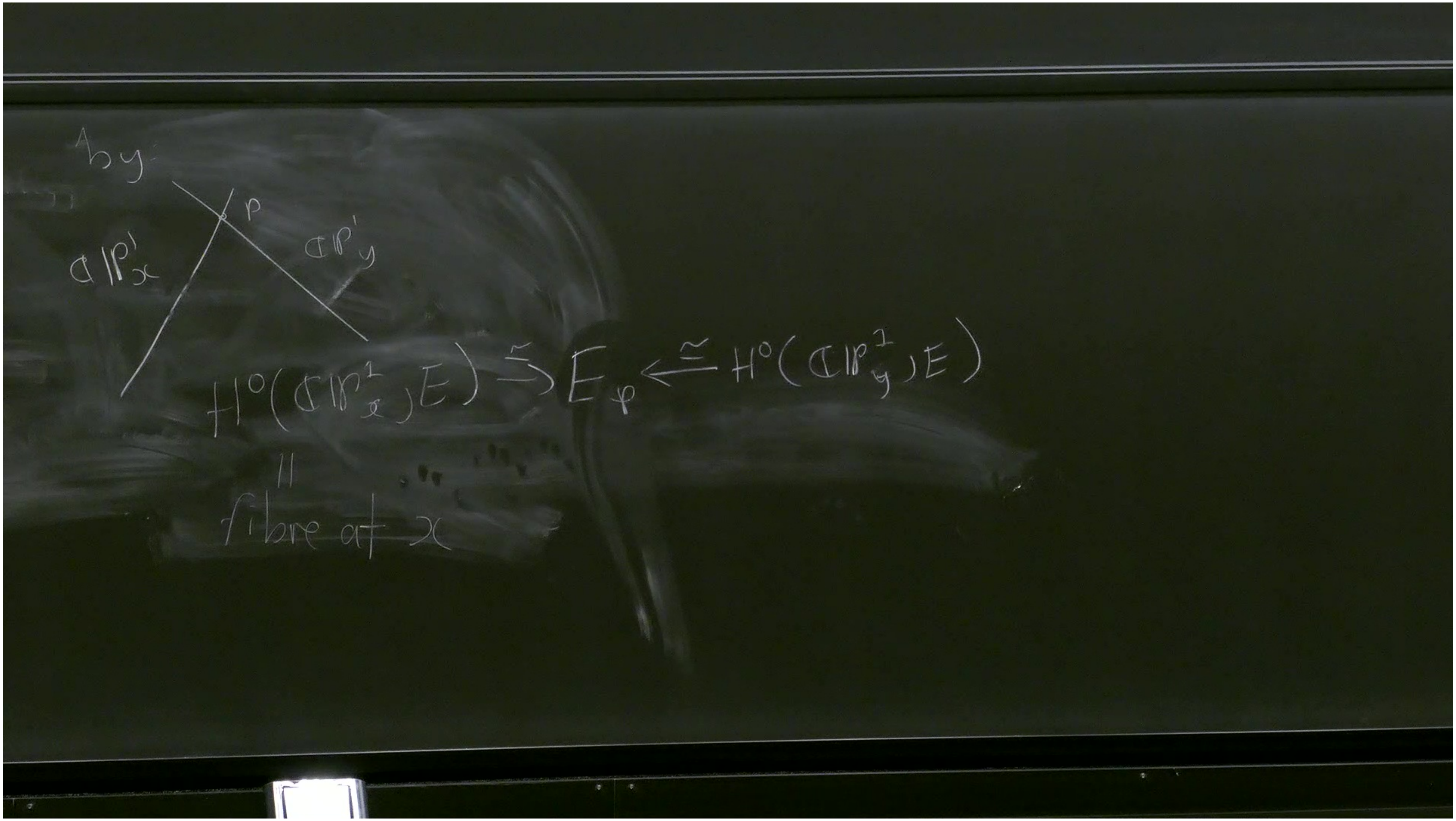
E hol. bundle on $\mathbb{PT}$

$\Longrightarrow$ bundle on $\mathbb{R}^4$ with a self-dual connection.

Parallel transport in null directions was given by:



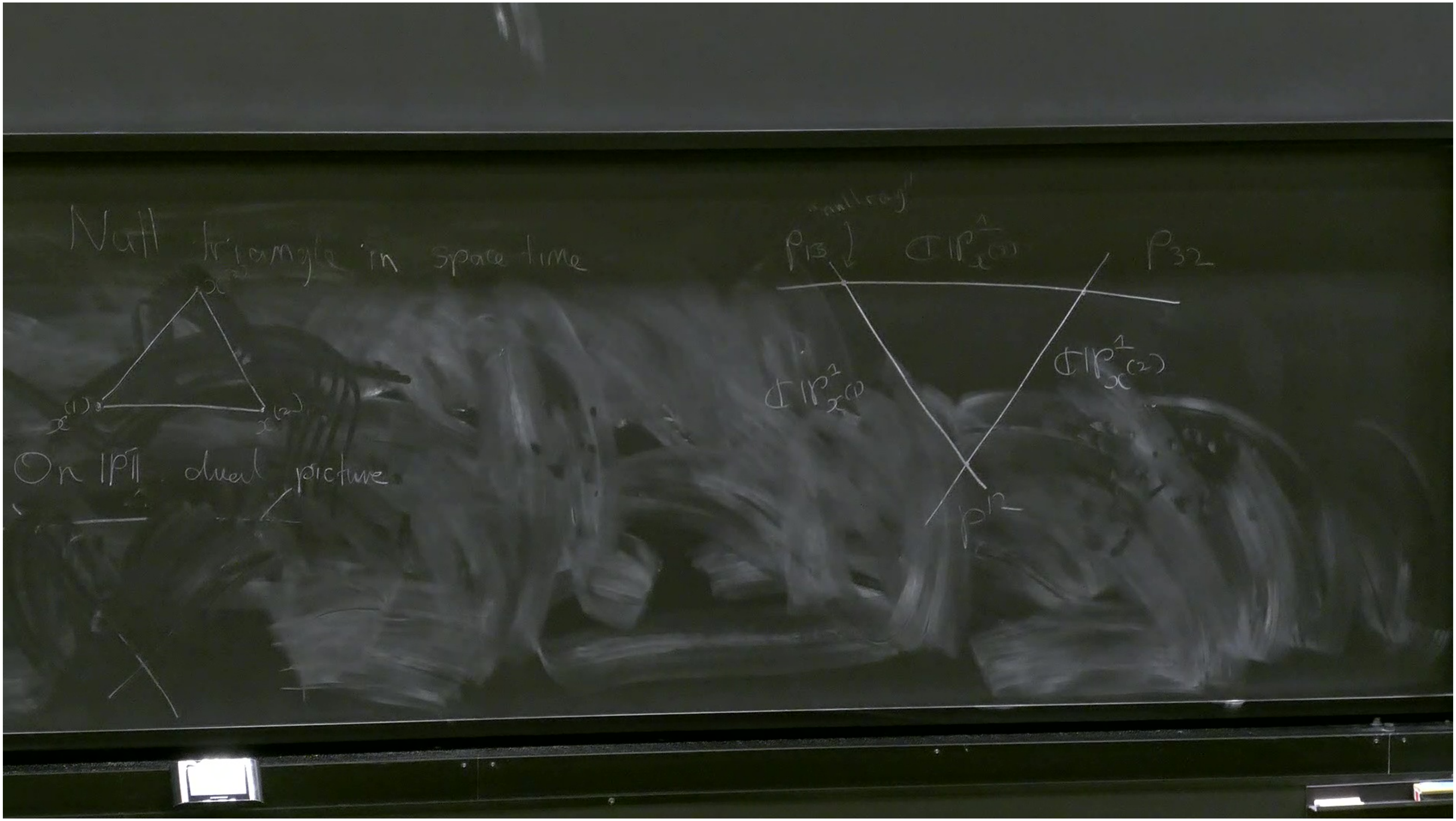

$H^0(\mathbb{CP}^1_x, E) \xrightarrow{\sim} E_p \leftarrow$


by:
$p$
$\mathbb{CP}^1_x$
$\mathbb{CP}^1_y$
$H^0(\mathbb{CP}^1_x, E) \xrightarrow{\simeq} E_p \xleftarrow{\simeq} H^0(\mathbb{CP}^1_y, E)$
$\|$
fibre at $x$

Why is the bundle not trivial?

On space time

Answer: we can compute
the parallel transport on a
null polygon, this is not zero.


Null triangle in space time
"null ray"
$P_{13}$
$\mathbb{CP}^1_{x^{(3)}}$
$P_{32}$
$\mathbb{CP}^1_{x^{(1)}}$
$\mathbb{CP}^1_{x^{(2)}}$
$x^{(1)}$
$x^{(2)}$
On $\mathbb{PT}$ dual picture

Parallel transport along null polygon is

$$\begin{tikzcd}
H^0(\mathbb{CP}^1_{x^{(1)}}, E) \arrow[r, "\simeq"] & E_{p_{12}} \arrow[r, "\simeq"] & H^0(\mathbb{CP}^1_{x^{(2)}}, E) \arrow[d] \\
E_{p_{13}} \arrow[u] & H^0(\mathbb{CP}^1_{x^{(3)}}, E) \arrow[l] & E_{p_{23}} \arrow[l]
\end{tikzcd}$$

Want a formula for the null Wilson line
in terms of this class

$A \in \Omega^{0,1}(\mathbb{PT})$ corresponds to $L$

$\mathbb{CP}^1_0$ at $v_{\dot{\alpha}} = 0$

$H^0(\mathbb{C}$

$L_0$

$$
\begin{tikzcd}
H^0(\mathbb{CP}^1, L) \arrow[r, "\cong"] \arrow[d, "r"'] & L_z \\
L_0 \arrow[ur, dashed] &
\end{tikzcd}
$$

what is that map?

A section of $L$ is a function $f(z, \bar{z})$ s.t.

$$\frac{\partial f}{\partial \bar{z}} + A_{\bar{z}} f = 0$$

Want

Solve this equation with initial condition $f(0) = 1$

Then, if $f(z)$ is the sol$^n$ the map

$L_0 \longrightarrow L_z$ is just $f$.

Soln.

$$f(z) = \exp\left(\frac{1}{2\pi i}\int_{\omega} \frac{d\omega}{z-\omega} A_{\bar{\omega}}(\omega)\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial}{\partial \bar{z}} \int_{\omega} \frac{1}{2\pi i} \frac{d\omega}{z-\omega} A\right) f$$

$$\frac{\partial}{\partial \bar z}\int \frac{dw}{z-w} A$$

$$= \int \frac{\partial}{\partial \bar w}\left(\frac{dw}{z-w}\right) A$$

$$= \int \delta_{w=z} A = A_{\bar z}(z)$$

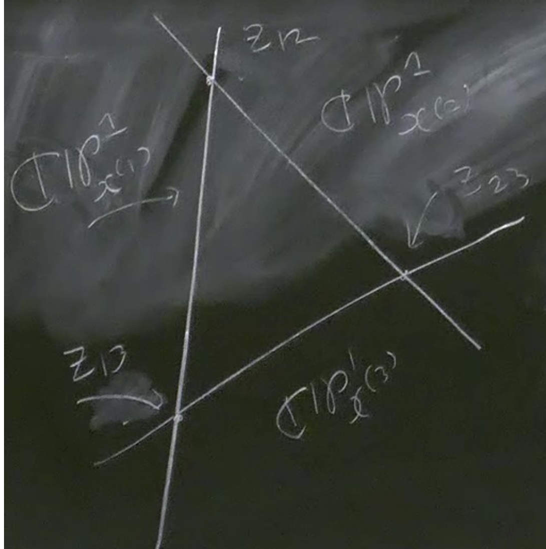
$\dfrac{f(z)}{f(0)}$ transition matrix

is $\exp\left(\frac{1}{2\pi i}\int A_{\bar w}(w)\left(\frac{dw}{z-\dots}\right.\right.$

$\frac{f(z)}{f(0)}$ transition matrix

is $\exp\left(\frac{1}{2\pi i}\int A_{\bar{w}}(w)\left(\frac{dw}{z-w}+\frac{dw}{w}\right)\right)$

# Formula




Holonomy around null triangle is

$$\exp\left(\frac{1}{2\pi i}\int_{\mathbb{CP}^1_{x(1)}}\left(\frac{dw}{w-z^{13}}-\frac{dw}{w-z^{12}}\right)+\int_{\mathbb{CP}^1_{x(2)}}\frac{dw}{w-z^{23}}-\frac{dw}{w-\cdots}\right.$$

Holonomy around null triangle

is

$$\exp\left(\frac{1}{2\pi i}\int_{\mathbb{CP}^1_{x(1)}}\left(\frac{d\omega}{\omega - z^{13}} - \frac{d\omega}{\omega - z^{12}}\right)A + \int_{\mathbb{CP}^1_{x(2)}}\left(\frac{d\omega}{\omega - z^{23}} - \frac{d\omega}{\omega - z^{21}}\right)A + \int_{\mathbb{CP}^1_{x(3)}}\left(\frac{d\omega}{\omega - z^{13}} - \frac{d\omega}{\omega - z^{23}}\right)A\right)$$

## How to build bundles on $\mathbb{PT}$

If $E_1, E_2$ are hol. bundles on a complex manifold and $f: E_1 \longrightarrow E_2$ is a holomorphic map of hol. bundles i.e. in a local frame $f$

is a matrix of size $r_1 \times r_2$

whose entries are hol. functions

Assume rank $f$ = maximal,

i.e. rank $f = \dim E_2$

Then $\operatorname{Ker} f \subseteq E_1$ is a new hol. bundle.

Similarly if

$$g : E_0 \to E_1$$

is a hol map of hol. bundles and rank $g = \dim E_0$

then, $E_1 / \mathrm{Im}\, g$ is a new hol. bundle.

If

$E_0 \xrightarrow{g}$

If

$$E_0 \xrightarrow{g} E_1 \xrightarrow{f} E_2$$

are maps of hol. bundles, and

– Rank $f = \dim E_2$

Rank $g = \dim E_0$

$f \circ g = 0$

Then,

$\dfrac{\mathrm{Ker} f}{\mathrm{Im} g}$ (cohomology) is a new hol. bundle

<u>Fact:</u>

Can write (as smooth bundles)

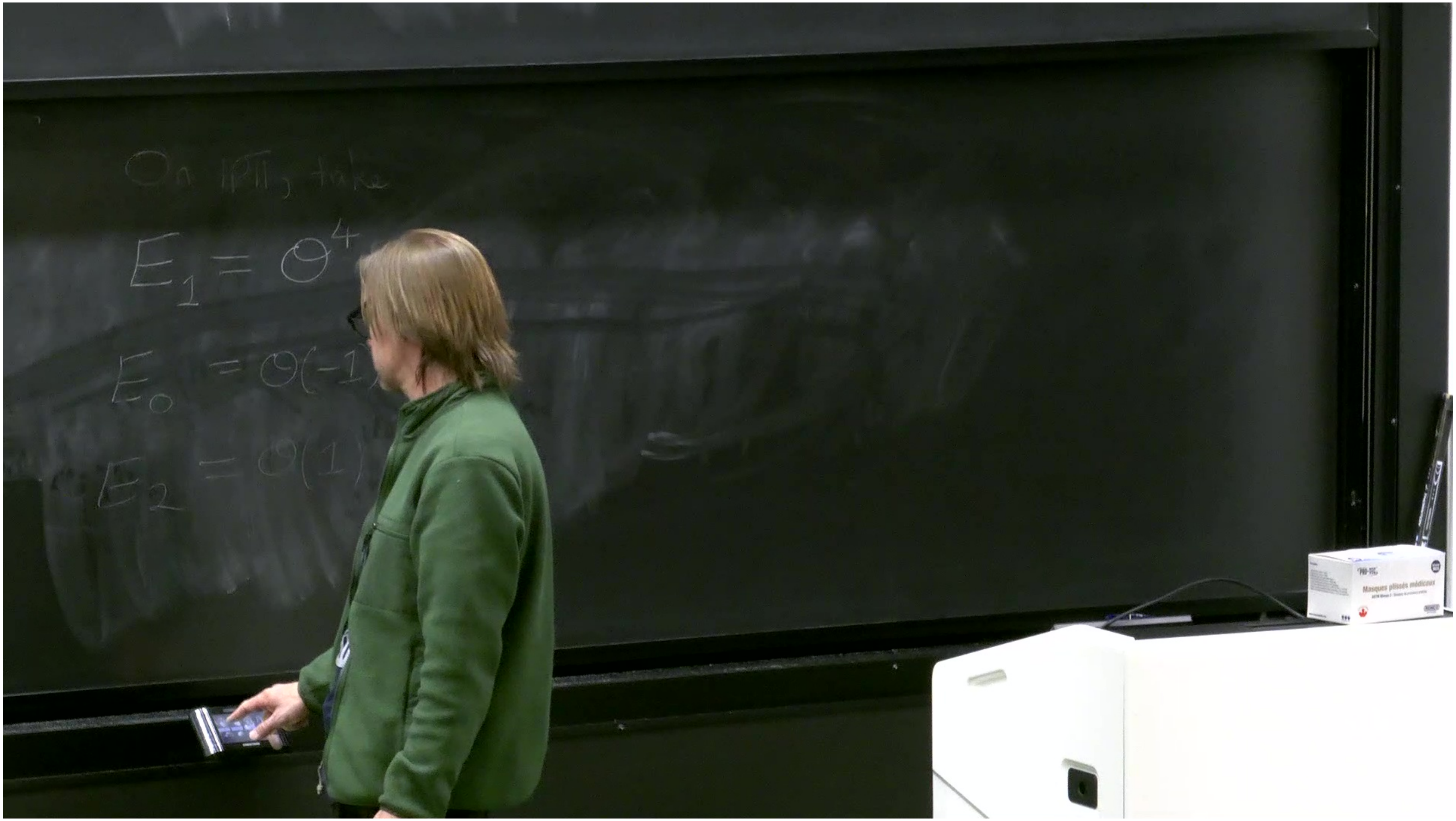
$$E_1 = \frac{\mathrm{Ker} f}{\mathrm{Im} g} \oplus E_0 \oplus E_2$$

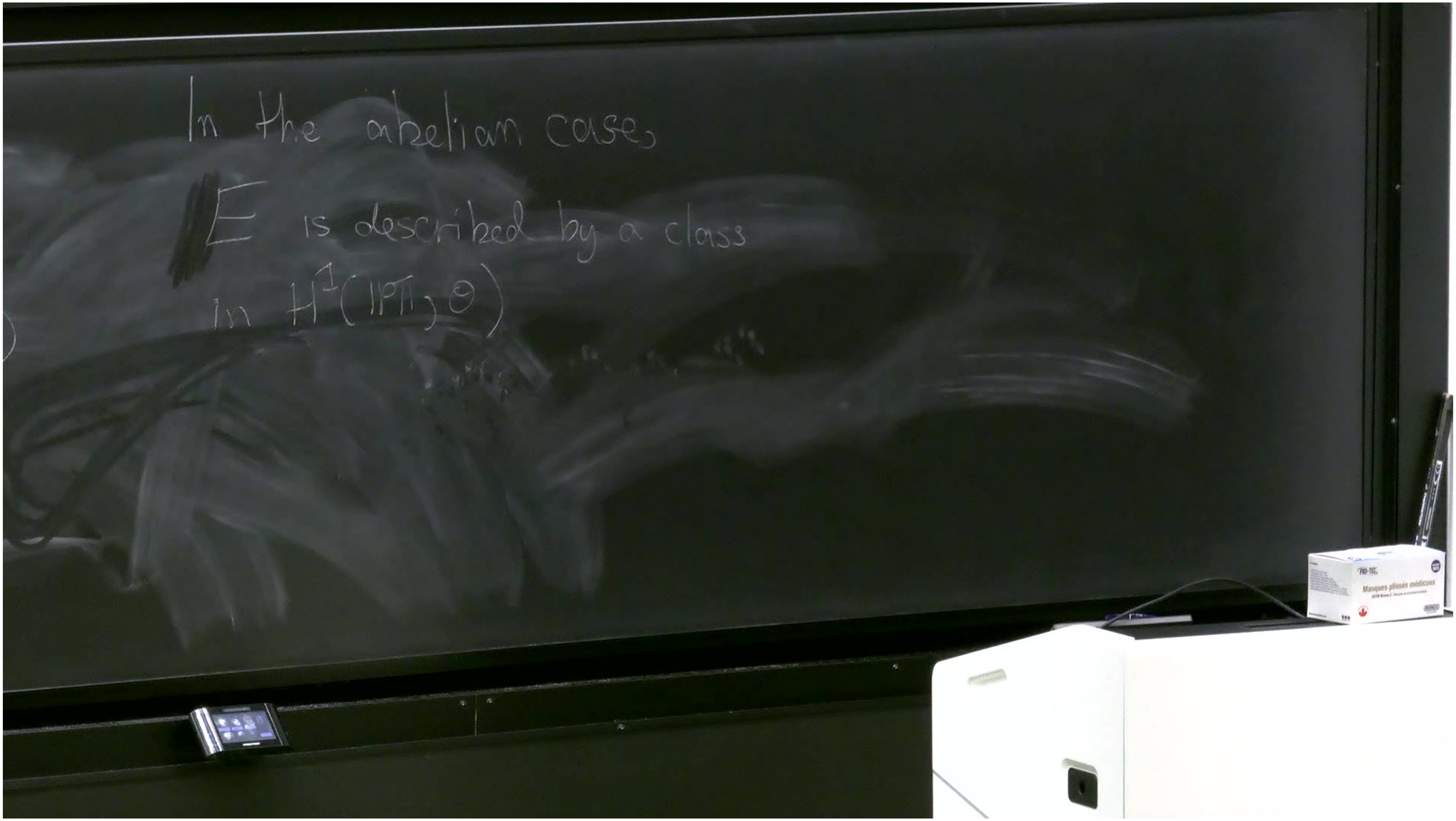
$$\mathrm{Tr}_{\frac{\mathrm{Ker} f}{\mathrm{Im} g}} F^n = \mathrm{Tr}_{E_1} F^n - \mathrm{Tr}_{E_0} F^n - \mathrm{Tr}_{E_0} F^n$$

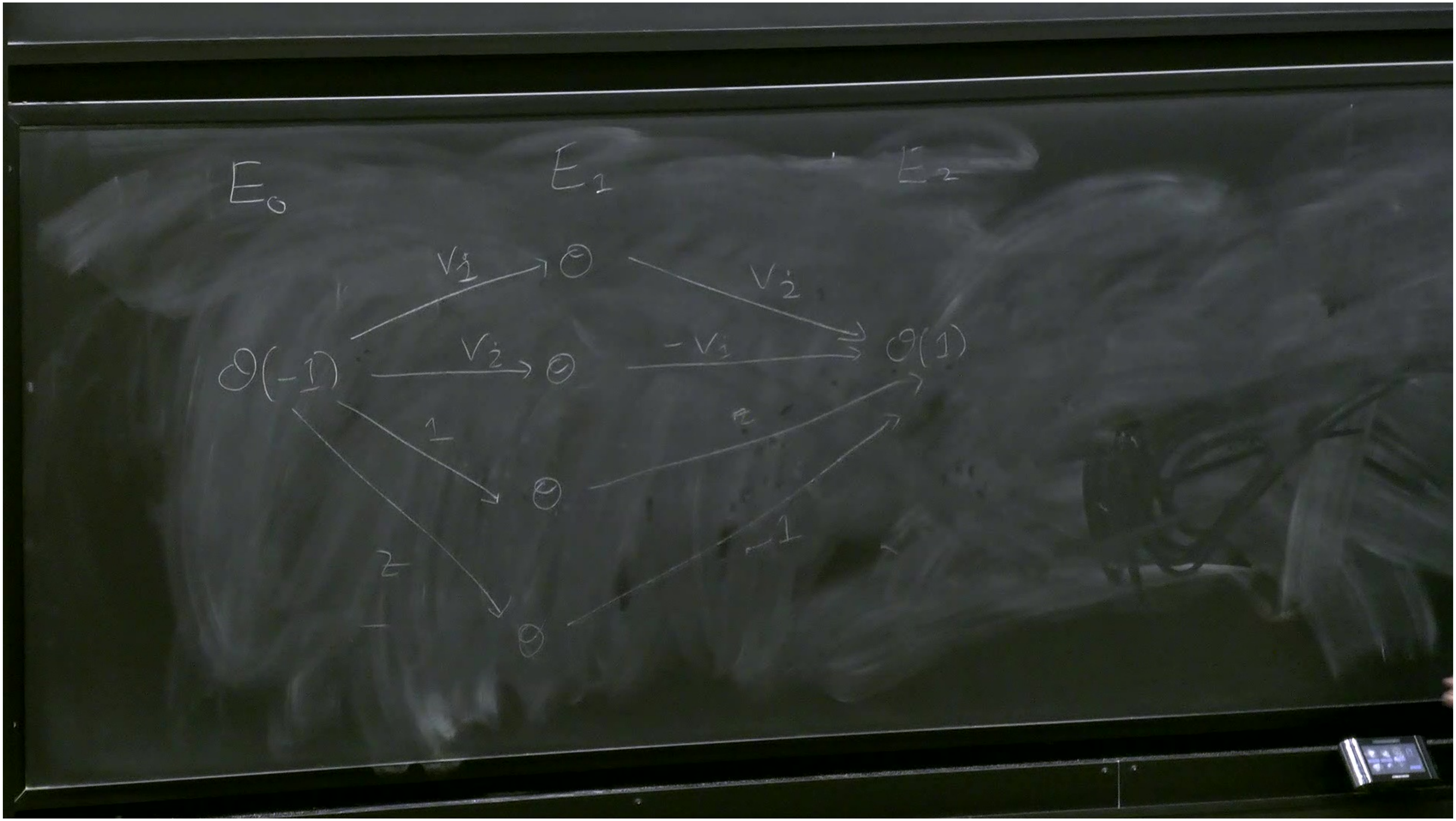
Simplest case of ADHM construction:

On $\mathbb{PT}$,

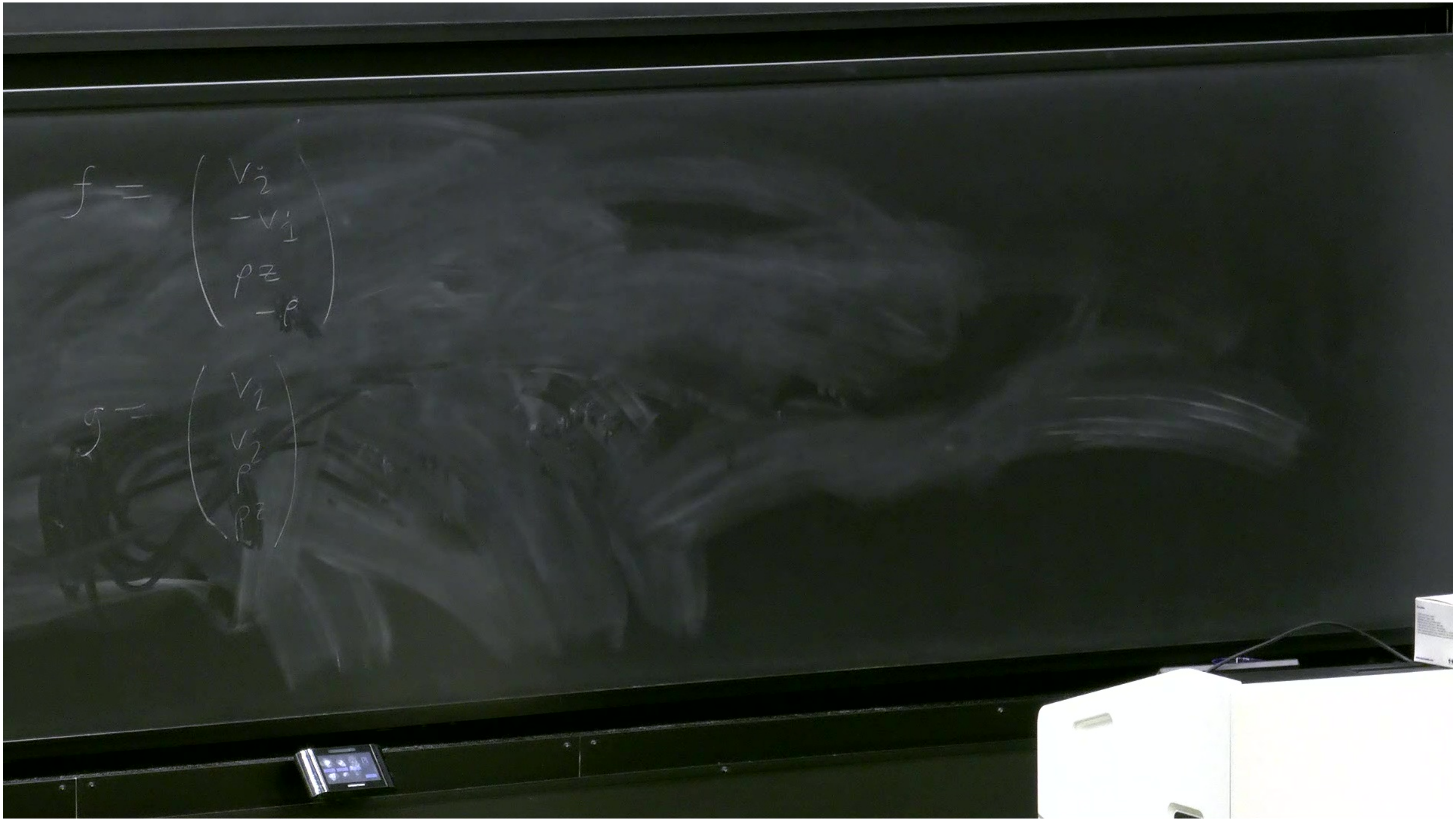
Want to build an $SU(2)$ (or $SL(2,\mathbb{C})$)
instanton of charge 1


On $\mathbb{PT}_3$ take
$E_1 = \mathcal{O}^4$
$E_0 = \mathcal{O}(-1)$
$E_2 = \mathcal{O}(1)$


In the abelian case,
E is described by a class
in $H^1(\mathbb{P}^1, \mathcal{O})$
PRO-TEC
Masques plissés médicaux



$E_0$
$E_1$
$E_2$
$\vartheta(-1)$
$\vartheta(1)$
$v_1$
$v_2$
$v_2$
$-v_1$
1
2
2
$-1$


$f = \begin{pmatrix} v_2 \\ -v_1 \\ \rho z \\ -\rho \end{pmatrix}$
$g = \begin{pmatrix} v_2 \\ v_2 \\ \rho \\ \rho^2 \end{pmatrix}$

$f \cdot g = 0$ ✓

Need $f, g$ both have rank 1

i.e. vectors

$$\begin{pmatrix} v_1 \\ v_2 \\ \rho \\ \rho z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_2 \\ -v_1 \\ \rho z \\ -\rho \end{pmatrix}$$

Can never vanish

( Instanton density

$$F \wedge F = \Box^2 \log(1+\rho^2)$$

First 2 entries zero at

$v_1, v_2 = 0$

Last two are not zero (if $\rho \neq 0$)

Coho. $\dfrac{\mathrm{Ker}\, f}{\mathrm{Im}\, g}$ is the hol. bundle

on $\mathbb{PT}$ which is instanton of charge 1

and size $\rho$.

ntries zero at

$= 0$

o are not zero (if $\rho \neq 0$)

$\dfrac{\mathrm{Ker}\, f}{\mathrm{Im}\, g}$ is the hol. bundle

T which is instanton of charge 1 and size $\rho$.

( Instanton density

$\mathrm{tr}\, F \wedge F = \Box^2 \log(\rho^2 + \|x\|^2)$ )

Need to check:

- Charge is 1
- Large $V_{\dot{\alpha}}$, at $\infty$ on space times, bundle is trivial

Charge $=$ $2^{nd}$ chern character

$$= \frac{1}{2} \operatorname{tr} F^2 \in H^4(\mathbb{P}\mathbb{T}, \text{rel. to } \infty)$$

$E_1$ does not contribute.

$E_0 = \mathcal{O}(-1)$ has charge

$$\frac{1}{2} c_1(\mathcal{O}(-1))^2 = \frac{1}{2} c_1(\mathcal{O}(1))^2$$

$\mathcal{O}(1)$ ha

which is instanton of charge 1
and size $\rho$.

$\mathcal{O}(1)$ has charge

$$\frac{1}{2} c_1(\mathcal{O}(1))^2$$

Total charge is

$$- c_1(\mathcal{O}(1))^2 \in H^4(\mathbb{P}\mathbb{T}, \text{rel.}\ \infty)$$

which is charge 1