Title: Lecture - Mathematical Physics, PHYS 777

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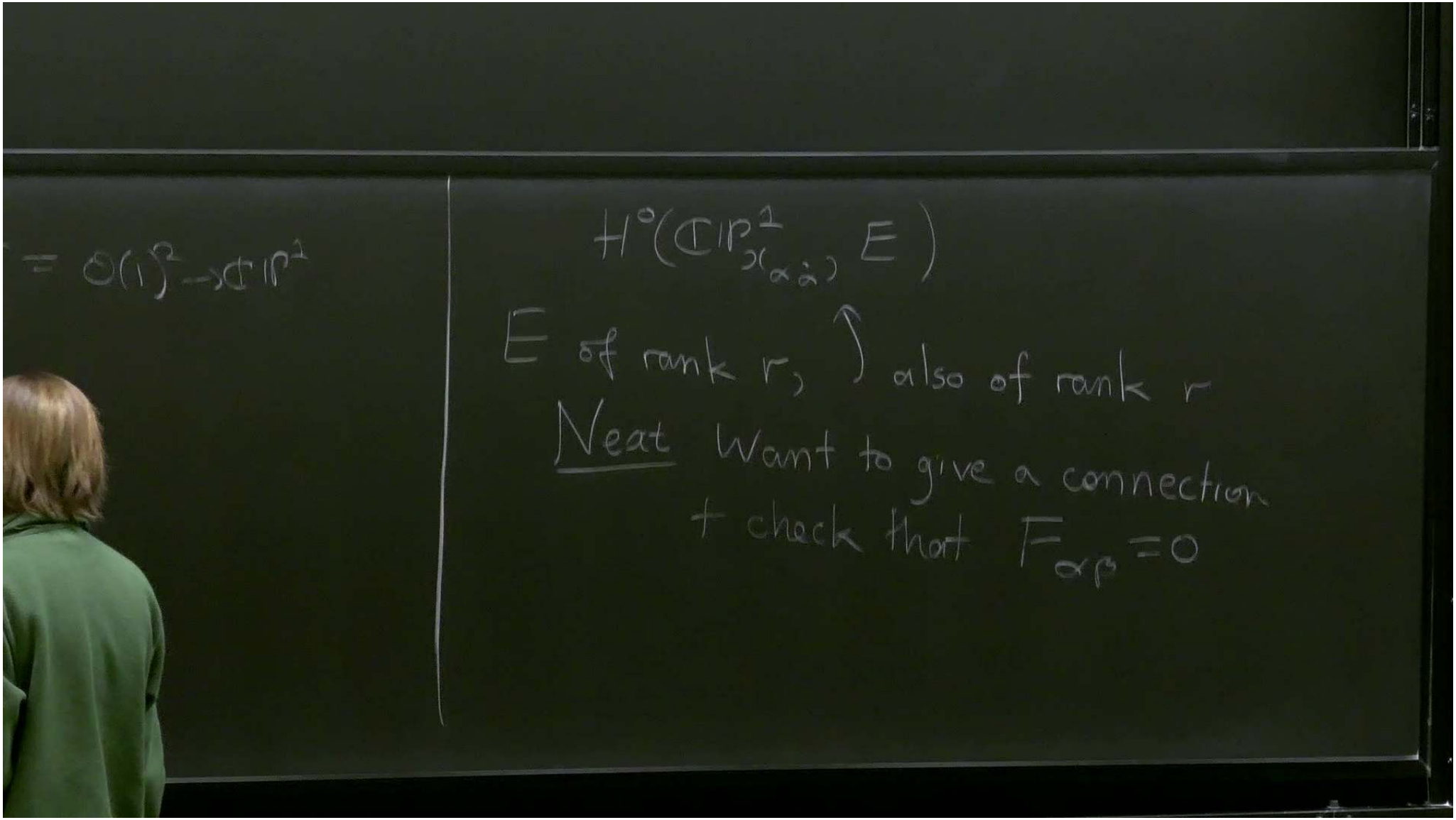
E, a hol. vector bundle on $\mathbb{PT} = \mathcal{O}(1)^2 \to \mathbb{CP}^1$

$E|_{\mathbb{CP}^1}$ is trivial

$\Rightarrow$ vector bundle on space time:

Given $x_{\alpha\dot{\alpha}} \in \mathbb{C}^4$,

$\mathbb{CP}^1_{x_{\alpha\dot{\alpha}}} \subseteq \mathbb{PT}$, fibre of v. bundle on space-time is


$= \mathcal{O}(1)^2 \to \mathbb{CP}^1$
$H^0(\mathbb{CP}^1_{x(\alpha\dot{\alpha})}, E)$
E of rank r, $\nearrow$ also of rank r
Neat Want to give a connection
+ check that $F_{\alpha\beta} = 0$

on space-time is

On any manifold with a vector bundle V a connection is data of infinitesimal parallel transport:

iso. from V_x to $V_{x+\varepsilon u}$

$x \in M$

$u \in T_x M$

on space-time is

On any manifold with a vector bundle V a connection is data of infinitesimal parallel transport:

iso from V_x to $V_{x+\epsilon u}$

$x \in M$

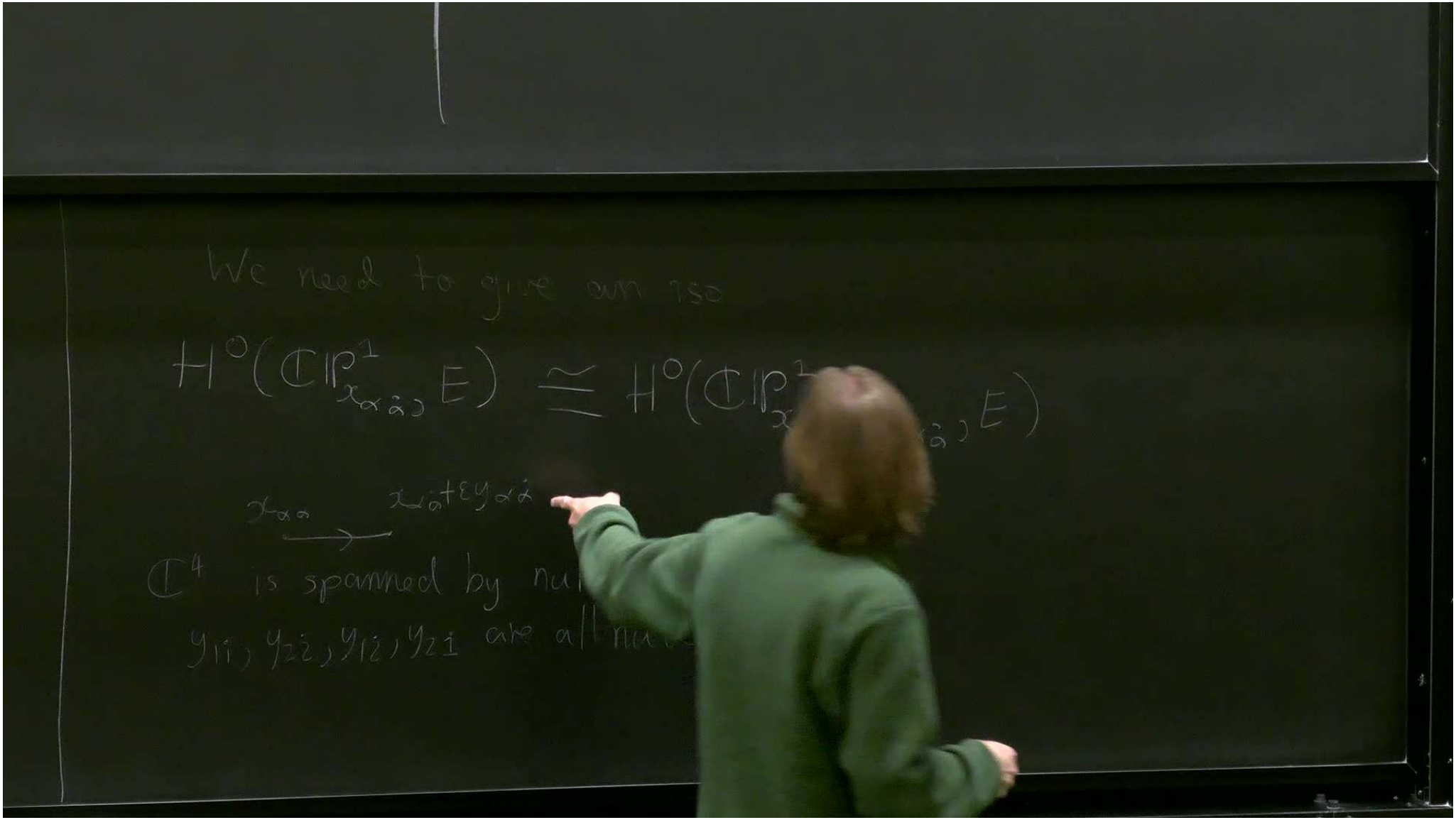
$u \in T_x M$

Wilson line

We need to give an iso

$$H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}}, E) \cong H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}}, E)$$

$$x_{\alpha\dot\alpha} \longrightarrow x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}$$


We need to give an iso
$H^0(\mathbb{CP}^1_{x_{\alpha\hat{\alpha}}}, E) \cong H^0(\mathbb{CP}^1$
$x_{\alpha\alpha}$
$x_{\alpha\alpha} + \varepsilon y_{\alpha\hat{\alpha}}$
$\mathbb{C}^4$ is spanned by nu
$y_{11}, y_{22}, y_{12}, y_{21}$ are all nu

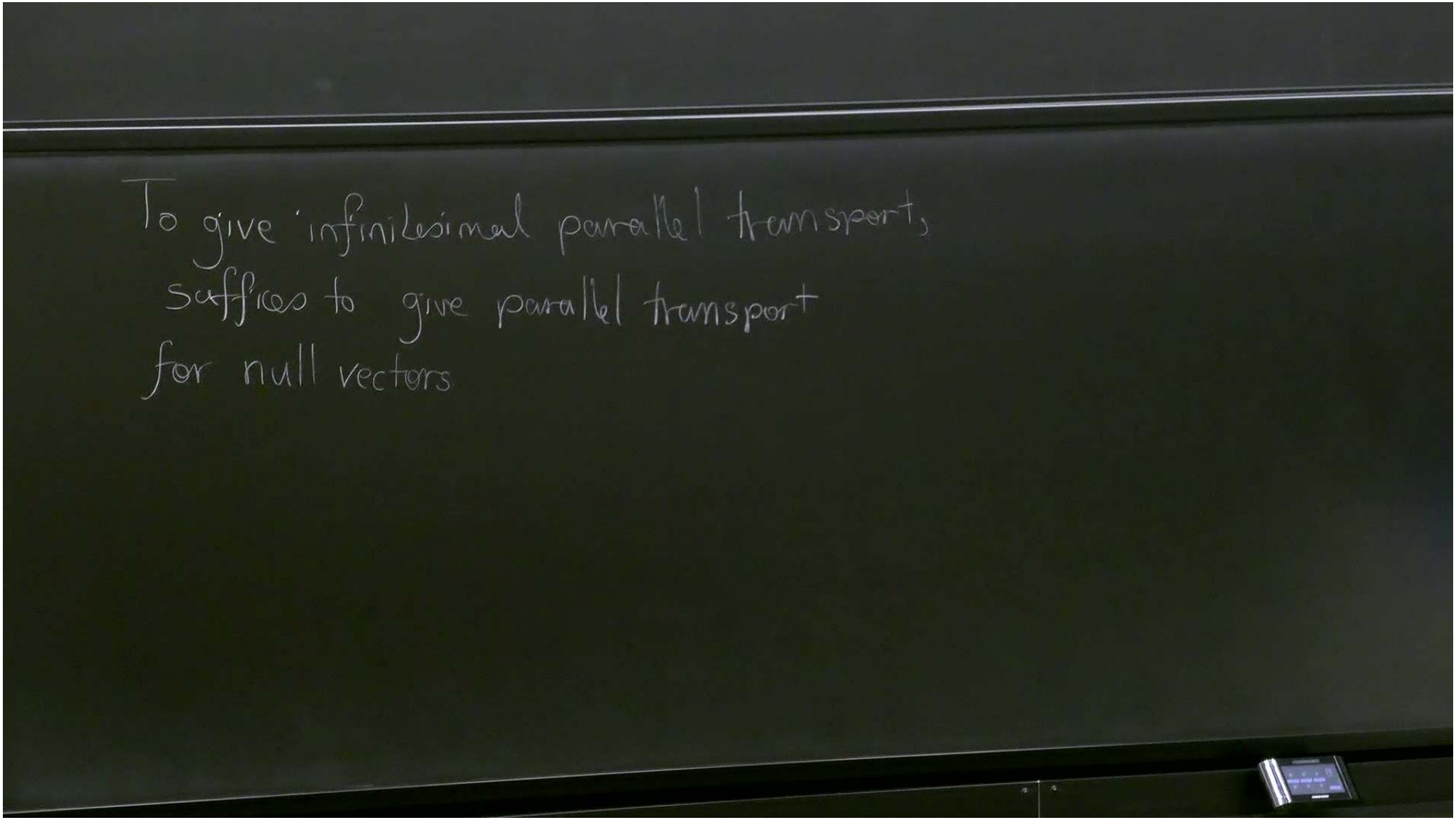
We need to give an iso

$$H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}}, E) \cong H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}}, E)$$

$x_{\alpha\dot\alpha} \longrightarrow x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}$

$\mathbb{C}^4$ is spanned by null vectors

$y_{11}, y_{22}, y_{12}, y_{21}$ are all null.


To give 'infinitesimal parallel transport',
suffices to give parallel transport
for null vectors

We need to give evn iso

$$H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}}, E) \cong H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}}, E)$$

$x_{\alpha\dot\alpha} \longrightarrow x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}$

$\mathbb{C}^4$ is spanned by null vectors

$y_{11}, y_{22}, y_{12}, y_{21}$ are all null.

$\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta} dy_{\alpha\dot\alpha}\, dy_{\beta\dot\beta}$

We need to give an iso

$$H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}}, E) \cong H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}}, E)$$

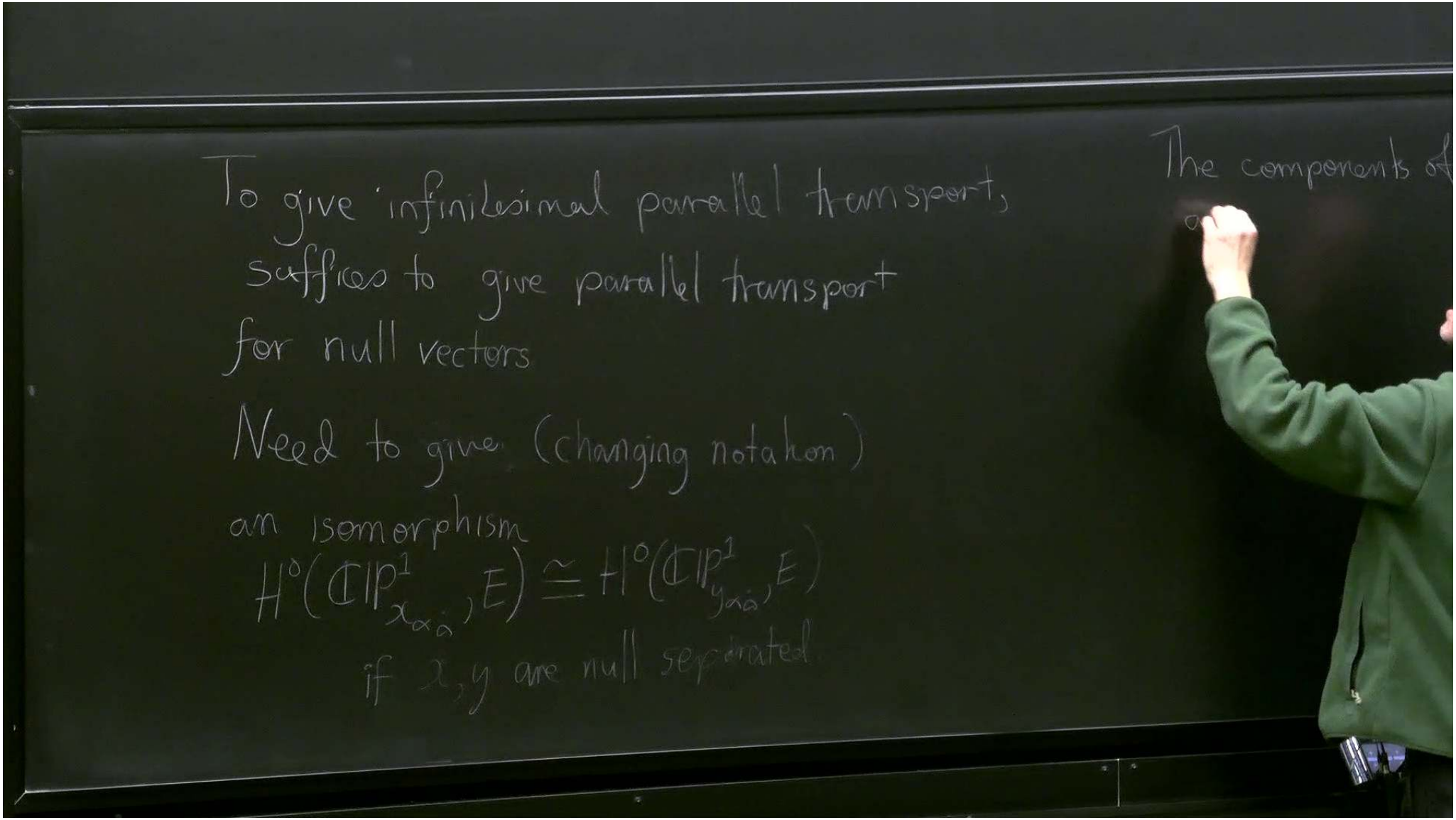
$x_{\alpha\dot\alpha} \longrightarrow x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}$

$y_{11}+y_{22}$ is not null

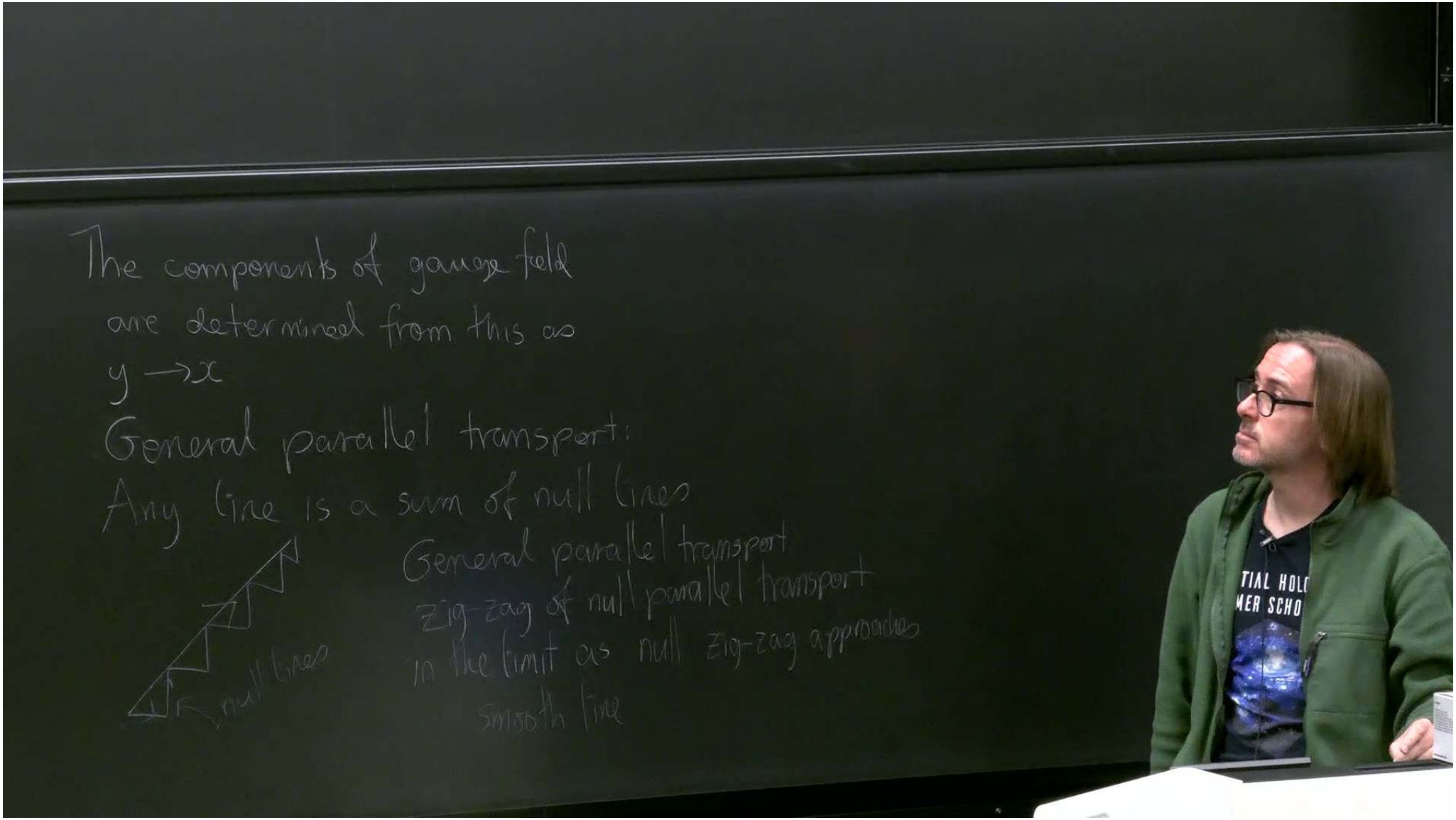
$\mathbb{C}^4$ is spanned by null vectors

$y_{11}, y_{22}, y_{12}, y_{21}$ are all null.

$\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\, dy_{\alpha\dot\alpha}\, dy_{\beta\dot\beta}$


To give 'infinitesimal parallel transport,
suffices to give parallel transport
for null vectors
Need to give (changing notation)
an isomorphism
$H^0(\mathbb{CP}^1_{x_{\alpha\dot{\alpha}}}, E) \cong H^0(\mathbb{CP}^1_{y_{\alpha\dot{\alpha}}}, E)$
if x, y are null separated
The components of

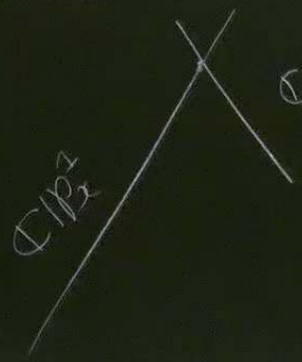
The components of gauge field
are determined from this as
$y \to x$
General parallel transport:
Any line is a sum of null lines


The components of gauge field
are determined from this as
y → x
General parallel transport:
Any line is a sum of null lines
null lines
General parallel transport
zig-zag of null parallel transport
in the limit as null zig-zag approaches
smooth line

if x, y are null separated

x, y null separated

$\Longleftrightarrow$ $\mathbb{CP}^1_x \cap \mathbb{CP}^1_y$ in a single point.

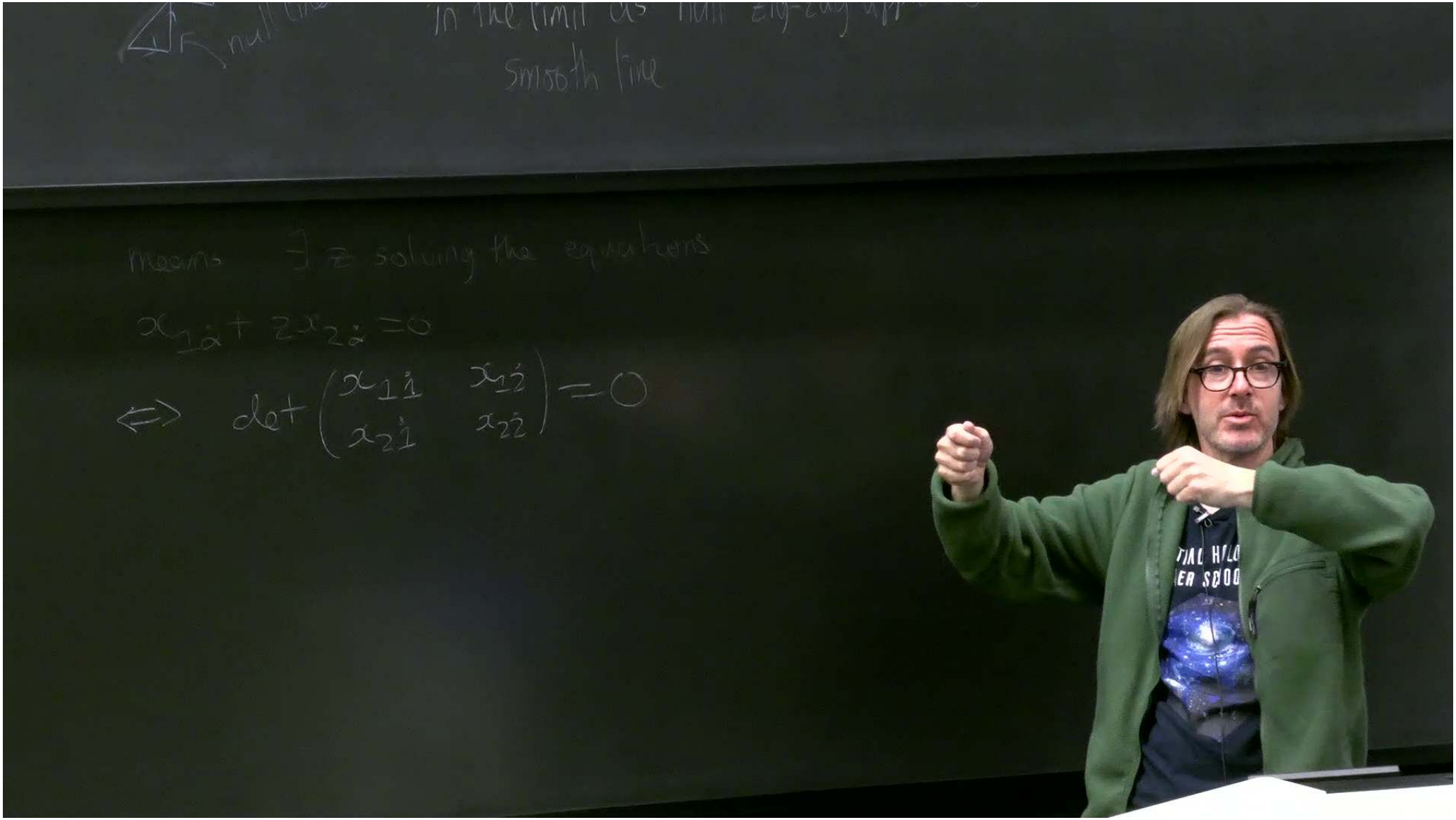


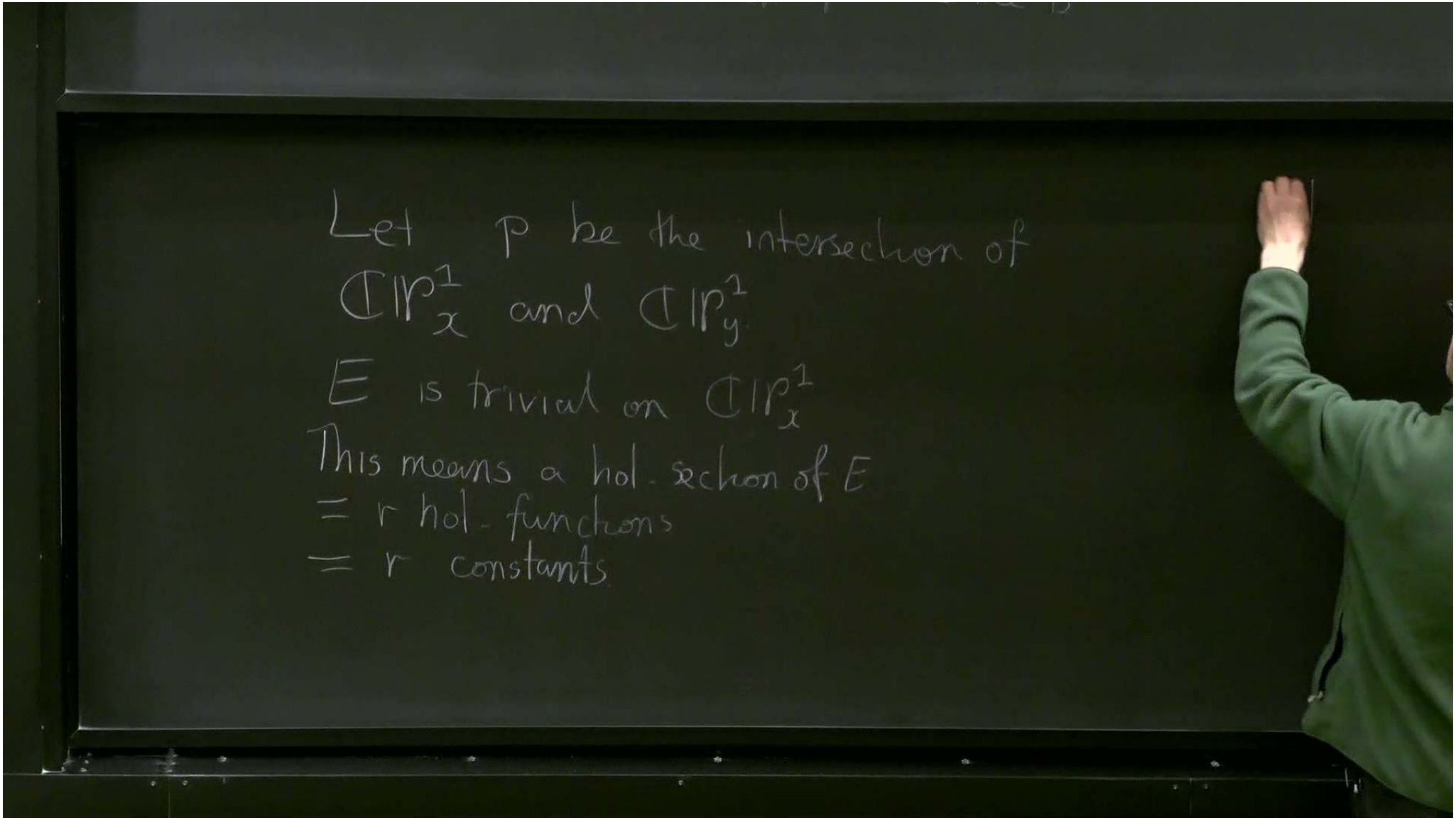
If $y=0$,

$\mathbb{CP}^1_x$ is locus

$V_{\dot\alpha} = x_{1\dot\alpha} + z x_{2\dot\alpha}$

Asking this intersects $V_{\dot\alpha}=0$


null line
In the limit as null zig-zag
smooth line
means ∃ z solving the equations
$x_{1\dot\alpha} + z x_{2\dot\alpha} = 0$
$\Leftrightarrow \det\begin{pmatrix} x_{1\dot1} & x_{1\dot2} \\ x_{2\dot1} & x_{2\dot2} \end{pmatrix} = 0$


Let P be the intersection of
$\mathbb{CP}^1_x$ and $\mathbb{CP}^1_y$
E is trivial on $\mathbb{CP}^1_x$
This means a hol. section of E
$\cong$ r hol. functions
$\cong$ r constants

This means the map

$H^0(\mathbb{CP}^2_x, E) \longrightarrow E_p$ is an isomorphism

Same for

$H^0(\mathbb{CP}^2_y, E) \to E_p$

Null parallel transport is the resulting map

$H^0(\mathbb{CP}^2_x, E) \cong H^0(\mathbb{CP}^2_y, E)$

This means the map

$H^0(\mathbb{CP}^1_x, E) \longrightarrow E_p$ is an isomorphism

Same for

$H^0(\mathbb{CP}^1_y, E) \to E_p$

Null parallel transport is the resulting map

$H^0(\mathbb{CP}^1_x, E) \cong H^0(\mathbb{CP}^1_y, E)$

$= r^4$ constants

$H^0(\mathbb{CP}^7_x, E) \cong H^0$

Want to check that $F_{\alpha\beta} = 0$

$\Longleftrightarrow$ field strength vanishes on any plane given by $\lambda^\alpha x_{\alpha\dot{\alpha}} = 0$ (also translates of such planes)

Given 3 points in $\mathbb{C}^4$, $x^{(1)}_{\alpha\dot{\alpha}}, x^{(2)}_{\alpha\dot{\alpha}}, x^{(3)}_{\alpha\dot{\alpha}}$,

when is the plane they lie on of this form?

parallel transport is the resulting map

$H^0(\mathbb{CP}^1_x, E) \cong H^0(\mathbb{CP}^1_y, E)$

Answer:

If and only if they are all null separated

so $\mathbb{CP}^1_{x(\sigma)} \cap \mathbb{CP}^1_{x(\tau)}$ is non-empty.

And they all intersect at the same point.

parallel transport is the resulting map

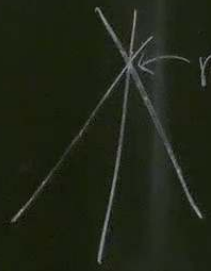
$H^0(\mathbb{CP}^1_x, E) \cong H^0(\mathbb{CP}^1_y, E)$

Answer:

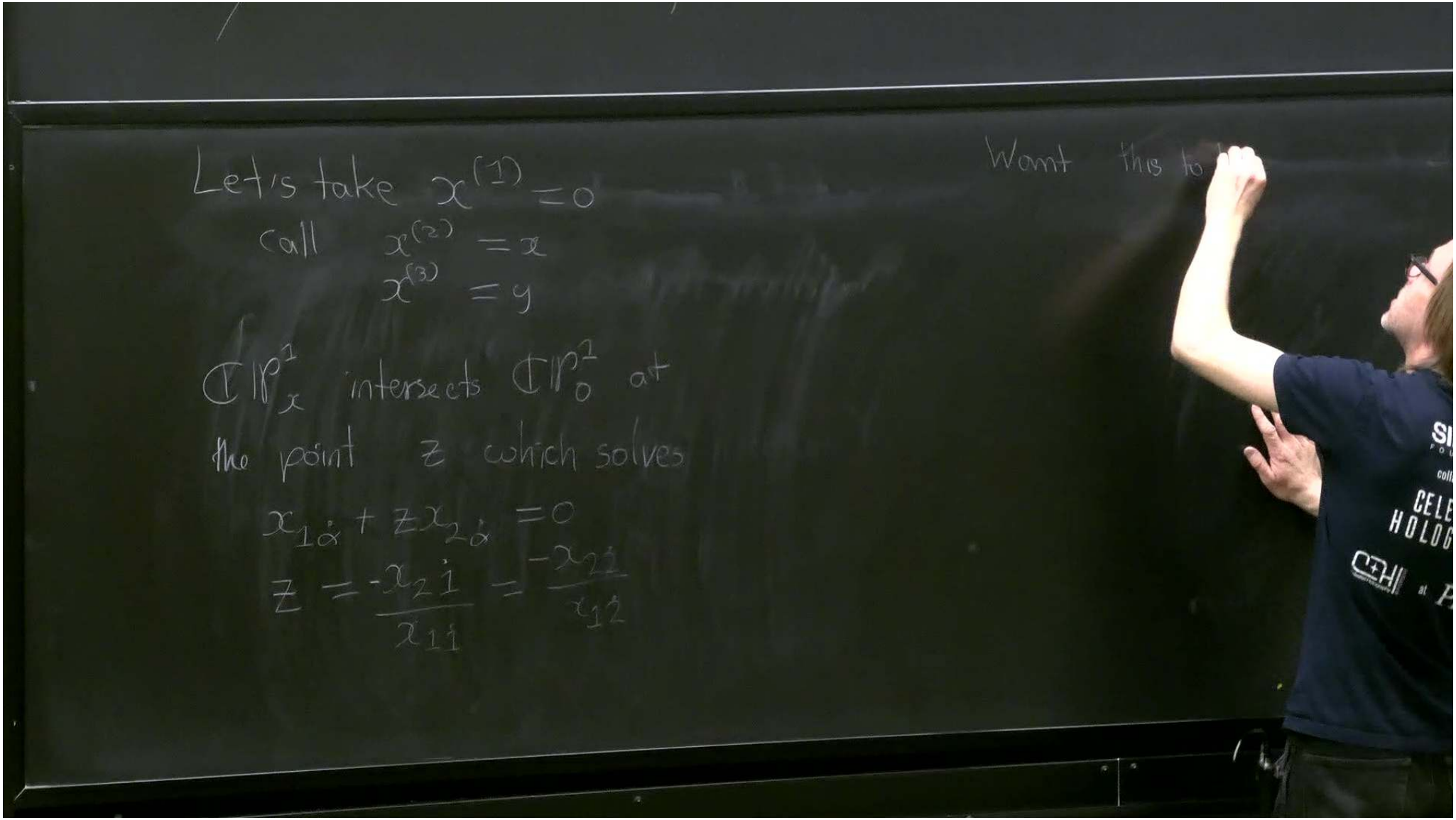
If and only if they are all null separated

so $\mathbb{CP}^1_{x(1)} \cap \mathbb{CP}^1_{x(j)}$ is non-empty.

And they all intersect at the same point.



$\nabla = \mathrm{Id}$


Let's take $x^{(1)} = 0$
call $x^{(2)} = x$
$x^{(3)} = y$
$\mathbb{CP}^1_x$ intersects $\mathbb{CP}^1_0$ at
the point z which solves
$x_{1\dot{\alpha}} + z x_{2\dot{\alpha}} = 0$
$z = \frac{-x_{21}}{x_{11}} = \frac{-x_{22}}{x_{12}}$
Want this to

Asking this intersects $V_a^\vee = 0$

Let's take $x^{(1)} = 0$

call $x^{(2)} = x$

$x^{(3)} = y$

$\mathbb{CP}^1_x$ intersects $\mathbb{CP}^2_0$ at the point z which solves

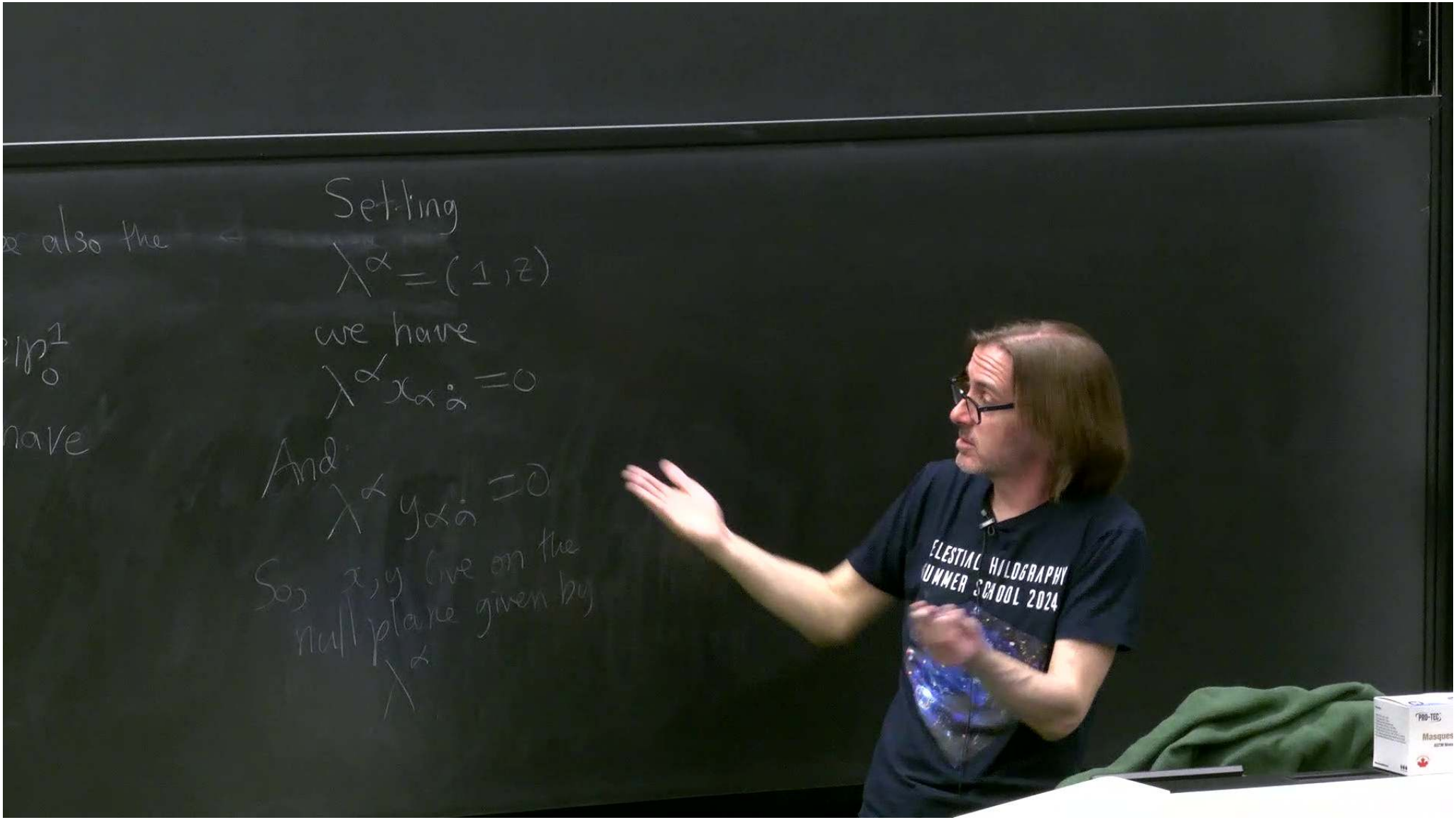
$$x_{1i} + z x_{2i} = 0$$

$$z = \frac{-x_{2i}}{x_{1i}} = \frac{-x_{2i}}{x_{1i}}$$

Want this to be also the intersection of $\mathbb{CP}^1_y$ and $\mathbb{CP}^1_0$

So, we must have

$$\frac{x_{2i}}{x_{1i}} = \frac{y_{2i}}{y_{1i}}$$


also the
Setting
$\lambda^\alpha = (1, z)$
we have
$\lambda^\alpha x_{\alpha\dot{\alpha}} = 0$
And
$\lambda^\alpha y_{\alpha\dot{\alpha}} = 0$
So x, y live on the null plane given by
λ^α
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Want to show that on the plane

$\lambda^\alpha x_{\alpha\dot{\alpha}} = 0$, the connection is flat.

Take 3 points on this plane $0, x, y$

$\mathbb{CP}^2_0 \cap \mathbb{CP}^1_x$ at $p = (3,0,0)$

$\mathbb{CP}^2_0 \cap \mathbb{CP}^1_y$ also at p.

$F = 0$ on this plane

$\Leftrightarrow$ parallel transport

along a closed loop is identity.

It suffices to take the triangle

$0 \to x$ is the composition

$H^0(\mathbb{CP}^1_0, E) \xrightarrow{\alpha_{0p}} E_p \xleftarrow{\alpha_{\infty p}} H^0(\mathbb{CP}^2_\infty, E)$

$\alpha_{\infty p}^{-1} \alpha_{0p}$ is the map $H^0(\mathbb{CP}^2_0, E) \to H^0(\mathbb{CP}^2_\infty, E)$

Other 2 maps are defined similarly.

If we identify $H^0(\mathbb{CP}^1_0, E) = E_p$

$H^0(\mathbb{CP}^2_\infty, E) = E_p$

$H^0(\mathbb{CP}^1_{\infty_1}, E) = E_p$

parallel transport is the resulting map

$H^0(\mathbb{CP}^1_x, E) \cong H^0(\mathbb{CP}^1_y, E)$

these maps are the identity,

so $F_{\alpha\beta} = 0$

$\nabla = \mathrm{Id}$

parallel transport is the resulting map

$H^0(\mathbb{CP}^1_x, E) \cong H^0(\mathbb{CP}^1_y, E)$

these maps are the identity,

so $F_{\alpha\beta} = 0$

(Actually, connection is gauge trivial on this plane)

$\nabla = d$

$H^0(\mathbb{CP}^1_0, E) \xrightarrow{\alpha_{0p}} E_p \xleftarrow{\alpha_{\infty p}} H^0(\mathbb{CP}^2_\infty, E)$

$\alpha_{\infty p}^{-1} \alpha_{0p}$ is the map $H^0(\mathbb{CP}^2_0, E) \to H^0(\mathbb{CP}^2_\infty, E)$

Other 2 maps are defined similarly.

If we identify $H^0(\mathbb{CP}^1_0, E) = E_p$

$H^0(\mathbb{CP}^1_\infty, E) = E_p$

$H^0(\mathbb{CP}^1_{q_1}, E) = E_p$

parallel transport is the resulting map

$H^0(\mathbb{CP}^1_x, E) \cong H^0(\mathbb{CP}^1_y, E)$

$x \to y$

$\alpha_{yp}^{-1} \alpha_{xp}$

is

$y \to 0$

$\alpha_{op}^{-1} \alpha_{yp}$

$\alpha_{op}^{-1} \alpha_{yp} \alpha_{yp}^{-1} \alpha_{xp} \alpha_{xp}^{-1} \alpha_{op}$

$= Id$