Title: Lecture - Mathematical Physics, PHYS 777

Speakers: Kevin Costello

Collection/Series: Mathematical Physics (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Mathematical physics

Date: April 09, 2025 - 10:15 AM

URL: https://pirsa.org/25040050

Last time

$H^1(\mathbb{PT}, \mathcal{O}(-2)) = \left\{ \begin{array}{c} \text{complex valued harmonic} \\ \text{functions on } \mathbb{R}^4 \end{array} \right\}$

Two proofs

1) Slightly lazy and uses some black-boxed facts about Dolbeault cohomology

2) Better

rmonic
4

Proof 1

Recall any Dolbeault class is equivalent to one of the form

$$f(z, \bar{z}, v_{\dot{\alpha}})\, d\bar{z}$$

- No $d\bar{v}_{\dot{\alpha}}$
- $\dfrac{\partial f}{\partial \bar{v}_{\dot{\alpha}}} = 0$

Recall if $\mathbb{R}^4$ has coords

$$u_1 = x_1 + i x_2$$

$$u_{\dot 2} = x_3 + i x_4$$

$$V_{\dot\alpha} = u_{\dot\alpha} + \varepsilon_{\dot\alpha\beta}\,\bar u^{\dot\beta} z$$

Let

$$F(u_\alpha, \bar u^{\dot\alpha}) =$$

$$- \frac{\partial f}{\partial v_{\dot\alpha}} = 0$$

coords

$^{\dot\beta}\, z$

Let

$$F(u_\alpha, \bar{u}^{\dot\alpha}) = \int_{z \in \mathbb{CP}^1} f(z, \bar{z}, u_{\dot\alpha} + \bar{u}^{\dot\beta} \varepsilon_{\dot\alpha\dot\beta} z)\, dz\, d\bar{z}$$

I.e each point in $\mathbb{R}^4$ (or $\mathbb{C}^4$) determines a $\mathbb{CP}^1$ in $\mathbb{PT}$,

F is integral of the Dolbeault class over this $\mathbb{CP}^1$

Claim: F is harmonic

i.e.

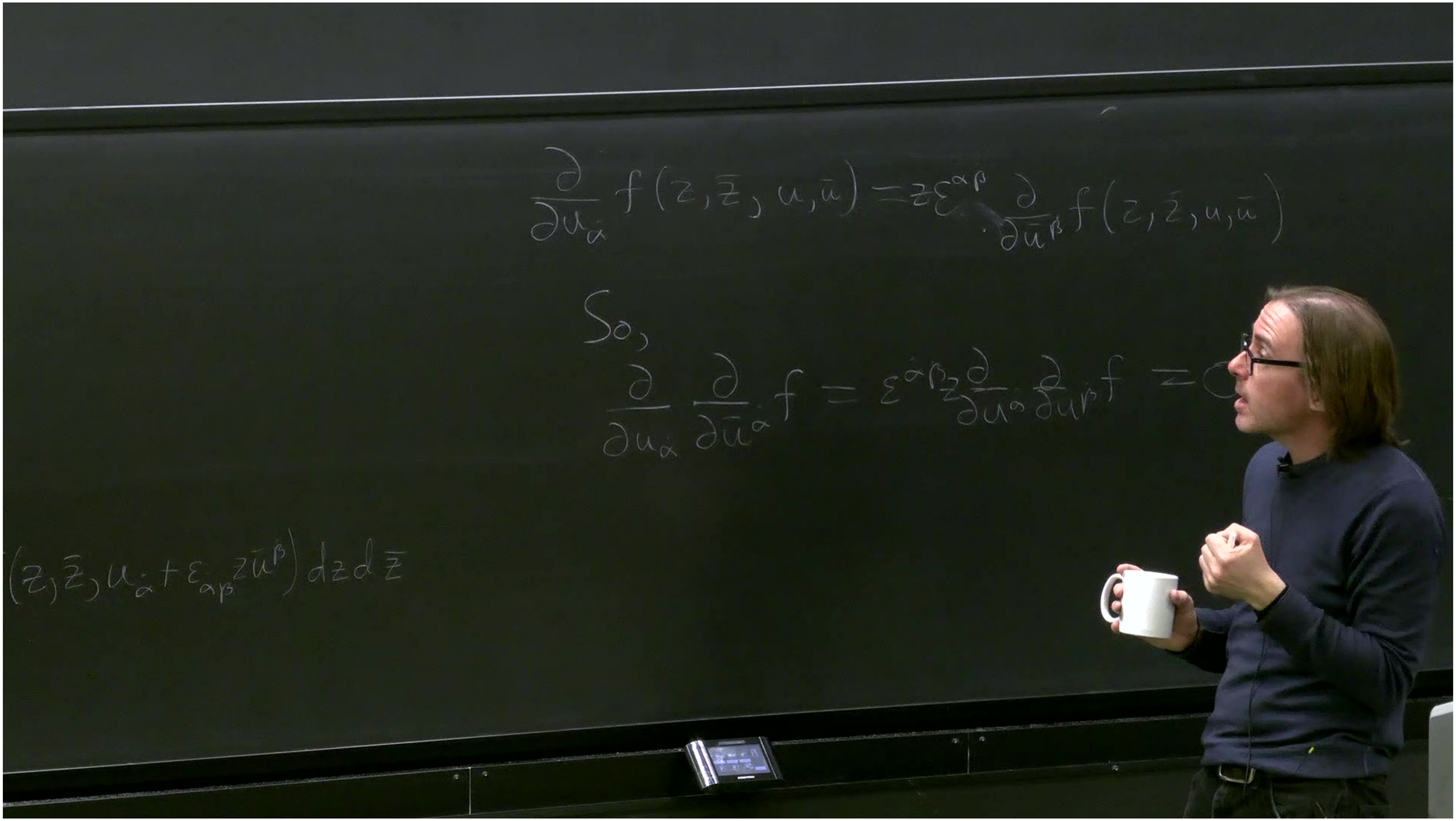
$$\sum \frac{\partial^2}{\partial x_i} F = 0$$

Equivalently

$$\frac{\partial}{\partial u_{\dot\alpha}} \frac{\partial}{\partial \bar u^{\dot\alpha}} F = 0$$

$$\frac{\partial}{\partial u_{\dot\alpha}} \frac{\partial}{\partial \bar u^{\dot\alpha}} F(u, \bar u) = \int_Z \frac{\partial}{\partial u_{\dot\alpha}} \frac{\partial}{\partial \bar u^{\dot\alpha}} f(z, \bar z, u_{\dot\alpha} + \varepsilon_{\alpha\beta} z \bar u^{\beta})\, dz\, d\bar z$$

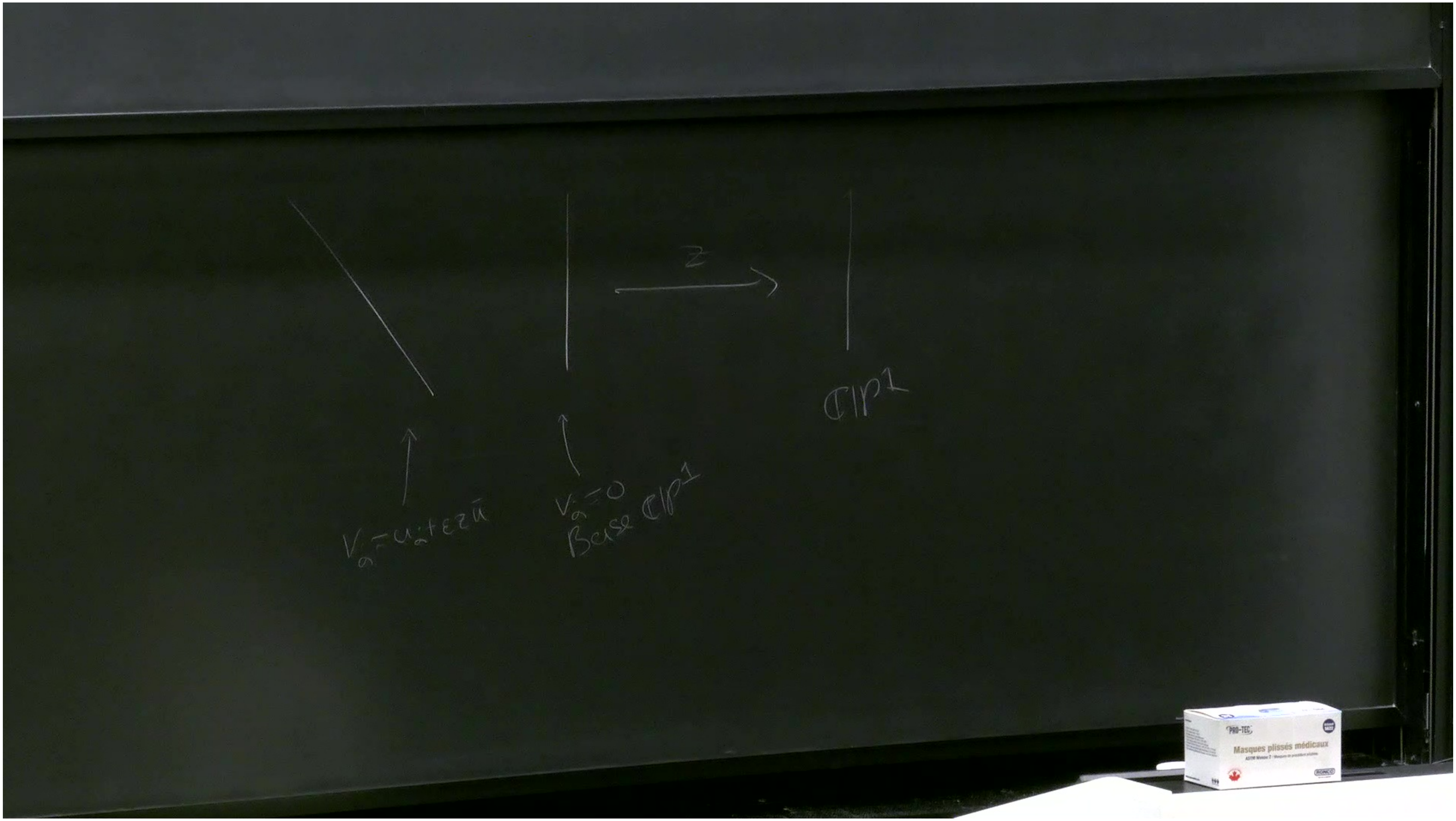
$$\frac{\partial}{\partial u_{\dot\alpha}}$$


$\frac{\partial}{\partial u_{\dot\alpha}} f(z,\bar z,u,\bar u) = z\varepsilon^{\alpha\beta}\frac{\partial}{\partial \bar u^\beta} f(z,\bar z,u,\bar u)$
So,
$\frac{\partial}{\partial u_{\dot\alpha}}\frac{\partial}{\partial \bar u^{\dot\alpha}} f = \varepsilon^{\dot\alpha\beta} z \frac{\partial}{\partial u^\alpha}\frac{\partial}{\partial u^\beta} f = 0$
$(z, \bar z, u_{\dot\alpha} + \varepsilon_{\alpha\beta} z \bar u^\beta)\, dz\, d\bar z$

KEY POINT

Any f that only depends on $u_1 + z\bar{u}_2$, $u_2 - z\bar{u}_1$ (fixed z) is already harmonic

As, f is holomorphic in complex structure z


z
CP1
Vα = uα + εzū
Vα = 0
Base CP1
PRO-TEC
Masques plissés médicaux

If $f \, dz \, d\bar{z} = \bar{\partial}\left(g(z, \bar{z}, v_a) \, dz \right)$

Then

$$\int_{\mathbb{CP}^1} \bar{\partial}(g \, dz) = 0$$

as it's a total derivative.

How big is $H^1(\mathbb{PT}, \mathcal{O}(-2))$?

$$d\bar{z}\, dz\, f(z, \bar{z}, v_{\dot\alpha}) = \sum v_{\dot\alpha_1} \cdots v_{\dot\alpha_n} f_{\dot\alpha_1 \cdots \dot\alpha_n}(z, \bar{z})\, dz\, d\bar{z}$$

$v_{\dot\alpha}$ transform as $(dz)^{1/2}$

$f_{\dot\alpha_1 \cdots \dot\alpha_n}(z, \bar{z})\, dz\, d\bar{z}$ can think of it as

$f^{\dot{\alpha}_1 \cdots \dot{\alpha}_n}(z, \bar{z})\, d\bar{z}\, dz^{1+n/2}$

$dz\, d\bar{z}$

i.e. this is in $H^1(\mathbb{CP}^1, \mathcal{O}(-2-n))$

$=$ the representation of $SL_2(\mathbb{C})$

of spin $n/2$ and dimension $n+1$

$$f^{\dot\alpha_1 \ldots \dot\alpha_n}(z, \bar z)\, d\bar z\, dz^{1+n/2}$$

i.e. this is in $H^1(\mathbb{CP}^1, \mathcal{O}(-2-n))$

= the representation of $SL_2(\mathbb{C})$

of spin $n/2$ and dimension $n+1$.

$Spin(4, \mathbb{C}) = SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$

Under the $SL_2(\mathbb{C})$ rotates $\dot\alpha$, $f^{\dot\alpha_1 \ldots \dot\alpha_n}$ is symmetric in $\dot\alpha$ indices

If $[j]_L$ denotes the rep

of $SL_2(\mathbb{C})$ rotates $\dot{\alpha}_j$ of spin j

$[j]_R$ same for other $SL_2(\mathbb{C})$,

we see

$$f^{\dot{\alpha}_1 \dots \dot{\alpha}_n} \in [n/2]$$

Conclusion: $H^1(\mathbb{PT}, \mathcal{O}(-2)) = \sum_{n=0}^{\infty} \left[\frac{n}{2}\right]_L \otimes \left[\frac{n}{2}\right]_R$

Piece in $\left[\frac{n}{2}\right]_L \otimes \left[\frac{n}{2}\right]_R$ is of dimension $-n$

(i.e. a polynomial order exactly n on $\mathbb{R}^4$)

To complete the proof we need to show

Harmonic fns on $\mathbb{R}^4 = \sum \left[\frac{n}{2}\right]_L \otimes \left[\frac{n}{2}\right]_R$

Better proof of the Theorem.

Claim is If

If we don't use the fact we black-boxed, we see a general element of $\Omega^{0,1}(\mathbb{P}\Pi, \mathcal{O}(-2))$ looks like

$$f(z,\bar{z}, v_{\dot\alpha}, \bar{v}^{\dot\alpha})\, d\bar{z} + g_{\dot\alpha}(z,\bar{z}, v, \bar{v})\, d\bar{v}^{\dot\alpha}$$

We build F by $\int$ over a $\mathbb{CP}^1$

If $f\, dz\, d\bar{z} = \bar{\partial}\left(g(z,\bar{z},v_{\underline{a}})\, dz\right)$

Then

$$\int_{\mathbb{CP}^1} \bar{\partial}(g\, dz) = 0$$

as it's a total derivative.

Because f, g_α are no longer holomorphic in v, we see

$$\frac{\partial}{\partial u_{\dot\alpha}} = \frac{\partial}{\partial v_{\dot\alpha}} + \text{something involving } \frac{\partial}{\partial \bar v}$$

$$\frac{\partial}{\partial \bar u^{\dot\alpha}} = z \frac{\partial}{\partial v_\beta} \varepsilon_{\dot\alpha\beta} + \text{something involving } \frac{\partial}{\partial \bar v}$$

Lemma

On any complex manifold

if α is an element of $\Omega^{0,1}$

and $\bar{\partial}\alpha = 0$,

and if $V = \sum h_i \frac{\partial}{\partial z_i}$ is a v. field in $\frac{\partial}{\partial z}$ directions

then $\mathcal{L}_V \alpha = \bar{\partial}(V \lrcorner \alpha)$

I.e. if $\alpha = \sum \alpha^i \, d\bar{z}_i$

$V = \sum h_i \frac{\partial}{\partial \bar{z}_i}$

$V \vee \alpha = \sum h_i \alpha^i$

$\mathcal{L}_V \alpha = \sum \Big(h_j \frac{\partial \alpha^i}{\partial \bar{z}_j}$

Proof of Lemma

$\bar{\partial} \alpha = 0$ so

$$\frac{\partial \alpha^i}{\partial \bar{z}_j} = \frac{\partial \alpha^j}{\partial \bar{z}_i}$$

$$= \sum \alpha^i \, d\bar{z}_i$$

$$h_i \frac{\partial}{\partial \bar{z}_i}$$

$$\sum h_i \alpha^i$$

emma

$$\alpha^j$$

$$\mathcal{L}_V \alpha = \sum \left(h_j \frac{\partial \alpha^i}{\partial \bar{z}_j} \right) d\bar{z}_i + \sum \alpha^i \frac{\partial}{\partial \bar{z}_j} h_i \, d\bar{z}_j$$

$$= \sum h_j \frac{\partial \alpha^j}{\partial \bar{z}_i} d\bar{z}_i + \alpha^j \frac{\partial}{\partial \bar{z}_i} h_j \, d\bar{z}_i$$

$$= \bar{\partial} (h_j \alpha^j)$$

$\left.\frac{\alpha^i}{\bar z_j}\right) d\bar z_i + \sum \alpha^i \frac{\partial}{\partial \bar z_j} h_i \, d\bar z_j$

$\frac{\alpha^j}{\bar z_1} d\bar z_i + \alpha^j \frac{\partial}{\partial \bar z_j} h_j \, d\bar z_i$

If $\alpha \in \Omega^{0,1}(\mathbb{P}\mathbb{T}, \mathcal{O}(-2))$ $\quad \bar\partial \alpha = 0$

$F(u, \bar u) = \int \alpha$

$V_\alpha = u_\alpha + z \bar u^\beta \varepsilon_{\alpha\beta}$

$\frac{\partial F}{\partial u_\alpha} = \int \left(\mathcal{L}_{\frac{\partial}{\partial V_\alpha}} + \mathcal{L}_{\frac{\partial}{\partial \bar V}} \right) \alpha$

doesn't matter as it is $\bar\partial$ exact.

$\partial z_j \qquad \partial z_i$

So.

$$\frac{\partial F}{\partial u_{\dot\alpha}} = \int_{\mathbb{CP}^1} \mathcal{L}_{\frac{\partial}{\partial v_\alpha}} \alpha$$

$$\varepsilon_{\alpha\beta} \frac{\partial F}{\partial \bar{u}^{\dot\beta}} = \int \mathcal{L}_{z\frac{\partial}{\partial v_\alpha}} \alpha$$

$$\frac{\partial}{\partial \bar{u}^{\dot\alpha}} \frac{\partial F}{\partial u_\alpha} = \int \varepsilon_{\dot\alpha\dot\beta} \mathcal{L}_{z\partial_{v_\alpha}} \mathcal{L}_{\partial_{v_\beta}} \alpha$$

$$= 0 \quad \text{as} \quad \mathcal{L}_{z\partial_{v_{\dot\alpha}}} \alpha = z \mathcal{L}_{\partial_{v_\alpha}} \alpha$$

$\mathcal{L}_{\partial_{v_\beta}} \alpha$

$\mathcal{L}_{z\partial_{v_a}} \alpha = z \mathcal{L}_{\partial_{v_a}} \alpha$

Generalizations

Introduce indices $\dot{\alpha}$, as before

α, for the fun. rep of $SL(2,\mathbb{C})_R$

If S_+ = fun. rep of $SL(2,\mathbb{C})_L$

S_- " " $SL(2,\mathbb{C})_R$

S_+, S_- are the two spin reps. of $Spin(4)$

The vector rep is $S_+ \otimes S_-$

A spinor is $\tilde{\psi}_{\dot\alpha}$ or ψ_α

The Dirac eq^n for $\tilde{\psi}_{\dot\alpha}$ is

$$\frac{\partial}{\partial x_{\alpha\dot\alpha}} \tilde{\psi}_{\dot\alpha} = 0$$

For ψ_α, it is

$$\frac{\partial}{\partial x_{\alpha\dot\alpha}} \psi_\alpha = 0$$

Lemm

On a

if α

and ∂

and if V

then $\mathcal{L}_V \alpha$