Title: Lecture - Mathematical Physics, PHYS 777

Speakers: Kevin Costello

Collection/Series: Mathematical Physics (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Mathematical physics

Date: April 11, 2025 - 10:15 AM

URL: https://pirsa.org/25040051

Last time:

$$H^1(\mathbb{PT}, \mathcal{O}(n-2))$$

$$= \Big\{ \text{fields} \quad T^{\dot\alpha_1 \cdots \dot\alpha_n}$$

of spin $n/2$, satisfy

$$\frac{\partial}{\partial x_{\alpha\dot\alpha}} T^{\dot\alpha\dot\alpha_2 \ldots \dot\alpha_n} = 0$$

$n=4,$

T is symmetric

$n=1$: $\psi^{\dot\alpha}$, Dirac eq^n

$n=2$. $F_+^{\dot\alpha\dot\beta}$; field strength of a SD gauge field

$n=4$, $W^{\dot\alpha_1\ldots\dot\alpha_4}$

SD Weyl tensor

Weyl tensor in dim^n 4
has 10 components

$W_+^{\dot\alpha_1\ldots\dot\alpha_4}$, $W_-^{\alpha_1\ldots\alpha_4}$

$$g = \varepsilon\partial\partial(e^{ip\cdot x}) + \delta_{\mu\nu}$$

$$W_+^{\dot\alpha_1\ldots\dot\alpha_4} = \partial_{\dot\alpha_1\alpha}\partial^{\alpha}_{\dot\alpha_2}\partial_{\dot\alpha_3\beta}\partial^{\beta}_{\dot\alpha_3} e^{ip\cdot x}$$

eq^n

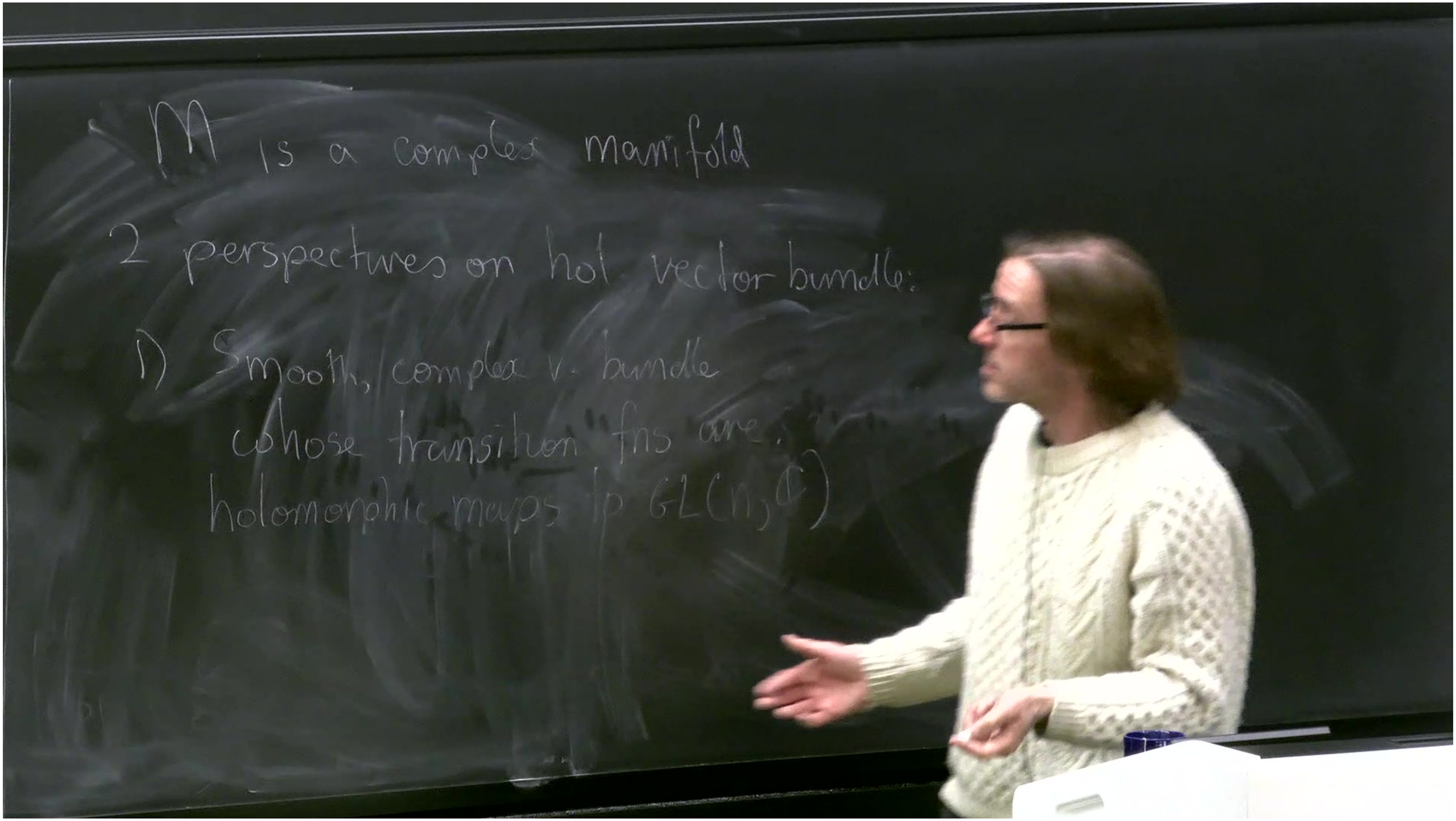
strength

uge field

Today:

State the theorem of Ward:

Holomorphic v. bundles E on $\mathbb{PT}$ of rank n $\Longleftrightarrow$ $U(n)$ gauge fields on $\mathbb{R}^4$, with $F_{-} = 0$.


M is a complex manifold
2 perspectures on hol. vector bundle:
1) Smooth, complex v. bundle
whose transition fns are
holomorphic maps to GL(n, ℂ)

2) A hol. v. bundle is a smooth complex vector bundle with a covariant $\bar{\partial}$ operator.

If $\Omega^{0,0}(X,E)$ = sections of v. bundle

a covariant $\bar{\partial}$ operator is

a map

$\bar{\partial}_E : \Omega^{0,0}(X,E) \to \Omega^{0,1}(X,E)$

(Recall a connection
is a map
$d_E : \Omega^{0,0}(X,E) \to \Omega^1(X,E)$)

$\bar{\partial}_E$ satisfies similar Leibniz rule to a
connection.

In a local patch with coords z_i, a covariant $\bar{\partial}$ operator gives covariant derivatives

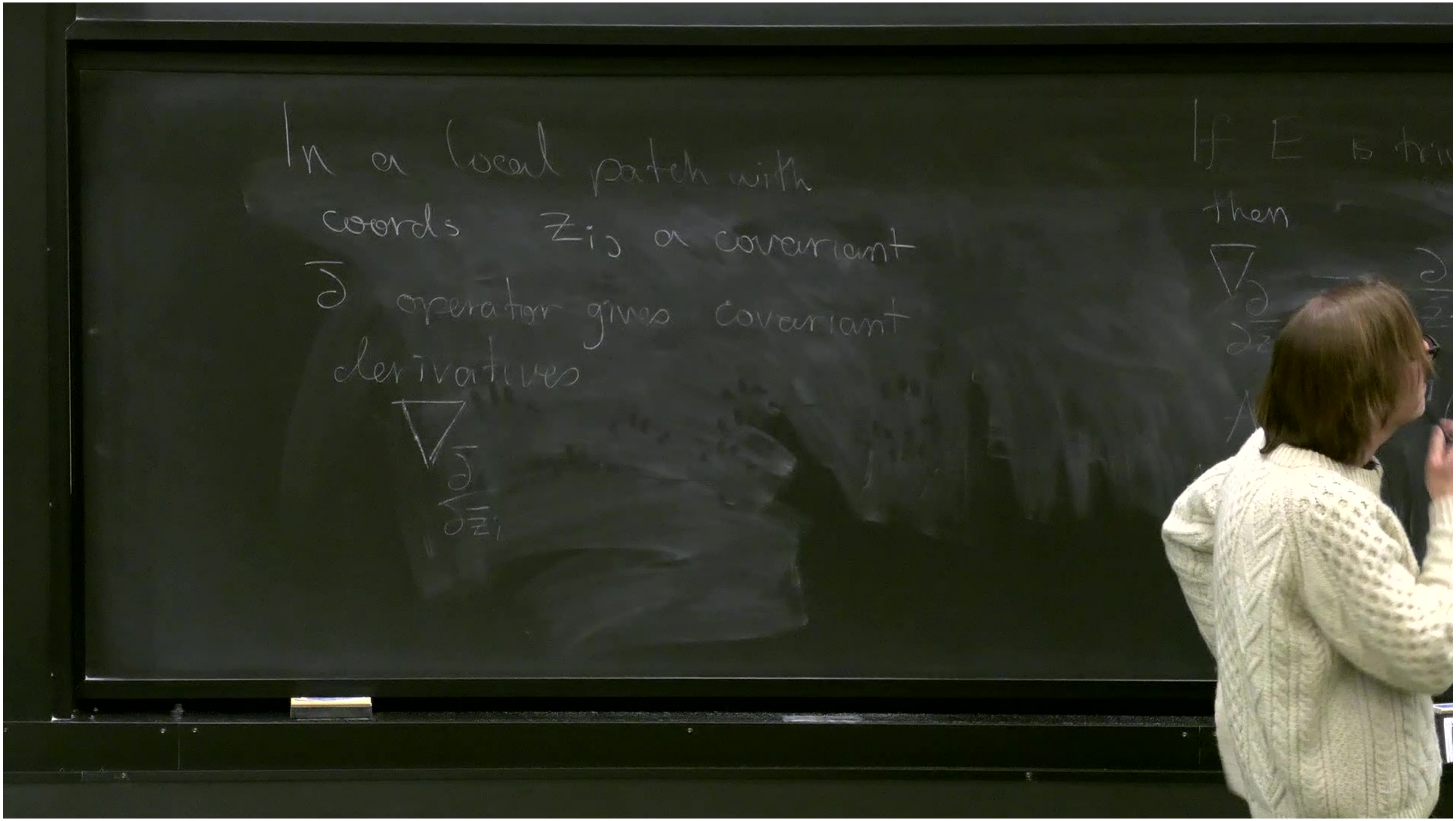
$$\nabla_{\frac{\partial}{\partial \bar{z}}}$$

...a local patch, is holomorphic if, locally

$$\nabla_{\frac{\partial}{\partial \bar{z}_i}} s = 0 \quad \forall i$$

$$\iff \frac{\partial s}{\partial \bar{z}_i} + A^i s = 0$$

used to characterize ...


In a local patch with
coords z_i, a covariant
$\bar\partial$ operator gives covariant
derivatives
$\nabla_{\frac{\partial}{\partial \bar z_i}}$
If E is
then

of a SD gauge field $W_+^{\dot\alpha_1 \dots \dot\alpha_4} = \partial_{\dot\alpha_1 \alpha} \partial_{\dot\alpha_2} \partial_{\dot\alpha_3 \beta} \partial_{\dot\alpha_3} e^{ip\cdot x}$

n a local patch,

is holomorphic if, locally

$$\nabla_{\frac{\partial}{\partial \bar z_i}} s = 0 \quad \forall i$$

$$\iff \frac{\partial s}{\partial \bar z_i} + A^i s = 0$$

Want there to be a basis of holomorphic sections locally

$(\mathbb{C}))$

is

For this to hold, we need

$$\left[\nabla_{\frac{\partial}{\partial \bar{z}_i}}, \nabla_{\frac{\partial}{\partial \bar{z}_j}} \right] = 0.$$

Because,

$$\left[\nabla_{\frac{\partial}{\partial \bar{z}_i}}, \nabla_{\frac{\partial}{\partial \bar{z}_j}} \right] s = 0$$

on a hol. section s.

$$\left[\nabla_{\frac{\partial}{\partial \bar{z}_i}},\right.$$

If this holds then locally

$\exists$ a gauge trans $X : \mathcal{U} \to GL(n, \mathbb{C})$

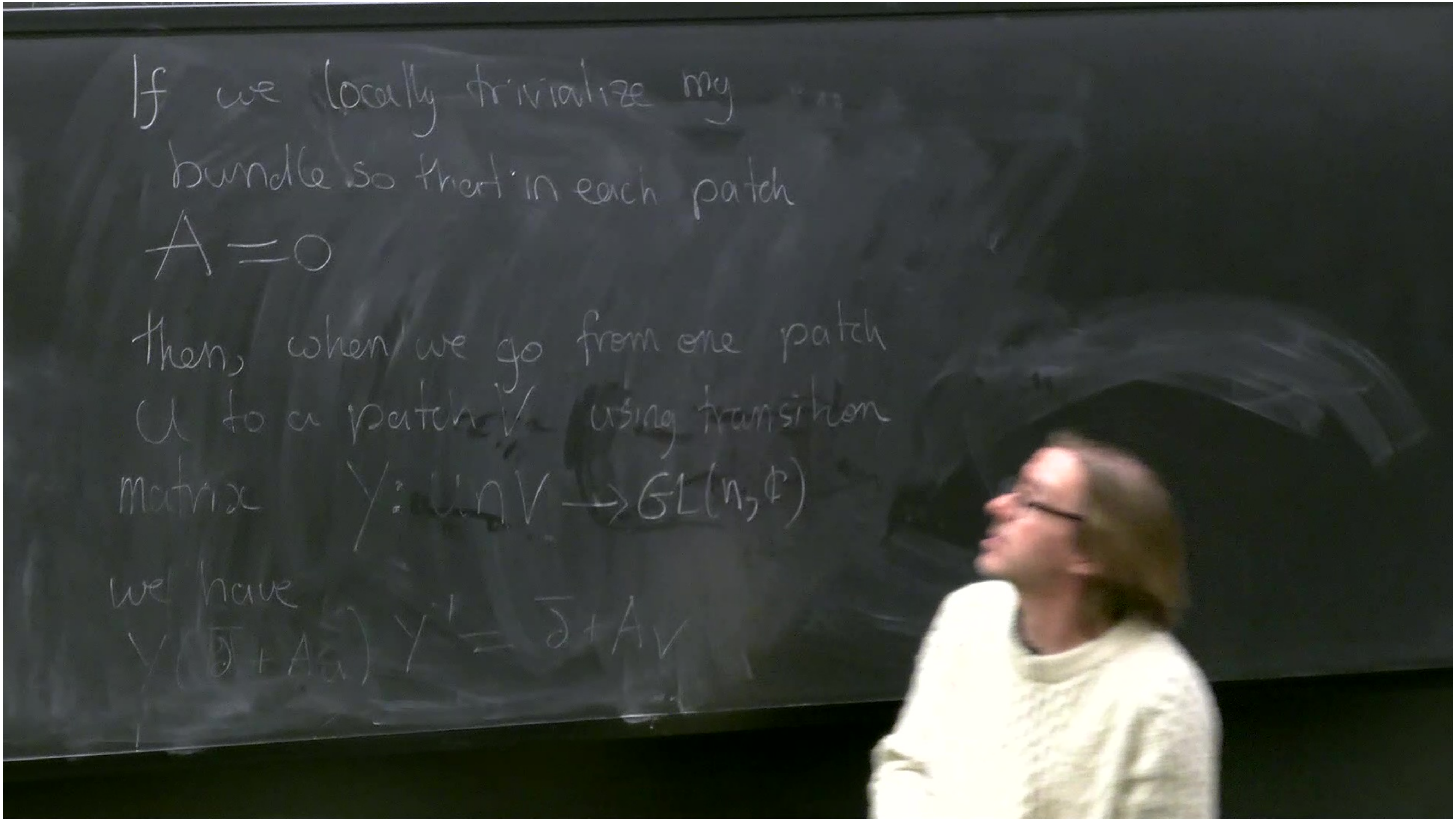
so that $X \bar{\partial} X^{-1} = \bar{\partial} + A$

i.e. A is locally gauge trivial

$$\left[\nabla_{\frac{\partial}{\partial \bar{z}_i}}, \nabla_{\frac{\partial}{\partial \bar{z}_j}}\right] d\bar{z}_i \, d\bar{z}_j$$

$$= F^{0,2}(A) \in \Omega^{0,2}(X, \mathfrak{gl}_n(\mathbb{C}))$$

A hol. bundle = smooth bundle + a $(0,1)$ connection A so that $F^{0,2}(A) = 0$


If we locally trivialize my
bundle so that in each patch
$A = 0$
then, when we go from one patch
U to a patch V using transition
matrix $\gamma : U \cap V \to GL(n, \mathbb{C})$
we have
$\gamma(\bar\partial + A_U)\gamma^{-1} = \bar\partial + A_V$

If $A_u = 0$, $A_v = 0$ then

$$y \bar{\partial} y^{-1} = 0$$

$$\Rightarrow y : u \cap v \to GL(n, \mathbb{C})$$

is a holomorphic transition matrix.

What are the possible structures of holomorphic line bundle on the trivial smooth bundle $\mathbb{C} \times X$?

Here, $A^i d\bar{z}_i$ is a $\underline{1} \times \underline{1}$ matrix

$A \in \Omega^{0,1}(X)$ i.e. it's a $(0,1)$ form.

If A, A' are related by a gauge transformation

$$g\,\bar\partial g^{-1} = A - A'$$

where $g: X \longrightarrow \mathbb{C}^\times$

is gauge transformation.

If we assume $\log g$ is single valued

(eg. if $H_1(X,\mathbb{Z}) = 0$) then

$$A - A' = \bar\partial \log g$$

So

$H^1(X,\mathcal{O}) = \left\{ \begin{array}{l} \text{holomorphic line} \\ \text{bundles that are} \\ \text{topologically trivial} \end{array} \right\}$ up to gauge trans

If $H^1(X,\mathbb{Z}) \neq 0$

correct statement involves

$H^1(X,\mathcal{O}) / H^1(X,\mathbb{Z})$

Precise statement of the Ward correspondence

E is trivial (

<u>Theorem</u>

There is a bijection between:

- Holomorphic vector bundles E on $\mathbb{PT}$, such that

E is trivial (as a hol. bundle)
on every $\mathbb{CP}^1 \subseteq \mathbb{PT}$

up to gauge equivalence;

— $GL(n, \mathbb{C})$ bundles on $\mathbb{R}^4$ + a connection
A, where $F^{\alpha\beta}(A) = 0$
up to gauge equivalence

Rank one case

$E|_{\mathbb{CP}^1}$ is trivial

$\Rightarrow c_1(E)|_{\mathbb{CP}^1} = 0$

$\Rightarrow c_1(E) = 0$ as

$H^2(\mathbb{PT}) \to H^2(\mathbb{CP}^1)$

is an isomorphism.

So, E is topologically trivial.

In rank one case, the data on $\mathbb{PT}$ is

$$H^1(\mathbb{PT}, \mathcal{O})$$

$$= \left\{ F_+^{\alpha\dot\beta},\ \frac{\partial}{\partial x_{\alpha\dot\alpha}} F_+^{\alpha\dot\beta} = 0 \right\}$$

$$= \left\{ \text{Complex Maxwell fields with } F_- = 0 \right\} / \text{Gauge}$$