Title: Lecture - Mathematical Physics, PHYS 777

Speakers: Kevin Costello

Collection/Series: Mathematical Physics (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Mathematical physics

Date: April 08, 2025 - 10:15 AM

URL: https://pirsa.org/25040048

Twistor space:

Coord $v_{\dot\alpha}, z$

Space-time

Complex coords $u_{\dot\alpha}$

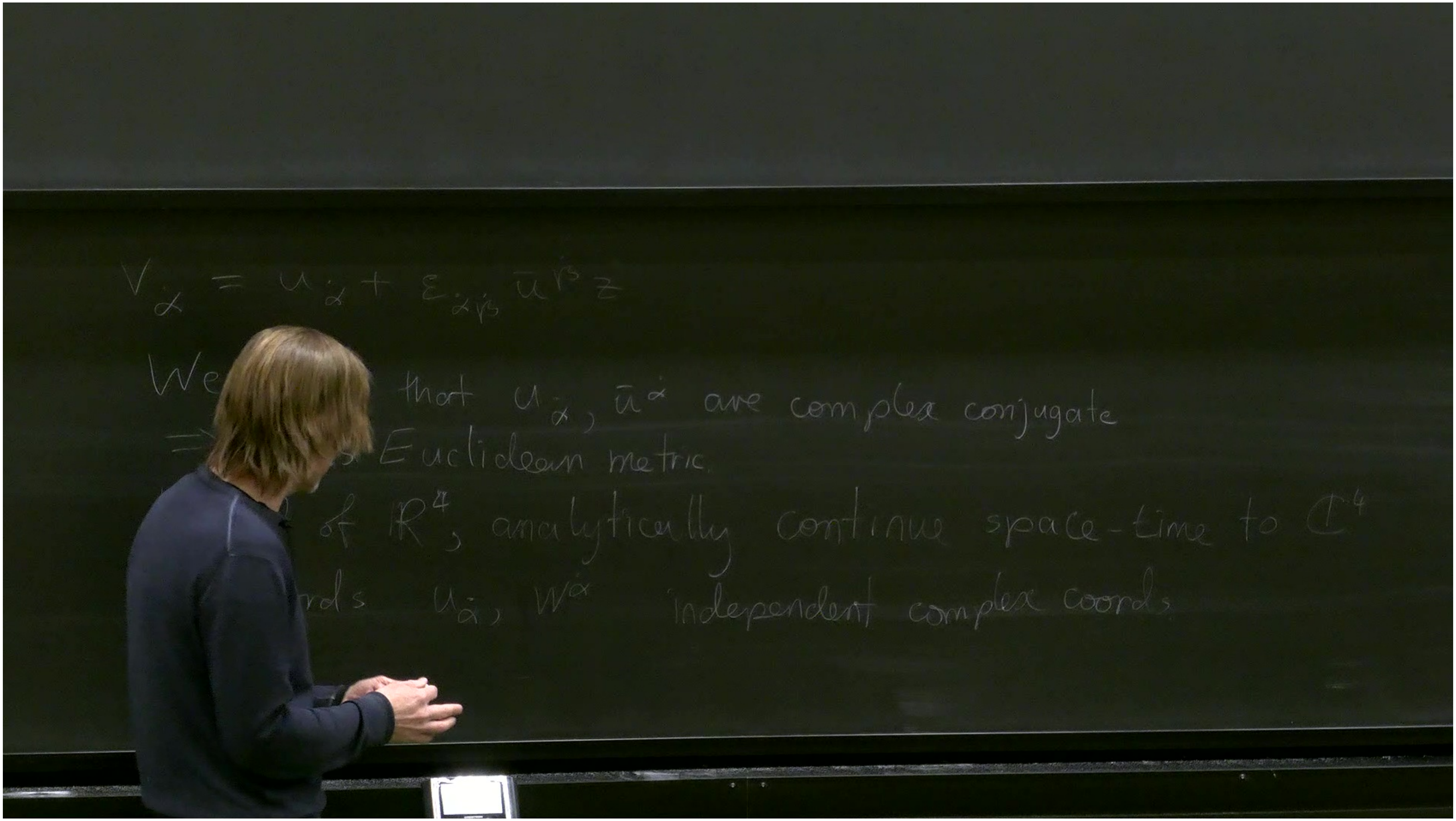
Complex conjugates $\bar{u}^{\dot\alpha}$

$g = du_{\dot\alpha}\, d\bar{u}^{\dot\alpha}$

$v_{\dot\alpha} = u_{\dot\alpha} + \varepsilon_{\dot\alpha\dot\beta}\,\bar{u}^{\dot\beta} z$

We asked that $u_{\dot\alpha}, \bar{u}^{\dot\alpha}$ are complex c

$\Rightarrow g$ is Euclidean metric


$V_{\dot\alpha} = u_{\dot\alpha} + \varepsilon_{\dot\alpha\dot\beta}\,\bar{u}^{\dot\beta} z$
We
that $u_{\dot\alpha}$, $\bar{u}^{\dot\alpha}$ are complex conjugate
Euclidean metric
of $\mathbb{R}^4$, analytically continue space-time to $\mathbb{C}^4$
$u_{\dot\alpha}$, $w^{\dot\alpha}$ independent complex coords

Complex conjugates $\bar{u}^{\dot\alpha}$ with coords $u_{\dot\alpha}$, w^{α}

$g = du_{\dot\alpha}\, d\bar{u}^{\dot\alpha}$

Euclidean:

$w^{\alpha} = \bar{u}_{\alpha}$

Lorentzian

u_1, w^1 are real

$u_{\dot 2}, w^2$ are complex conjugate

$g = du_{\dot\alpha}\, dw^{\dot\alpha}$

Twistor correspondence:

$V_{\dot\alpha} = u_{\dot\alpha} + \varepsilon_{\dot\alpha\beta} z\, w^{\beta}$

Given a point $(u, w) \in$

we get a copy of

with coords $u_{\dot\alpha}$, $w^{\dot\alpha}$ independent complex coords

Twistor correspondence:

$$v_{\dot\alpha} = u_{\dot\alpha} + \varepsilon_{\dot\alpha\dot\beta} z w^{\beta}$$

Given a point $(u,w) \in \mathbb{C}^4$ we get a copy of $\mathbb{CP}^1$ in $\mathbb{PT}$ given by the two equations

Express v in terms of z

$$\mathbb{CP}^1 \longrightarrow \mathbb{PT}$$

$$z \longmapsto (z,\ u_1 + z$$

$\dot\alpha$, $w^{\dot\alpha}$ independent complex coords

ondence:

$\varepsilon_{\dot\alpha\dot\beta} z w^{\dot\beta}$

oint $(u,w) \in \mathbb{C}^4$

opy of $\mathbb{CP}^1$

by the two equations

in terms of z

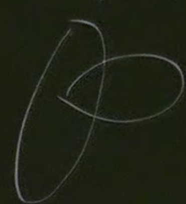
$z \longmapsto (z, u_{\dot 1} + z w^{\dot 2}, u_{\dot 2} - z w^{\dot 1})$

KEY POINT

If (u,w) and $(\tilde u, \tilde w)$

the corresponding $\mathbb{CP}^1$'s intersect

$\Longleftrightarrow$ $(u-\tilde u, w - \tilde w)$ is null.



Let's check this with $(\tilde{u}, \tilde{w}) = 0$

Want to show

curve

$$V_{\dot\alpha} = u_{\dot\alpha} + z\,\varepsilon_{\dot\alpha\beta} w^{\beta}$$

intersects the locus $V_{\dot\alpha} = 0$

$$\iff \quad u_{\dot\alpha} w^{\dot\alpha} = 0$$

$$u_{\dot\alpha} + z\,\varepsilon_{\dot\alpha\beta} w^{\beta} = 0$$

has a solution if and only if

$$\det \begin{pmatrix} u_1 & w^2 \\ u_2 & -w^1 \end{pmatrix} = 0$$

$$= -u_1 w^1 - u_2 w^2$$

Twistor geometry encodes
light cones i.e metric
up to a conformal factor

Twistor geometry encodes
light cones i.e metric
up to a conformal factor

Lorentz group, after complexifying
is $SO(4,\mathbb{C}) = \left\{ \begin{array}{c} 4\times 4 \text{ complex matrices } M \\ MM^T = \mathbb{1} \end{array} \right\}$

$\mathrm{Spin}(4,\mathbb{C}) = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$

Lie algebra $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$

On $\mathbb{C}^4$, coords $u_{\dot\alpha}, w^{\dot\alpha}$

One copy of $sl(2,\mathbb{C})$ rotates the $\dot\alpha$ index

Other copy: exchanges u, w

Complex conjugates $\bar{u}^{\dot\alpha}$ with coords $u_{\dot\alpha}$, $w^{\dot\alpha}$ ind

$g = du_\alpha \, d\bar{u}^{\dot\alpha}$

A vector in $\mathbb{C}^4 =$ 2x2 matrix $\begin{pmatrix} u_{\dot 1} & w_{\dot 2} \\ u_{\dot 2} & -w_{\dot 1} \end{pmatrix}$

One copy of $SL(2,\mathbb{C})$ acts by mult. on left

Other, by mult. on the right

Both preserve determinant, as if $M \in SL(2,\mathbb{C})$

$\det M = 1$

$w^{\dot\alpha}$ independent complex coords

$$v_{\dot\alpha} = u_{\dot\alpha} + \varepsilon_{\dot\alpha\dot\beta}\, z\, w^{\dot\beta}$$

One copy of sl_2 rotates $v_{\dot\alpha}$
by vector fields

$$v_{\dot 1}\frac{\partial}{\partial v_{\dot 2}}, \quad v_{\dot 2}\frac{\partial}{\partial v_{\dot 1}}, \quad v_{\dot 1}\frac{\partial}{\partial v_{\dot 1}} - v_{\dot 2}\frac{\partial}{\partial v_{\dot 2}}$$

Other $sl(2,\mathbb{C})$ acts by Möbius transformations on z

$V_{\dot\alpha}$ transforms as $(dz)^{1/2}$

Taking this into account, other sl_2 is

$$\frac{\partial}{\partial z}, \quad z\frac{\partial}{\partial z} + \frac{1}{2} V_{\dot\alpha}\frac{\partial}{\partial V_{\dot\alpha}}, \quad z^2\frac{\partial}{\partial z} + V_{\dot\alpha}\frac{\partial}{\partial V_{\dot\alpha}}$$

Other $sl(2,\mathbb{C})$ acts by

Möbius transformations on z

$V_{\dot\alpha}$ transforms as $(dz)^{1/2}$

Taking this into account, other sl_2 is

$$\frac{\partial}{\partial z}, \quad z\frac{\partial}{\partial z} + \frac{1}{2} V_{\dot\alpha}\frac{\partial}{\partial V_{\dot\alpha}}, \quad z^2\frac{\partial}{\partial z} + V_{\dot\alpha}\frac{\partial}{\partial V_{\dot\alpha}}$$

$\frac{\partial}{\partial u}$

4

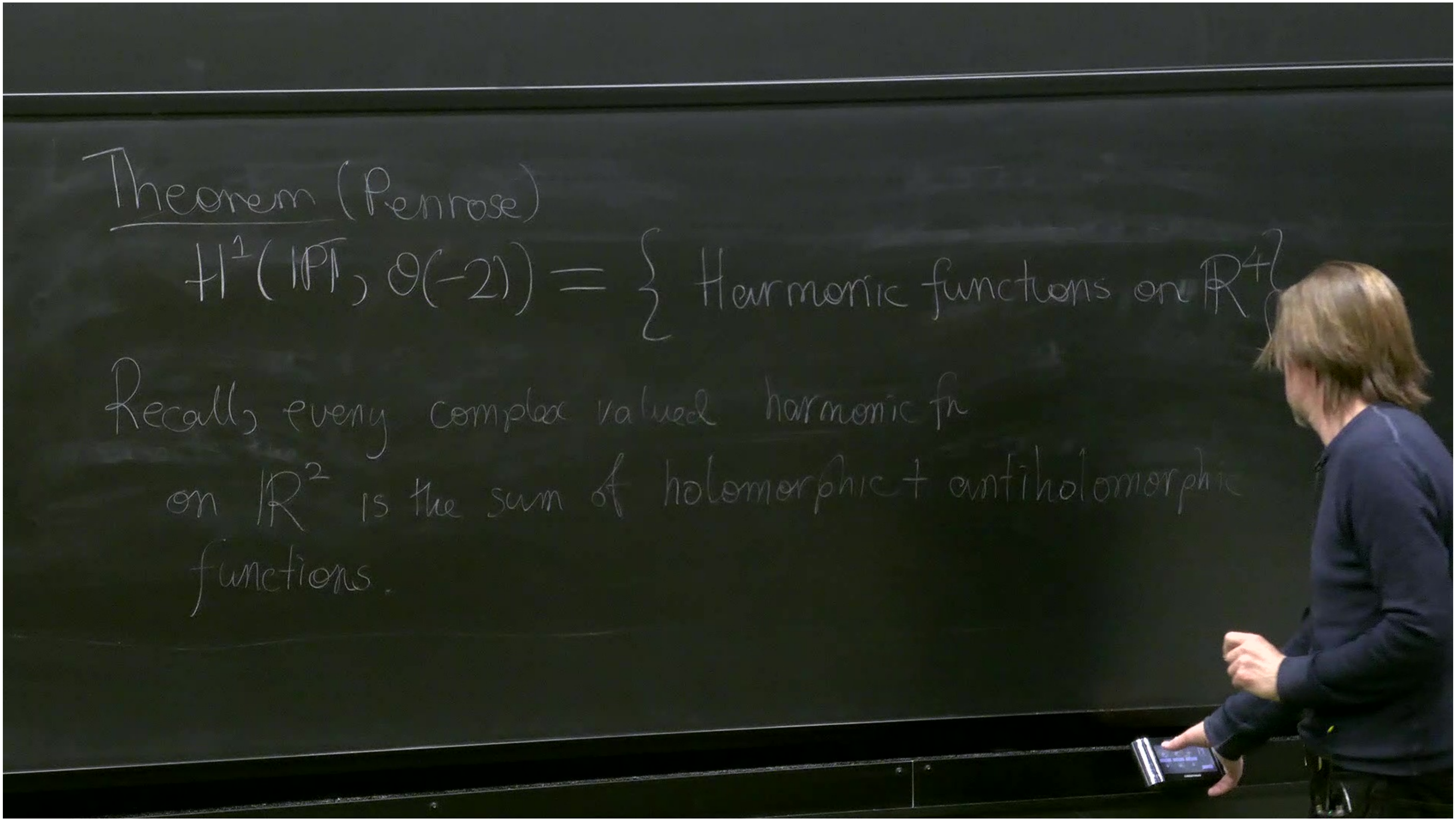
What about translations?

$V_{\dot\alpha}$ has a pole at $z=\infty$

$\frac{\partial}{\partial V_{\dot\alpha}}$ has a zero at $z=\infty$

$$\frac{\partial}{\partial u_{\dot\alpha}} \Longleftrightarrow \frac{\partial}{\partial V_{\dot\alpha}} \qquad \frac{\partial}{\partial w^{\dot\alpha}} \Longleftrightarrow \varepsilon_{\dot\alpha\dot\beta}\, z \frac{\partial}{\partial V_{\dot\beta}}$$

4 translations are $\frac{\partial}{\partial V_{\dot\alpha}}$, $z\frac{\partial}{\partial V_{\dot\alpha}}$


Theorem (Penrose)
$H^1(\mathbb{PT}, \mathcal{O}(-2)) = \{$ Harmonic functions on $\mathbb{R}^4\}$
Recall, every complex valued harmonic fn
on $\mathbb{R}^2$ is the sum of holomorphic + antiholomorphic
functions.

$\mathbb{R}^4$ has a continuous family of complex structures.
This result of Penrose is a similar statement, but sum becomes an $\int$

A little more Dolbeault cohomology.

What is $\mathcal{O}(-2)$ on $\mathbb{PT}$?

On $\mathbb{CP}^1$, $\mathcal{O}(-2)$ = tensors that transform as dz.

This makes sense on $\mathbb{PT}$ as well.

(there is a map $\mathbb{PT} \to \mathbb{CP}^1$)

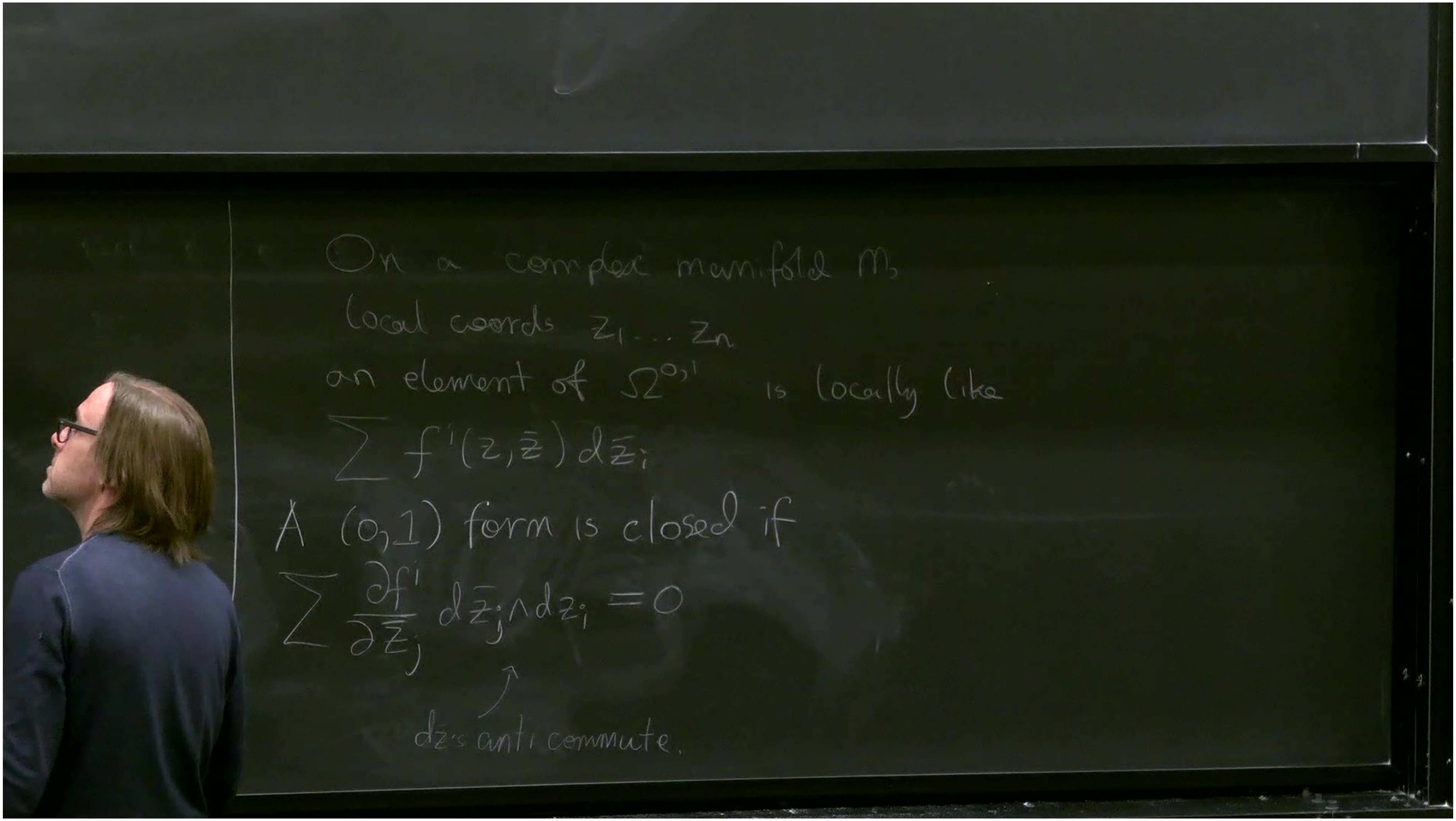
Or

$d v_{\dot{\alpha}}$ transforms as $(dz)^{1/2}$

So, $d v_{\dot{1}}\, d v_{\dot{2}}\, dz$ transforms as $(dz)^2$

$dv_{\dot{1}} dv_{\dot{2}} dz \in \mathcal{O}(-4)$

A section of $\mathcal{O}(-2)$, is a tensor transforming as $(d v_{\dot{1}}\, d v_{\dot{2}}\, dz)^{1/2}$


On a complex manifold M,
local coords $z_1 \dots z_n$
an element of $\Omega^{0,1}$ is locally like
$\sum f^i(z,\bar{z})\, d\bar{z}_i$
A (0,1) form is closed if
$\sum \frac{\partial f^i}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_i = 0$
dz's anti commute.

A section of $\mathcal{O}(-2)$, is a tensor transforming as $(dv_1\, dv_2\, dz)$

Dolbeault $H^1 = \dfrac{\operatorname{Ker} \bar\partial : \Omega^{0,1} \to \Omega^{0,2}}{\operatorname{Im} \bar\partial}$

$=$ closed $(0,1)$ forms $/ \operatorname{Im} \bar\partial$

This makes sense for expressions like

$$(dz_1 \wedge \dots \wedge dz_n)^k \left(\sum f^i \, d\bar z_i \right)$$

In particular, we now have a defn. for

$dv_2\, dz)$ $\sum \partial z_j$

dz's anti commute

$H^1(\mathbb{P}^1, \mathcal{O}(-2))$

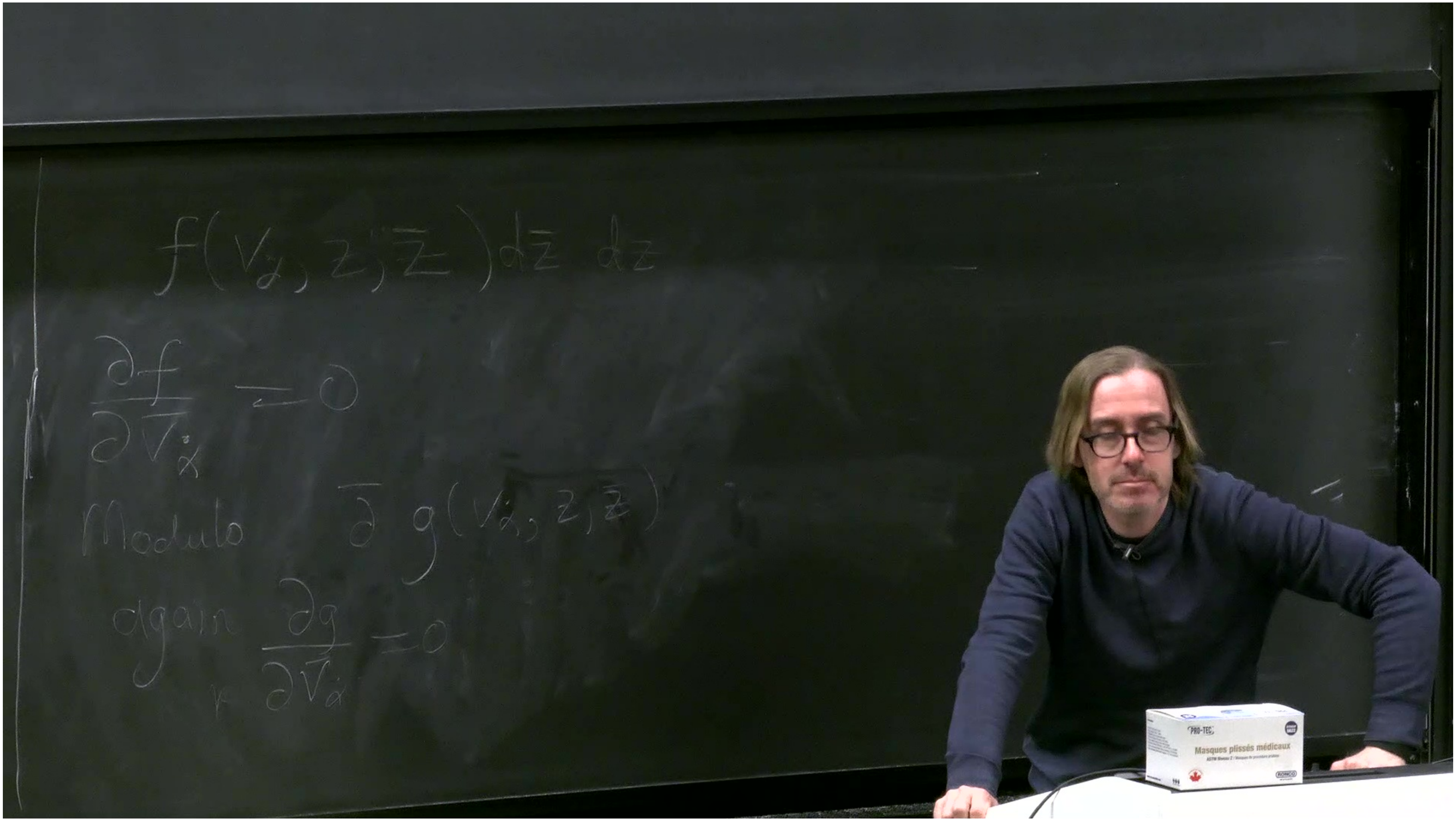
FACTS:

1) $H^1_{\bar\partial}(\mathbb{C}^n) = 0$

2) If X, Y are complex manifolds coords z, w
$X \xrightarrow{f} Y$ is a map, X is a vector bundle over Y
any Dolbeault class is represented by $f^j(z, w, \bar{w})\, d\bar{w}_j$

- f hol in fibre coordinates
- all $(0,1)$ forms come from base

On $\mathbb{PT}$, every element of $H^1(\mathbb{PT}, \mathcal{O}(-2))$ is an expression like


$f(V_\alpha, z, \bar{z})\, d\bar{z}\, dz$
$\frac{\partial f}{\partial V_\alpha} = 0$
Modulo $\bar{\partial}\, g(V_\alpha, z, \bar{z})$
again $\frac{\partial g}{\partial V_\alpha} = 0$
PRO-TEC
Masques plissés médicaux