Title: Lecture - Mathematical Physics, PHYS 777

Speakers: Kevin Costello

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Last time

$H^1(\mathbb{PT}, \mathcal{O}(-2)) =$ Harmonic functions

Want to show

$$H^1(\mathbb{PT}, \mathcal{O}(-1)) = \left\{ \begin{array}{l} \text{Spinors } \psi_{\dot\alpha} \text{ satisfy} \\ \dfrac{\partial}{\partial x_{\dot\alpha \alpha}} \psi_{\dot\alpha} = 0 \end{array} \right\}$$

$\mathcal{O}(-3)$ spinors

$\mathcal{O}$. gauge fields

$H^1(\mathbb{PT}, \mathcal{O}(-1))$

$=$ expressions like

$(dz)^{1/2}\left(d\bar{z} f + d\bar{v}^{\dot{\alpha}} g_{\dot{\alpha}} \right)$

With $\mathcal{O}(-2)$, we had dz: could integrate.

Not with $(dz)^{1/2}$

If $\alpha = (dz)$

Note $dv_1 \frac{\partial}{\partial v_1}$ makes sense

dv_1 transforms as $(dz)^{1/2}$

$(dz)^{1/2} \frac{\partial}{\partial v_1}$ also makes sense gauge fields

Note $dv_{\dot 1} \frac{\partial}{\partial v_{\dot 1}}$ makes sense

$dv_{\dot 1}$ transforms as $(dz)^{1/2}$

$(dz)^{1/2} \frac{\partial}{\partial v_{\dot 1}}$ also makes sense

If $\varphi = (dz)^{1/2} \left(f\, d\bar{z} + g_{\dot\alpha}\, d\bar{v}^{\dot\alpha} \right)$

then, $(dz)^{1/2} \frac{\partial \varphi}{\partial v_{\dot\alpha}} = dz(\ldots\ldots)$

Let

$$\psi^{\dot\alpha} = \int_{v_{\dot\alpha} = u_{\alpha} + \varepsilon_{\alpha\beta} \bar{u}^{\beta} z} \frac{\partial}{\partial v_{\dot\alpha}} \varphi$$

If $\varphi = (dz)^{1/2}\left(f\, d\bar{z} + g_{\dot\alpha}\, d\bar{v}^{\dot\alpha}\right)$

then, $(dz)^{1/2} \dfrac{\partial \varphi}{\partial v_{\dot\alpha}} = dz\left(\cdots \right)$

Let

$$\psi^{\dot\alpha} = \int \frac{\partial}{\partial v_{\dot\alpha}} \varphi$$

$v_{\dot\alpha} = u_\alpha + \varepsilon_{\alpha\beta}\bar{u}^\beta z$

Equivalently,

$\dfrac{\partial}{\partial v_{\dot\alpha}}$ is a map $H^1(\mathbb{PT}, \mathcal{O}(-1))$

$\longrightarrow H^1(\mathbb{PT}, \mathcal{O}(-2))$

We need to check

$$\varepsilon_{\dot\alpha\dot\beta} \frac{\partial}{\partial x_{\dot\alpha\beta}} \psi^{\dot\beta} = 0$$

$$\frac{\partial}{\partial x_{\dot\alpha 1}} = \frac{\partial}{\partial u_{\dot\alpha}}$$

$$\frac{\partial}{\partial x_{\dot\alpha 2}} = \varepsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial \bar{u}^{\dot\beta}}$$

Need

$$\frac{\partial \psi^{\dot\beta}}{\partial \bar{u}^{\dot\beta}} = 0$$

$$\varepsilon_{\dot\alpha\dot\beta}\frac{\partial \psi^{\dot\beta}}{\partial u_{\dot\alpha}} = 0$$

$$\varepsilon_{\dot\alpha\dot\beta} \frac{\partial \psi^{\dot\beta}}{\partial u_{\dot\alpha}} = \int \frac{\partial}{\partial v_{\dot\alpha}} \frac{\partial}{\partial v_{\dot\beta}} \varphi = 0 \checkmark$$

$$\frac{\partial \psi^{\dot\beta}}{\partial \bar{u}^{\dot\beta}} = \int z \frac{\partial}{\partial v_{\dot\alpha}} \frac{\partial}{\partial v_{\dot\beta}} \varphi \, \varepsilon_{\dot\alpha\dot\beta} = 0$$

$$H^1(\mathbb{PT}, \mathcal{O}(-1)) = \bigoplus_{n>0} \left[\frac{n}{2}\right]_L \otimes \left[\frac{n-1}{2}\right]_R$$

$$= \left\{ \begin{array}{c} \text{Spherical harmonics for a} \\ \text{spinor field on } S^3 \end{array} \right\}$$

$=$ Cauchy data for sol^n to Dirac eq^n

$$H^1(\mathbb{PT}, \mathcal{O}(-1)) = \bigoplus_{n>0} \left[\tfrac{n}{2}\right]_L \otimes \left[\tfrac{n-1}{2}\right]_R$$

$$= \left\{ \text{Spherical harmonics for a spinor field on } S^3 \right\}$$

= Cauchy data for sol^n to Dirac eq^n

A two form F can be written

$$F = F^{\alpha\dot{\alpha}\beta\dot{\beta}}\, dx_{\alpha\dot{\alpha}} \wedge dx_{\beta\dot{\beta}}$$

$$F^{\alpha\dot{\alpha}\beta\dot{\beta}} = -F^{\beta\dot{\beta}\alpha\dot{\alpha}}$$

We can write

$$F^{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon^{\alpha\beta} F_+^{\dot{\alpha}\dot{\beta}} + \varepsilon^{\dot{\alpha}\dot{\beta}} F_-^{\alpha\beta}$$

F_+, F_- are symmetric in spinor indices.

$\mathbb{C}^4 = S_+ \otimes S_-$

basis $x_{1\dot{1}}, x_{1\dot{2}}, x_{2\dot{1}}, x_{2\dot{2}}$

metric $\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}$

A SD gauge field (for $U(1)$) on $\mathbb{R}^4$

has F where $F_- = 0$

$$F = F_+^{\dot\alpha\beta}\,\varepsilon^{\alpha\beta}$$

Bianchi identity says $dF = 0$

This $\Rightarrow$ YM equation, $d{*}F = 0$

$$\varepsilon_{\dot\alpha\beta}\frac{\partial\psi^{\beta}}{\partial u_{\dot\alpha}} = \int \frac{\partial}{\partial v_{\dot\alpha}}\frac{\partial}{\partial v_{\beta}}\varphi = 0 \checkmark$$

$$\frac{\partial\psi^{\beta}}{\partial\bar u^{\beta'}} = \int z\frac{\partial}{\partial v_{\dot\alpha}}\frac{\partial}{\partial v_{\beta}}\varphi\,\varepsilon_{\dot\alpha\beta} = 0$$

$$\left\{ \begin{array}{l} \text{Sol}^{n}\text{s of SDYM eq}^{n} \quad F_{-} = 0, \\ \quad u(1) \text{ gauge field, up to gauge} \\ \quad\quad \text{transformations} \end{array} \right\} = H^{1}(\mathbb{PT}, \mathcal{O})$$

Proof:

Gauge invariant data is $F_{+}^{\dot{\alpha}\dot{\beta}}$ satisfying

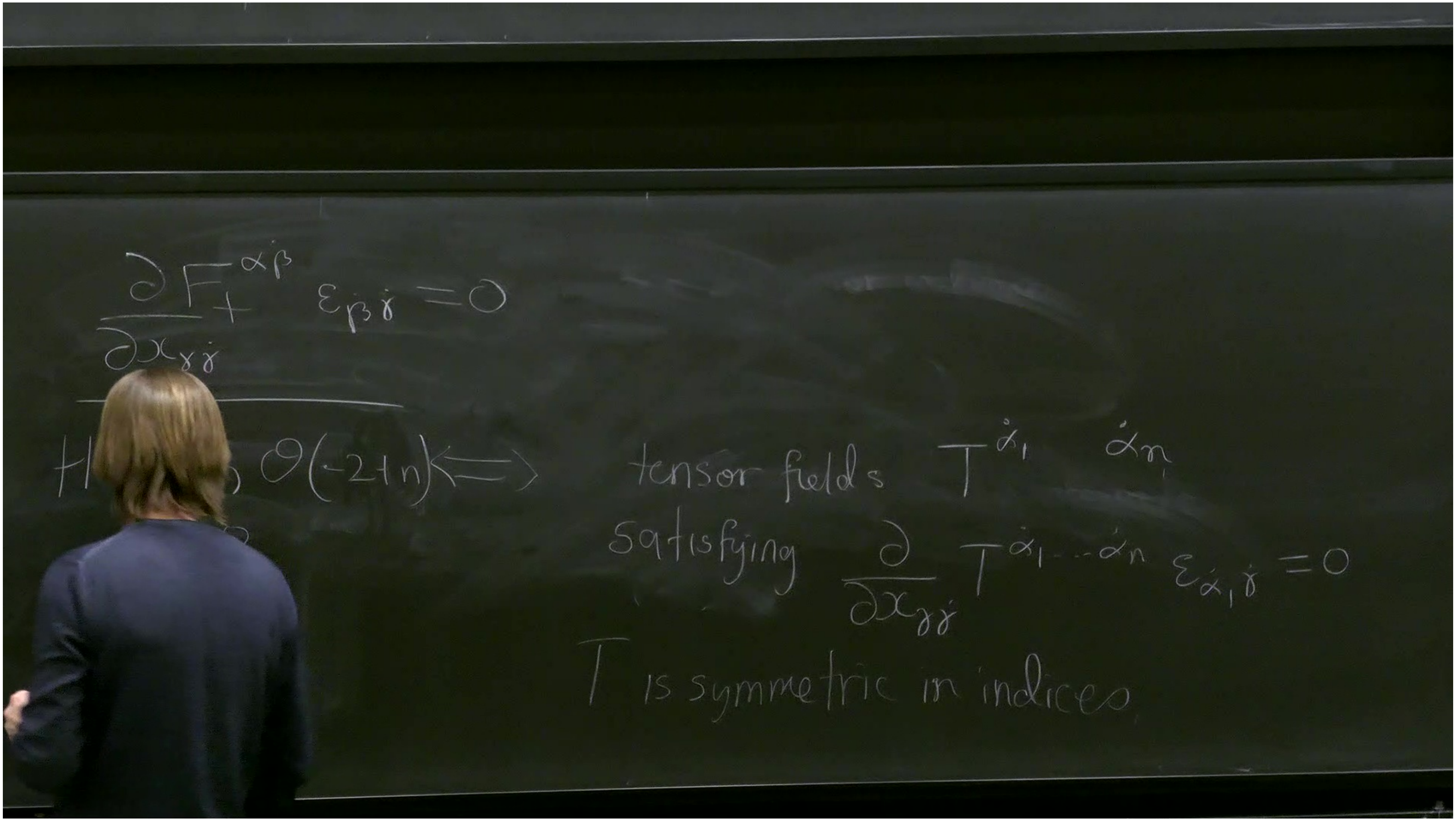
$$\frac{\partial}{\partial x_{\gamma\dot{\gamma}}} F_{+}^{\dot{\alpha}\dot{\beta}} \varepsilon_{\dot{\gamma}\dot{\beta}} = 0$$

On $\mathbb{PT}$, a class in $H^1(\mathbb{PT}, \mathcal{O})$ is like

$$\varphi = d\bar{z} f + d\bar{v}^{\dot{\alpha}} g_{\dot{\alpha}}$$

$$F_+^{\dot{\alpha}\dot{\beta}} = \int dz \frac{\partial}{\partial v_{\dot{\alpha}}} \frac{\partial}{\partial v_{\dot{\beta}}} \varphi$$

Same argument as before tells us that


$\frac{\partial F_+^{\alpha\dot\beta}}{\partial x_{\gamma\dot\gamma}} \varepsilon_{\dot\beta\dot\gamma} = 0$
$\mathcal{O}(-2+n) \Longleftrightarrow$
tensor fields $T^{\dot\alpha_1 \ldots \dot\alpha_n}$
satisfying $\frac{\partial}{\partial x_{\gamma\dot\gamma}} T^{\dot\alpha_1 \ldots \dot\alpha_n} \varepsilon_{\dot\alpha_1\dot\gamma} = 0$
T is symmetric in indices

E.g $n=4$, $T^{\dot{\alpha}_1 \cdots \dot{\alpha}_4}$

is a 5 component tensor

It's the SD part of the Weyl curvature tensor

$$H^1(\mathbb{PT}, \mathcal{O}(2))$$

$$= \left\{ \begin{array}{l} 1^{st} \text{ order perturbations of } \mathbb{R}^4 \\ \text{where ASD Weyl tensor} = 0 \\ \text{and Einstein eq}^{ns}\text{ hold} \end{array} \right\} \Big/ \mathrm{Diff}.$$