Title: Lecture - Mathematical Physics, PHYS 777

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On $\mathbb{CP}^1$:

2 patches coords z, w where $z = \frac{1}{w}$

A section of $\mathcal{O}(n)$ is a field $f(z,\bar{z})$ which picks up a factor of z^n as we go between patches

$z \neq$ ...

$f(z, \bar{z}$ ...

$z \neq \infty$ $\qquad$ $z \neq 0$

$f(z, \bar{z})$ $\qquad$ $g(w, \bar{w})$

and when $z \neq 0$, and $z \neq \infty$

$$f(z, \bar{z}) = z^{-n} g(1/z, 1/\bar{z})$$

If $w = 1/z$, $dz = -\frac{1}{w^2}\,dw$

So, a tensor that looks like dz locally has a Jacobian factor that makes it a section of $\mathcal{O}(-2)$

Sections of $\mathcal{O}(n)$

$= $ tensors that look like $(dz)^{-n/2}$

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A holomorphic section of $\mathcal{O}(n)$

is an expression, in local coords is

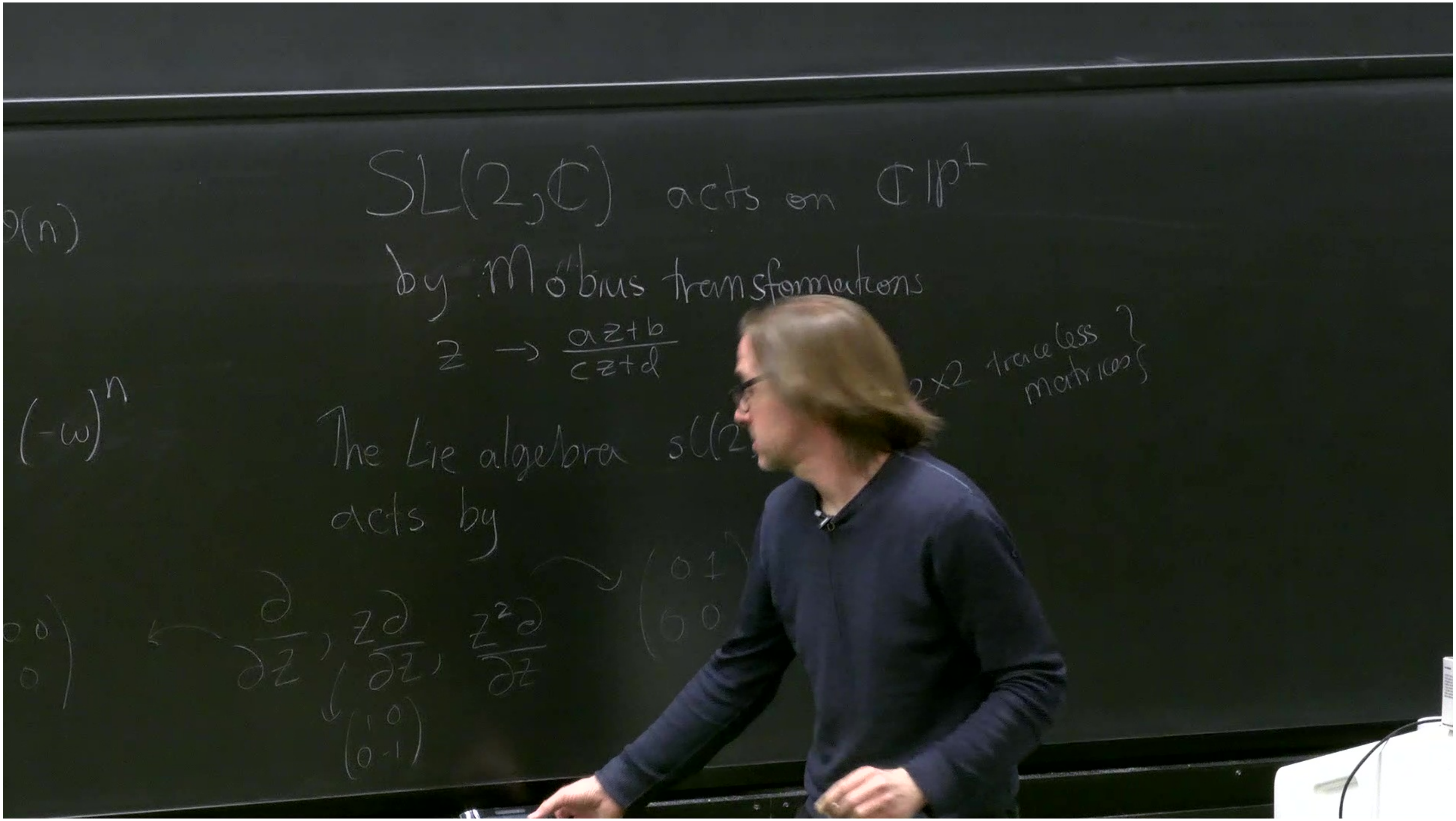
$$(dz)^{-n/2} f(z,\bar{z}) \text{ where } \frac{\partial f}{\partial \bar{z}} = 0$$

There are $n+1$ sections of $\mathcal{O}(n)$

$z^k (dz)^{-n/2}$ $\quad 0 \leq k \leq n$

On other patch, $\omega^{-k} \cdot (d\omega)^{-n/2} (-\omega)^n$

If $k > n$, pole at ∞


SL(2,ℂ) acts on ℂP¹
by Möbius transformations
z → (az+b)/(cz+d)
2×2 traceless matrices
The Lie algebra sl(2
acts by
∂/∂z, z∂/∂z, z²∂/∂z

$SL(2,\mathbb{C}) = Spin(3,\mathbb{C})$

Sections of $\mathcal{O}(n)$ transform in rep. of Spin $n/2$

$n=1$: basis is $(dz)^{-1/2}$, $z(dz)^{-1/2}$

$2z\frac{\partial}{\partial z}$ has eigenvalues $-1, 1$

$\mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\qquad \frac{\partial}{\partial z}$ sends $z(dz)^{-1/2} \to (dz)^{-1/2}$

$\Omega^{0,1}(\mathbb{CP}^1, \mathcal{O}(n))$

is the following

In patch $z \neq \infty$ $\qquad$ $z \neq 0$

$f(z,\bar{z})\, d\bar{z}$ $\qquad$ $g(z,\bar{z})\, d\bar{w}$

On overlap

$$f(z,\bar{z})\, d\bar{z} = z^{-n} g(1/z, 1/\bar{z})\, d(1/\bar{z})$$

Or

$\Omega^{0,1}(\mathbb{CP}^1, \mathcal{O}(n))$

= tensors that locally look like

$$f(z,\bar{z})(dz)^{-n/2}\, d\bar{z}$$

$\Omega^{0,0}(\mathbb{CP}^1, \mathcal{O}(n))$ = locally tensors like $(dz)^{-n/2} f(z,\bar{z})$

There's a map

$$\bar{\partial} : \Omega^{0,0}(\mathbb{CP}^1, \mathcal{O}(n)) \to \Omega^{0,1}(\mathbb{CP}^1, \mathcal{O}(n))$$

Locally

$$f(z,\bar{z})(dz)^{-n/2} \longrightarrow d\bar{z}\,\frac{\partial f}{\partial \bar{z}}\,(dz)^{-n/2}$$

This is independent of choice

of hol. coordinate z, because Jacobian factor for dz is holomorphic

Jacobian factors for $d\bar{z}$ and $\frac{\partial}{\partial \bar{z}}$ cancel

$$H^0(\mathbb{CP}^1, \mathcal{O}(n)) = \operatorname{Ker} \bar{\partial} = \text{holomorphic sections.}$$

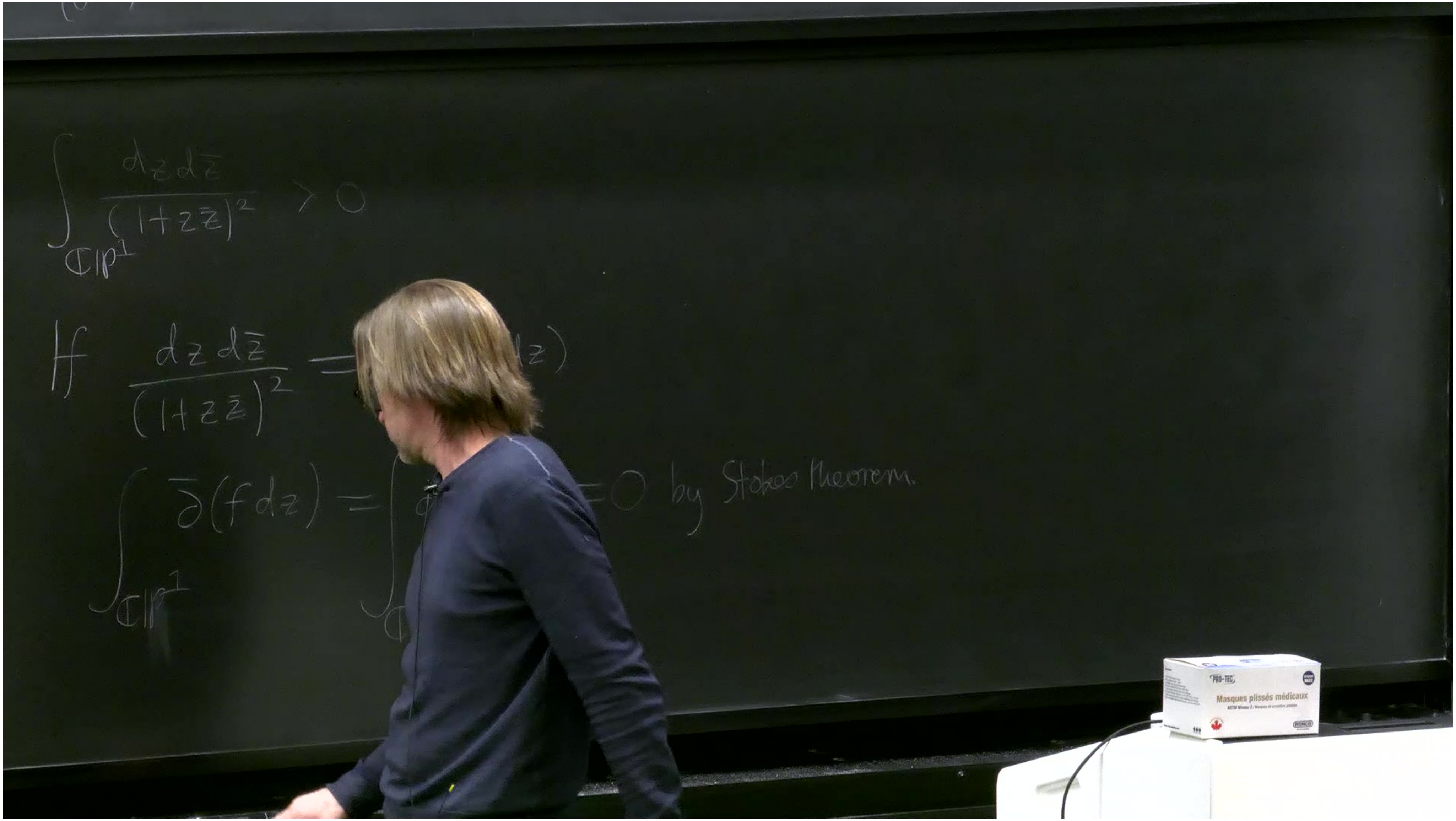
$$H^1(\mathbb{CP}^1, \mathcal{O}(n)) = \Omega^{0,1}(\mathbb{CP}^1, \mathcal{O}(n)) / \operatorname{Im} \bar{\partial}$$

Example:

$$\frac{dz\, d\bar{z}}{(1+z\bar{z})^2} \in \Omega^{0,1}(\mathbb{C}\mathbb{P}^1; \mathcal{O}(-2))$$

At $z = \infty$,

$$\frac{dz\, d\bar{z}}{(1+z\bar{z})^2} \longrightarrow \frac{d\omega\, d\bar{\omega}}{(\omega\bar{\omega})^2 (1+\omega^{-1}\bar{\omega}^{-1})^2} = \frac{d\omega\, d\bar{\omega}}{(1+\omega\bar{\omega})^2}$$


∫_CP¹ dz dz̄ / (1+zz̄)² > 0
If dz dz̄ / (1+zz̄)² = … dz)
∫_CP¹ ∂̄(f dz) = ∫ … = 0 by Stokes theorem.
PRO-TEC
Masques plissés médicaux

Fact:

$\frac{dz\,d\bar{z}}{(1+z\bar{z})^2}$ gives a basis of

$H^1(\mathbb{CP}^1, \mathcal{O}(-2))$

More General Fact

Fact:

$\dfrac{dz\,d\bar{z}}{(1+z\bar{z})^2}$ gives a basis of

$$H^1(\mathbb{CP}^1, \mathcal{O}(-2)) = \mathbb{C}$$

More General Fact

1) $H^1(\mathbb{CP}^1, \mathcal{O}(n)) = 0$ for $n \geq -1$

2) n

2) There's a natural way to pair an element of

$H^1(\mathbb{C}\mathbb{P}^1, \mathcal{O}(-2-n))$

with an element of

$H^0(\mathbb{C}\mathbb{P}^1, \mathcal{O}(n))$

2) There's a natural way to pair an element of

$H^1(\mathbb{CP}^1, \mathcal{O}(-2-n))$

with an element of

$H^0(\mathbb{CP}^1, \mathcal{O}(n))$

by

$$f(z)(dz)^{-n/2} \otimes g(z,\bar{z})\, d\bar{z}\, dz^{n/2+1}$$

$$\longrightarrow \int f\, g\, dz\, d\bar{z}$$

This gives an iso. between

H^1

$$H^1(\mathbb{CP}^1, \mathcal{O}(-2-n)) \cong H^0(\mathbb{CP}^1, \mathcal{O}(n))^*$$

$$\cong \text{spin } n/2 \text{ rep. of } SL(2,\mathbb{C}).$$

TWISTOR SPACE

$\mathbb{PT}$ is the space

$$\mathcal{O}(1)^2 \longrightarrow \mathbb{CP}^1$$

It has 2 coord. patches

On one, coords are $v_{\dot{1}}, v_{\dot{2}}, z$

On

On the other, coords are

$\tilde{v}_1, \tilde{v}_2, \tilde{z}$

Where the overlap when $z \neq 0$ and $\tilde{z} \neq 0$, then

$z = 1/\tilde{z}$

$v_{\dot\alpha} = \tilde{v}_{\dot\alpha}/\tilde{z}$

The $v_{\dot\alpha}$ coord. has a pole at $z = \infty$ (or equivalently $\tilde{z} = 0$)

$\mathbb{PT}$ is isomorphic

The Penrose transform

relates things on $\mathbb{PT}$ $\rightarrow$ things on $\mathbb{R}^4$

Give $\mathbb{R}^4$ coordinates x_1, x_2, x_3, x_4

$$u_1 = x_1 + i x_2$$

$$u_2 = x_3 + i x_4$$

$$\int_{\mathbb{CP}^1} \frac{dz\, d\bar{z}}{(1+z\bar{z})^2} > 0$$

If $\dfrac{dz\, d\bar{z}}{(1+z\bar{z})^2} = \bar{\partial}(f\, dz)$

$$\int_{\mathbb{CP}^1} \bar{\partial}(f\, dz) = \int_{\mathbb{CP}^1} d(f\, dz) = 0 \text{ by Stokes theorem.}$$

$$\int_{\mathbb{CP}^1} \frac{dz\, d\bar{z}}{(1+z\bar{z})^2} > 0$$

If $\dfrac{dz\, d\bar{z}}{(1+z\bar{z})^2} = \bar{\partial}(f\, dz)$

$$\int_{\mathbb{CP}^1} \bar{\partial}(f\, dz) = \int_{\mathbb{CP}^1} d(f\, dz) = 0$$ by Stokes theorem.

$$u_1 = \frac{1}{(1+z\bar{z})}\left(v_1 - z\bar{v}_2\right)$$

$$u_2 = \frac{1}{1+z\bar{z}}\left(v_2 + z v_1\right)$$

This is an iso. as real manifolds

$\mathbb{R}^4$ has more than one complex str

(more than one $J : \mathbb{R}^4 \to \mathbb{R}^4$ with $J^2 = -1$, J compatible w. metric)

Z = a $\mathbb{CP}^1$ of complex structures on $\mathbb{R}^4$.

Any 2 J's are related

$J_1 = X J_2 X^{-1}$, $X \in SO(4)$

If $X \in U(2)$

then $X J_2 X^{-1} = J_2$

So

$$\{\text{complex structures}\} = \frac{SO(4)}{U(2)}$$

$$= S^2 = \mathbb{CP}^1$$

w. coord z.

$$\begin{cases} v_1 = u_1 + z\bar{u}_2 \\ v_2 = u_2 - z\bar{u}_1 \end{cases}$$

holomorphic words in z complex structure