Title: Lecture - Mathematical Physics, PHYS 777

Speakers: Kevin Costello

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Subject: Mathematical physics

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E, a hol. vector bundle on $\mathbb{PT} = \mathcal{O}(1)^2 \to \mathbb{CP}^1$

$E|_{\mathbb{CP}^1}$ is trivial

$\Longrightarrow$ vector bundle on space time.

Given $x_{\alpha\dot{\alpha}} \in \mathbb{C}^4$,

$\mathbb{CP}^1_{x_{\alpha\dot{\alpha}}} \subseteq \mathbb{PT}$, fibre of v. bundle

on space-time is

$= \mathcal{O}(1)^2 \to \mathbb{C}P^1$

$H^0(\mathbb{C}P^1_{\lambda(\alpha\dot{\alpha})}, E)$

E of rank r, $\Big)$ also of rank r

<u>Neat</u> Want to give a connection + check that $F_{\alpha\beta} = 0$

on space-time is

On any manifold with a v. bundle V

a connection is data of infinitesimal parallel transport:

iso. from V_x to $V_{x+\varepsilon u}$

$x \in M$

$u \in T_x M$

on space-time is

On any manifold with a v. bundle V
a connection is data of infinitesimal parallel transport:

iso. from V_x to $V_{x+\varepsilon u}$

$x \in M$

$u \in T_x M$

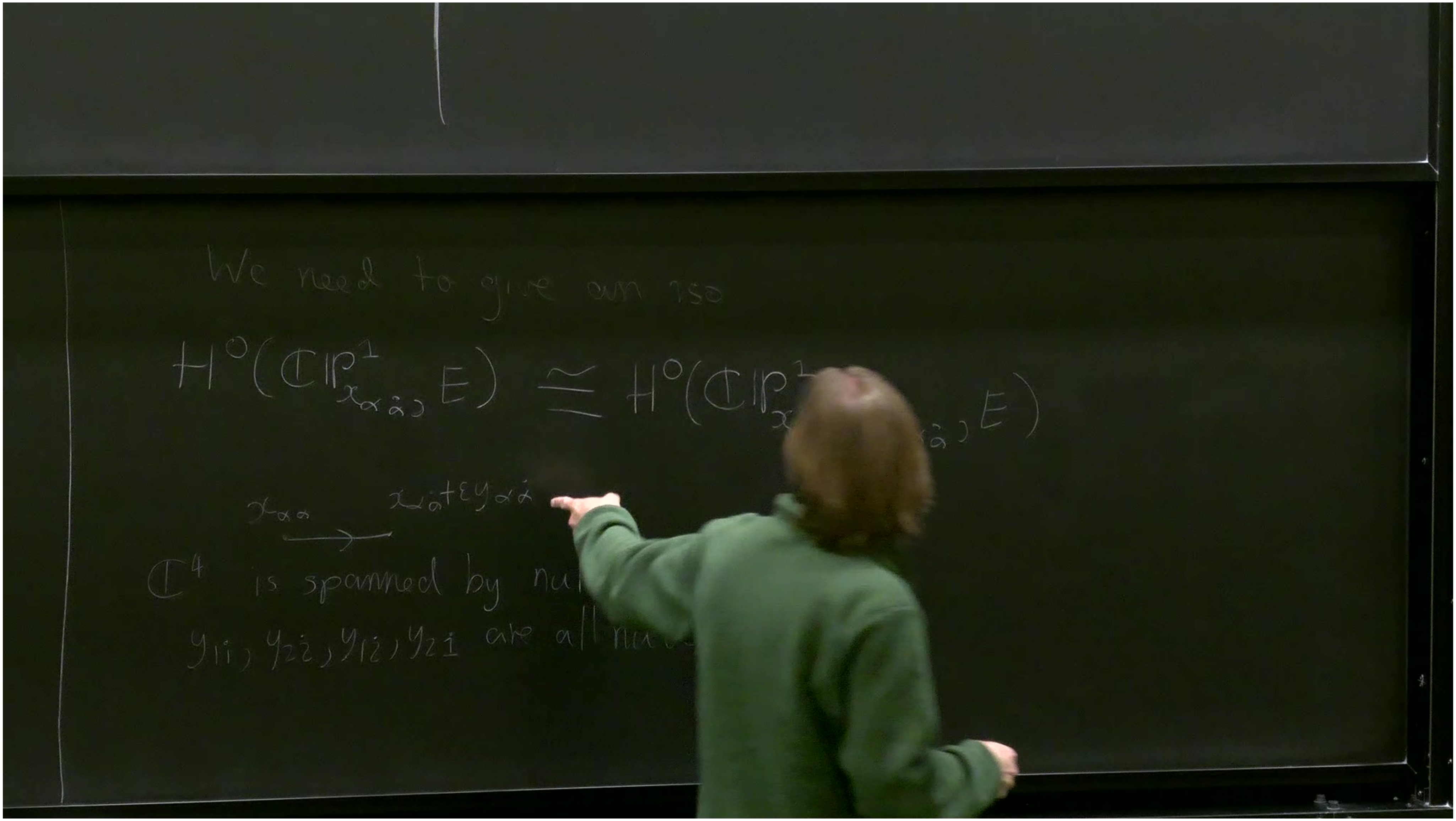
$x \longrightarrow x+\varepsilon u$

Wilson line

We need to give an iso

$$H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}}, E) \cong H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}}, E)$$

$$x_{\alpha\dot\alpha} \longrightarrow x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}$$


We need to give an iso
$H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}}, E) \cong H^0(\mathbb{CP}^1_{x}$
$\dot\alpha, E)$
$x_{\alpha\dot\alpha} \longrightarrow x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}$
$\mathbb{C}^4$ is spanned by nu
$y_{1\dot1}, y_{2\dot2}, y_{1\dot2}, y_{2\dot1}$ are all nu

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$$x_{\alpha\dot\alpha} \longrightarrow x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}$$

$\mathbb{C}^4$ is spanned by null vectors

$y_{1\dot1}, y_{2\dot2}, y_{1\dot2}, y_{2\dot1}$ are all null.

To give infinitesimal parallel transport,
suffices to give parallel transport
for null vectors

We need to give an iso

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$\mathbb{C}^4$ is spanned by null vectors

$y_{1\dot1}, y_{2\dot2}, y_{1\dot2}, y_{2\dot1}$ are all null.

$$\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta} dy_{\alpha\dot\alpha}\, dy_{\beta\dot\beta}$$

We need to give an iso

$$H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}}, E) \cong H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}}, E)$$

$$x_{\alpha\dot\alpha} \longrightarrow x_{\alpha\dot\alpha}+\varepsilon y_{\alpha\dot\alpha}$$

$y_{1\dot1}+y_{2\dot2}$ is not null

$\mathbb{C}^4$ is spanned by null vectors

$y_{1\dot1}, y_{2\dot2}, y_{1\dot2}, y_{2\dot1}$ are all null.

$$\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta} dy_{\alpha\dot\alpha} dy_{\beta\dot\beta}$$

To give infinitesimal parallel transport,
suffices to give parallel transport
for null vectors

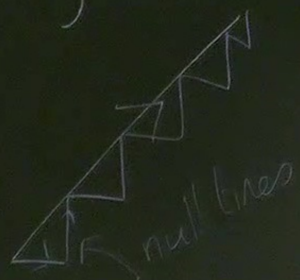
Need to give (changing notation)
an isomorphism

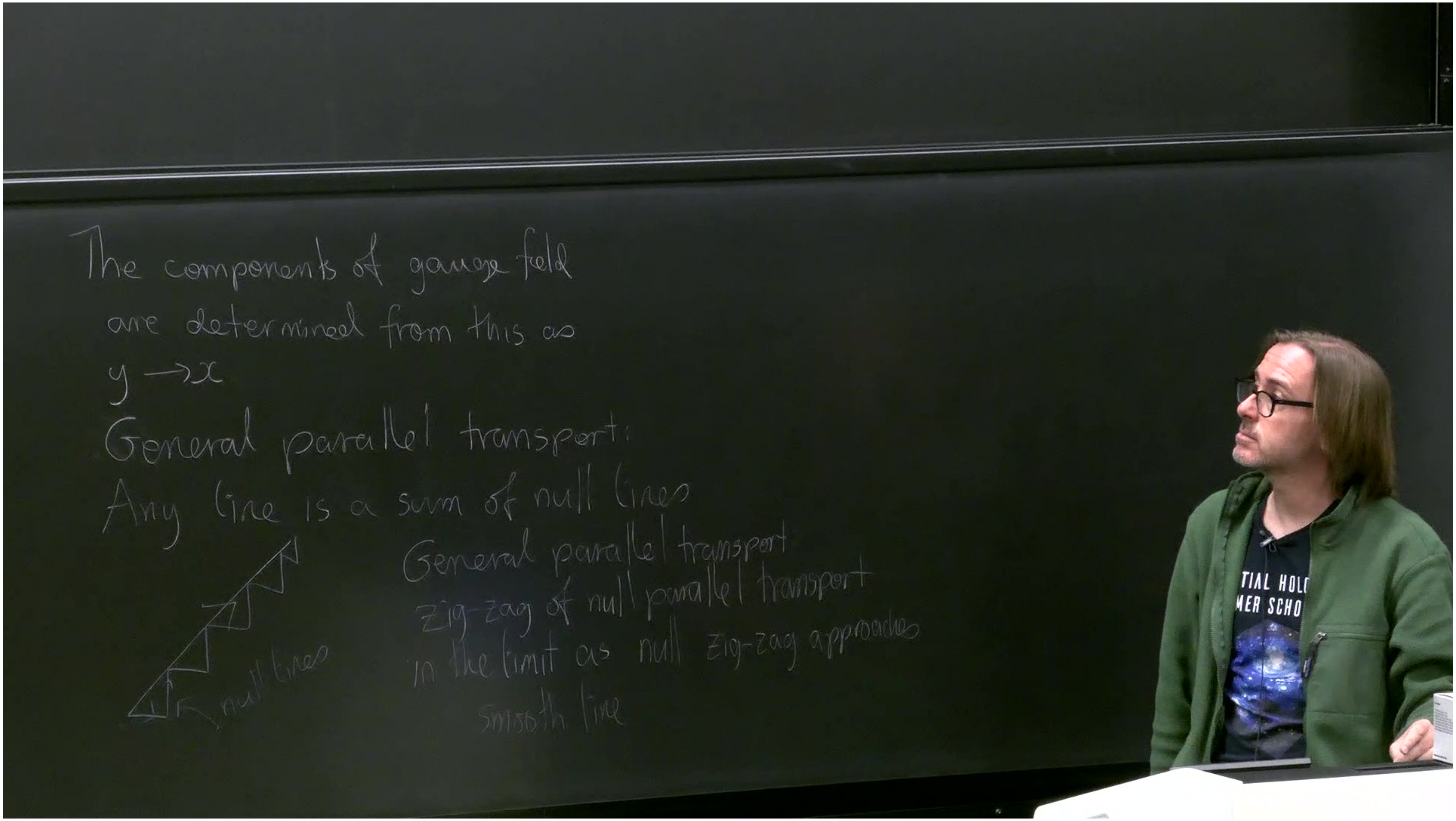
$$H^0(\mathbb{CP}^1_{x_{\alpha\dot\alpha}}, E) \cong H^0(\mathbb{CP}^1_{y_{\alpha\dot\alpha}}, E)$$

if x, y are null separated

The components of

The components of gauge field
are determined from this as
$y \to x$
General parallel transport:
Any line is a sum of null lines



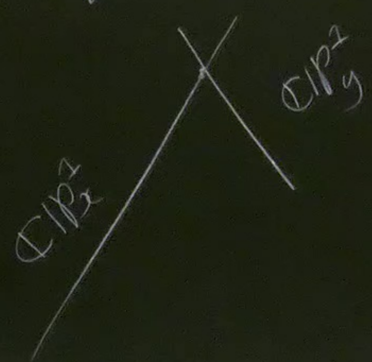

The components of gauge field
are determined from this as
y → x
General parallel transport:
Any line is a sum of null lines
General parallel transport
zig-zag of null parallel transport
in the limit as null zig-zag approaches
smooth line
null lines

if x, y are null separated

means

x, y null separated

$\iff \mathbb{CP}^1_x \cap \mathbb{CP}^1_y$ in a single point.

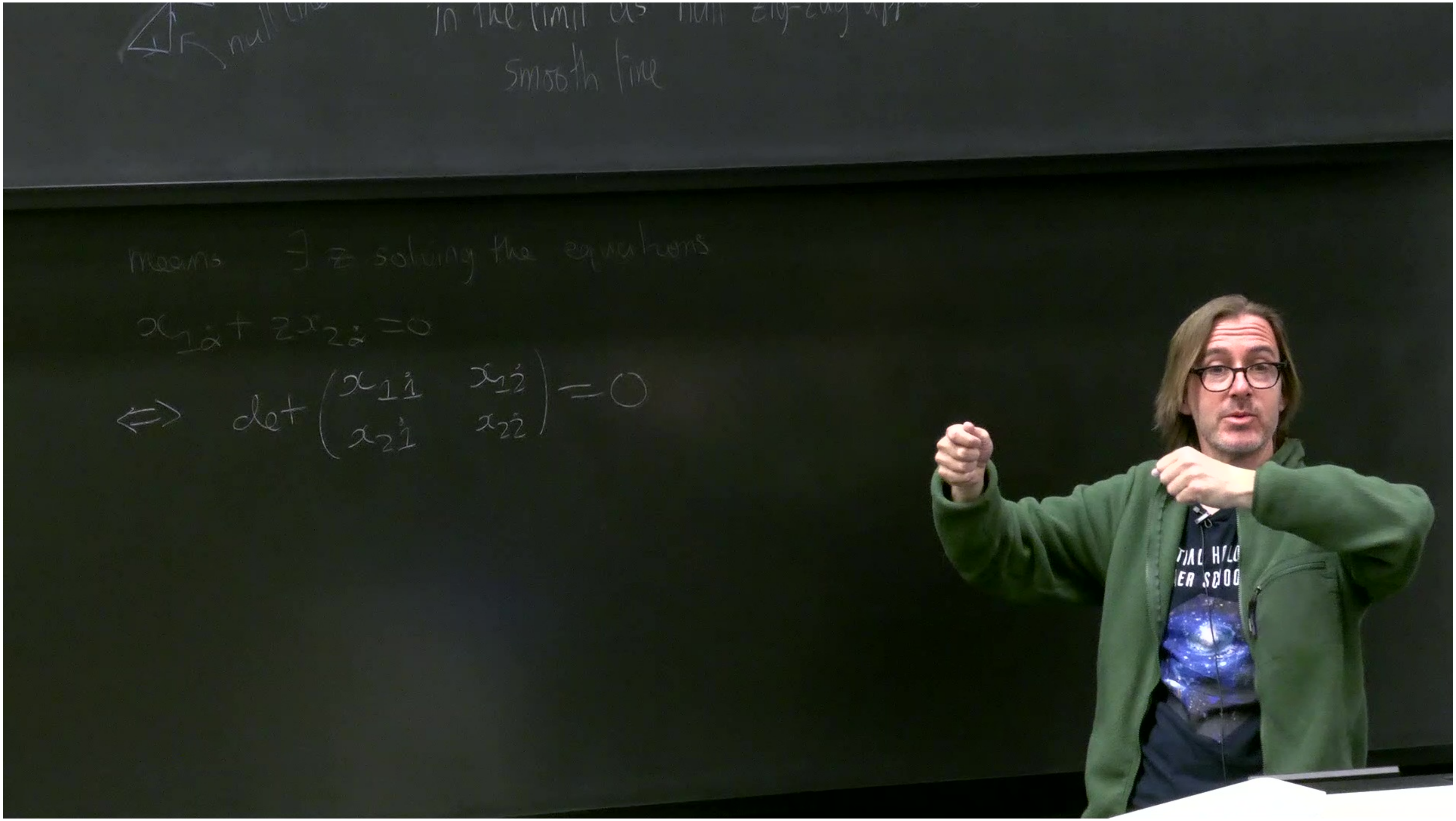


If $y=0$,

$\mathbb{CP}^1_x$ is locus

$$V_{\dot\alpha} = x_{1\dot\alpha} + z x_{2\dot\alpha}$$

Asking this intersects $V_{\dot\alpha} = 0$


smooth line
means $\exists z$ solving the equations
$x_{1\dot{\alpha}} + z x_{2\dot{\alpha}} = 0$
$\Leftrightarrow \det\begin{pmatrix} x_{1\dot{1}} & x_{1\dot{2}} \\ x_{2\dot{1}} & x_{2\dot{2}} \end{pmatrix} = 0$

Let p be the intersection of
$\mathbb{CP}^1_x$ and $\mathbb{CP}^1_y$.

E is trivial on $\mathbb{CP}^1_x$

This means a hol. section of E
$=$ r hol. functions
$=$ r constants

This means the map

$$H^0(\mathbb{CP}^1_x, E) \longrightarrow E_p \text{ is an isomorphism}$$

Same for

$$H^0(\mathbb{CP}^1_y, E) \to E_p$$

Null parallel transport is the resulting map

$$H^0(\mathbb{CP}^1_x, E) \cong H^0(\mathbb{CP}^1_y, E)$$

This means the map

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Same for

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Null parallel transport is the resulting map

$$H^0(\mathbb{CP}^1_x, E) \cong H^0(\mathbb{CP}^1_y, E)$$

$= \Gamma$ constants

$H^0(\mathbb{CP}^1_x, E) \cong H$

Want to check that $F_{\alpha\beta} = 0$

$\Longleftrightarrow$ field strength vanishes on any plane given by $\lambda^\alpha x_{\alpha\dot\alpha} = 0$ (also translates of such planes)

Given 3 points in $\mathbb{C}^4$, $x^{(1)}_{\alpha\dot\alpha}$, $x^{(2)}_{\alpha\dot\alpha}$, $x^{(3)}_{\alpha\dot\alpha}$,
when is the plane they lie on of this form?

...arallel transport is the resulting map

$$H^0(\mathbb{CP}^1_x, E) \simeq H^0(\mathbb{CP}^1_y, E)$$

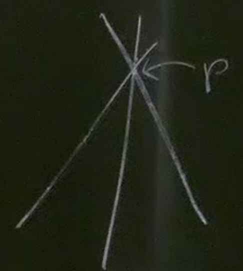
...ames)

Answer:

If and only if they are all null separated

so $\mathbb{CP}^1_{x^{(i)}} \cap \mathbb{CP}^1_{x^{(j)}}$ is non-empty.

And they all intersect at the same point.



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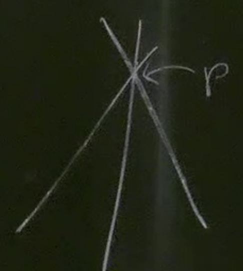


$$\nabla = \mathrm{Id}$$

Let's take $x^{(1)} = 0$

call $x^{(2)} = x$

$x^{(3)} = y$

$\mathbb{CP}^1_x$ intersects $\mathbb{CP}^1_0$ at the point z which solves

$$x_{1\dot\alpha} + z x_{2\dot\alpha} = 0$$

$$z = \frac{-x_{2\dot 1}}{x_{1\dot 1}} = \frac{-x_{2\dot 2}}{x_{1\dot 2}}$$

Want this to

Asking this intersects $V_{\dot\alpha} = 0$

Let's take $x^{(1)} = 0$

call $x^{(2)} = x$

$x^{(3)} = y$

$\mathbb{CP}^1_x$ intersects $\mathbb{CP}^1_0$ at the point z which solves

$$x_{1\dot\alpha} + z x_{2\dot\alpha} = 0$$

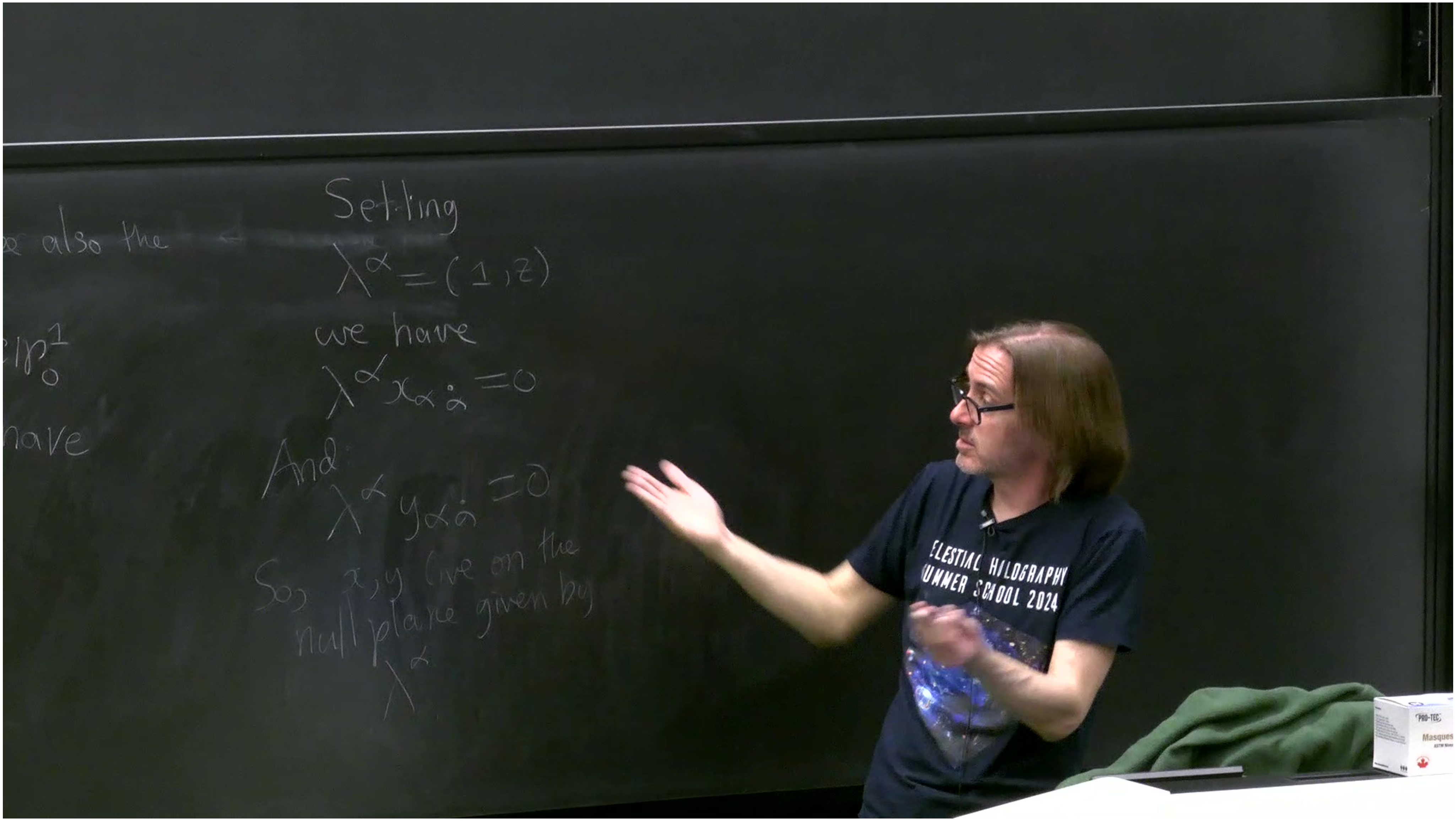
$$z = -\frac{x_{2\dot 1}}{x_{1\dot 1}} = \frac{-x_{2\dot 2}}{x_{1\dot 2}}$$

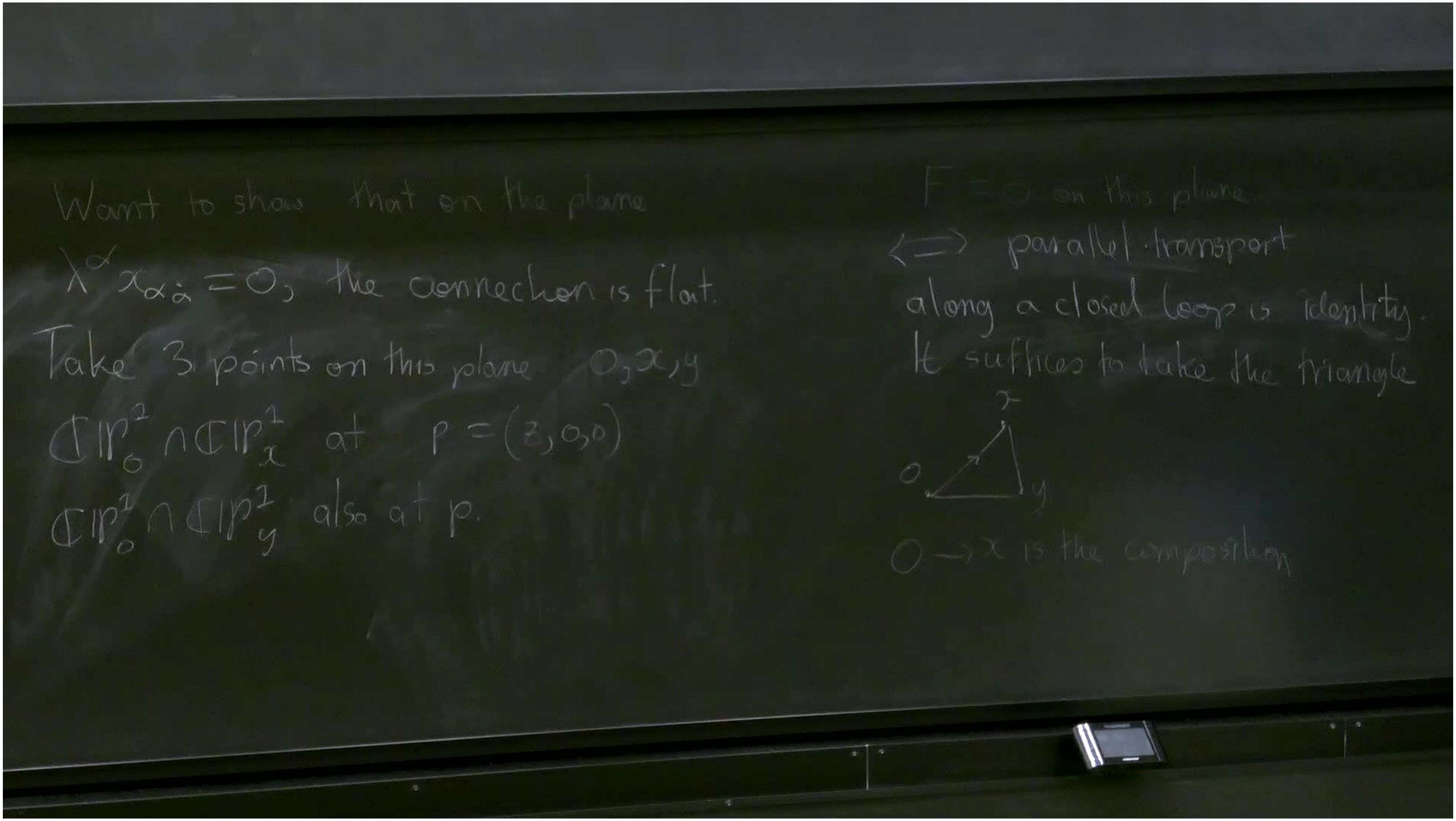
Want this to be also the intersection of

$\mathbb{CP}^1_y$ and $\mathbb{CP}^1_0$

So, we must have

$$\frac{x_{2\dot 1}}{x_{1\dot 1}} = \frac{y_{2\dot 1}}{y_{1\dot 1}}$$

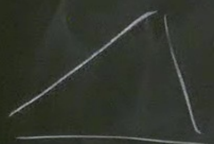

also the
have
Setting
λ^α = (1, z)
we have
λ^α x_{αα̇} = 0
And
λ^α y_{αα̇} = 0
So, x, y live on the
null plane given by
λ^α


Want to show that on the plane
$\lambda^\alpha x_{\alpha\dot\alpha} = 0$, the connection is flat.
Take 3 points on this plane 0, x, y
$\mathbb{CP}^1_0 \cap \mathbb{CP}^1_x$ at $p = (z,0,0)$
$\mathbb{CP}^1_0 \cap \mathbb{CP}^1_y$ also at p.
F = 0 on this plane
$\Longleftrightarrow$ parallel transport
along a closed loop is identity.
It suffices to take the triangle
x
0
y
0 → x is the composition

$$H^0(\mathbb{CP}^1_0, E) \xrightarrow{\alpha_{0p}} E_p \xleftarrow{\alpha_{xp}} H^0(\mathbb{CP}^1_x, E)$$

$\alpha_{xp}^{-1}\alpha_{0p}$ is the map $H^0(\mathbb{CP}^1_0, E)$

$\longrightarrow H^0(\mathbb{CP}^1_x, E)$



Other 2 maps are defined similarly.

If we identify $H^0(\mathbb{CP}^1_0, E) = E_p$

$H^0(\mathbb{CP}^1_x, E) = E_p$

$H^0(\mathbb{CP}^1_y, E) = E_p$

parallel transport is the resulting map

$$H^0(\mathbb{CP}^1_x, E) \simeq H^0(\mathbb{CP}^1_y, E)$$

These maps are the identity,

so $F_{\alpha\beta} = 0$

$\nabla = \mathrm{Id}$

...arallel transport is the resulting map

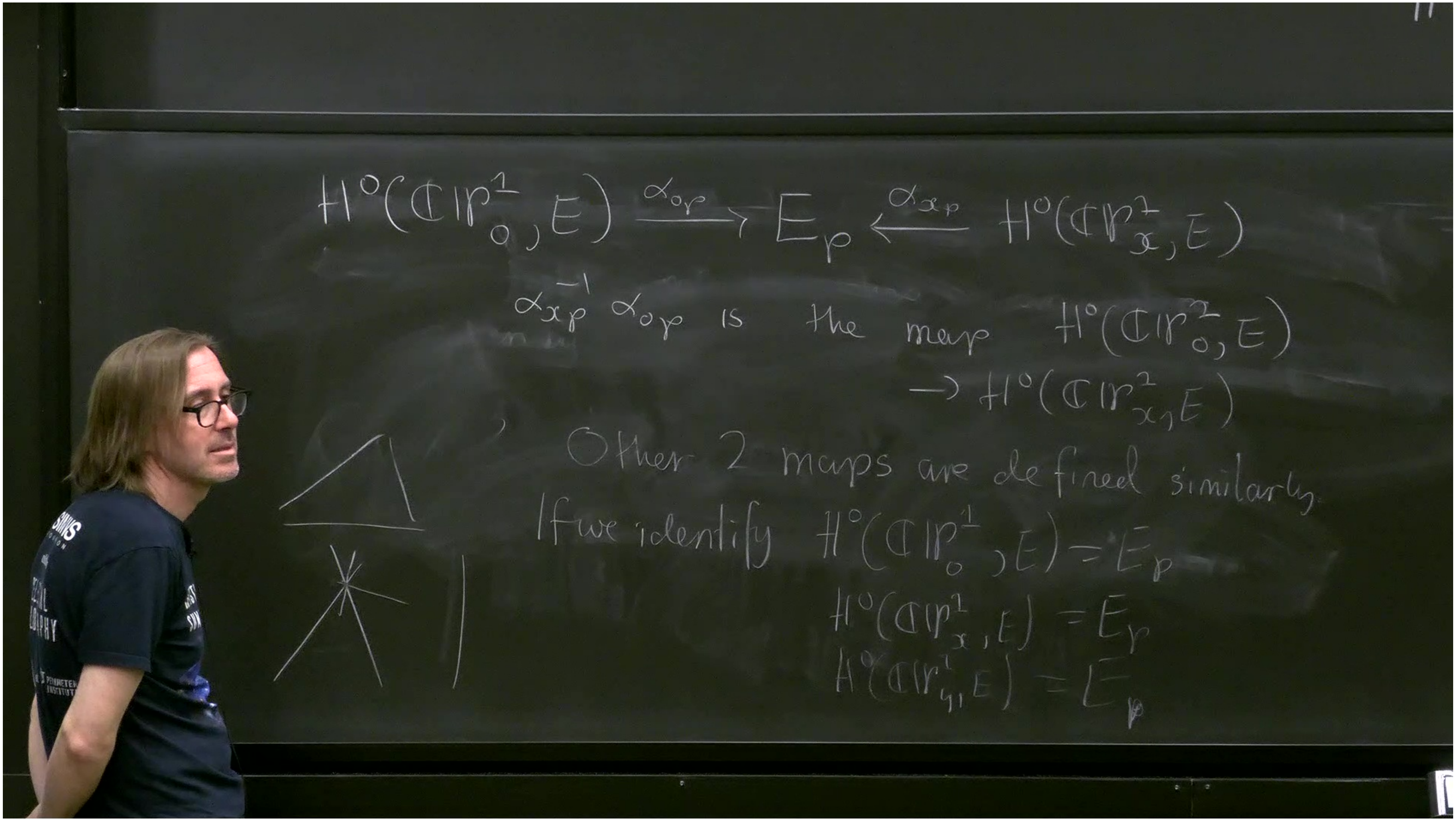
$$H^0(\mathbb{CP}^1_x, E) \cong H^0(\mathbb{CP}^1_y, E)$$

These maps are the identity,

so $F_{\alpha\beta} = 0$

(Actually, connection is gauge trivial on this plane)

$\nabla = \mathrm{Id}$



$H^0(\mathbb{CP}^1_0, E) \xrightarrow{\alpha_{0p}} E_p \xleftarrow{\alpha_{xp}} H^0(\mathbb{CP}^1_x, E)$
$\alpha_{xp}^{-1}\alpha_{0p}$ is the map $H^0(\mathbb{CP}^1_0, E)$
$\to H^0(\mathbb{CP}^1_x, E)$
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If we identify $H^0(\mathbb{CP}^1_0, E) = E_p$
$H^0(\mathbb{CP}^1_x, E) = E_p$
$H^0(\mathbb{CP}^1_y, E) = E_p$

parallel transport is the resulting map

$H^0(\mathbb{CP}^1_x, E) \cong H^0(\mathbb{CP}^1_y, E)$

$x \to y$

$\alpha_{yp}^{-1} \alpha_{xp}$

$0 \to x \to y$

is

x

y

$y \to 0$

$\alpha_{op}^{-1} \alpha_{yp}$

$\alpha_{op}^{-1} \alpha_{yp} \alpha_{yp}^{-1} \alpha_{xp} \alpha_{xp}^{-1} \alpha_{op}$

$= \mathrm{Id}$