Title: Lecture - Causal Inference, PHYS 777

Speakers: Robert Spekkens

Collection/Series: Causal Inference (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Quantum Foundations

Date: April 29, 2025 - 11:30 AM

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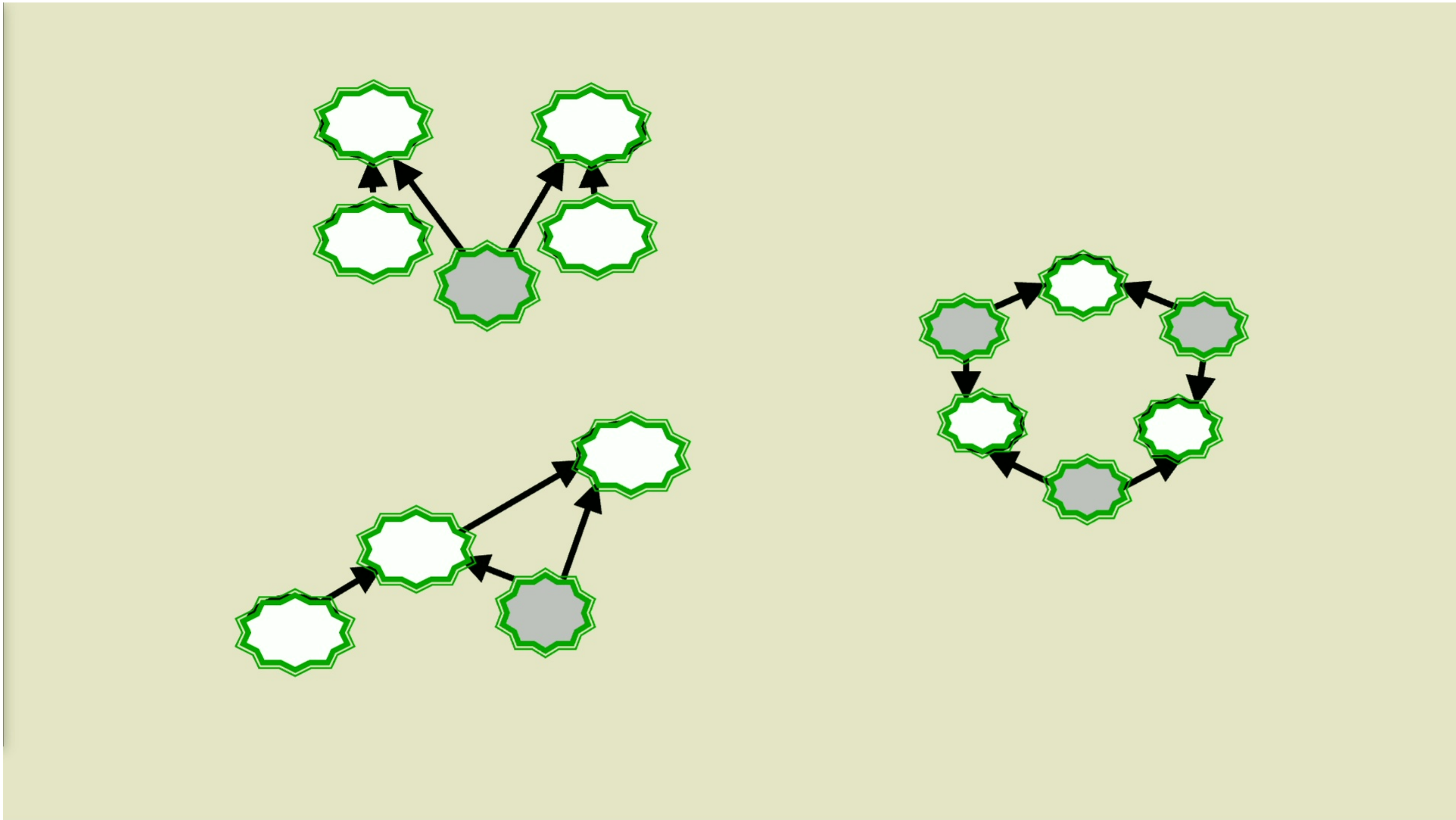
Causal compatibility in causal models with quantum latents and quantum visibles



Conventional assumption:
dimension of latent quantum system is arbitrary

Quantum marginal problem

Is there a quantum state on A, B, C that has the following marginals?

$$\rho_{AB} = \frac{1}{2}|0\rangle_A\langle 0| \otimes |0\rangle_B\langle 0| + \frac{1}{2}|1\rangle_A\langle 1| \otimes |1\rangle_B\langle 1|$$
$$\rho_{AC} = \frac{1}{2}|0\rangle_A\langle 0| \otimes |0\rangle_C\langle 0| + \frac{1}{2}|1\rangle_A\langle 1| \otimes |1\rangle_C\langle 1|$$
$$\rho_{BC} = \frac{1}{2}|0\rangle_B\langle 0| \otimes |0\rangle_C\langle 0| + \frac{1}{2}|1\rangle_B\langle 1| \otimes |1\rangle_C\langle 1|$$

Yes! $\rho_{ABC} = |\mathrm{GHZ}\rangle_{ABC}\langle \mathrm{GHZ}|$

$$|\mathrm{GHZ}\rangle_{ABC} = \frac{1}{\sqrt{2}}|000\rangle_{ABC} + \frac{1}{\sqrt{2}}|111\rangle_{ABC}$$

Is there a quantum state on A, B, C that has the following marginals?

$$\rho_{AB} = |\Phi^+\rangle_{AB}\langle\Phi^+|$$

$$|\Phi^+\rangle_{AB} = \tfrac{1}{2}|00\rangle_{AB} + \tfrac{1}{2}|11\rangle_{AB}$$

$$\rho_{AC} = \tfrac{1}{2}I_A \otimes \tfrac{1}{2}I_C$$

$$\rho_{BC} = \tfrac{1}{2}I_B \otimes \tfrac{1}{2}I_C$$

Yes! $\rho_{ABC} = |\Phi^+\rangle_{AB}\langle\Phi^+| \otimes \tfrac{1}{2}I_C$

Is there a quantum state on A, B, C that has the following marginals?

$$\rho_{AB} = |\Phi^+\rangle_{AB}\langle\Phi^+|$$
$$\rho_{AC} = |\Phi^+\rangle_{AC}\langle\Phi^+|$$
$$\rho_{BC} = |\Phi^+\rangle_{BC}\langle\Phi^+|$$

No!

Classical marginal inequality

$$0 \leq 1 - P_X - P_Y - P_Z + P_{XY} + P_{XZ} + P_{YZ} \leq 1$$

Quantum marginal inequality

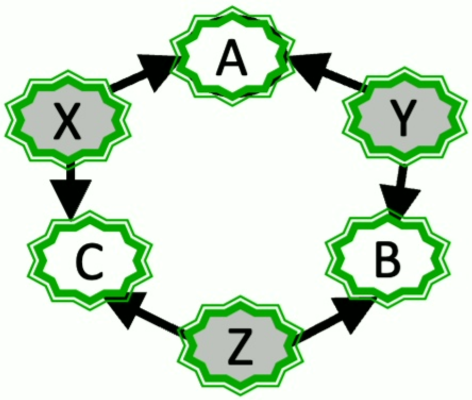
$$0 \leq I - \rho_A - \rho_B - \rho_C + \rho_{AB} + \rho_{AC} + \rho_{BC} \leq I$$

Butterley, Sudbery, Szulc, Found. Phys. 36, 83-101 (2006)

$$\text{where } A \geq 0 \text{ means } \forall|\phi\rangle : \langle\phi|A|\phi\rangle \geq 0$$

$$0 \leq I_{ABC} - \rho_A \otimes I_{BC} - \rho_B \otimes I_{AC} - \rho_C \otimes I_{AB} + \rho_{AB} \otimes I_C + \rho_{AC} \otimes I_B + \rho_{BC} \otimes I_A \quad \leq I_{ABC}$$

Triangle scenario



$\rho_{A|X_A Y_A}$

$\rho_{B|Y_B Z_B}$

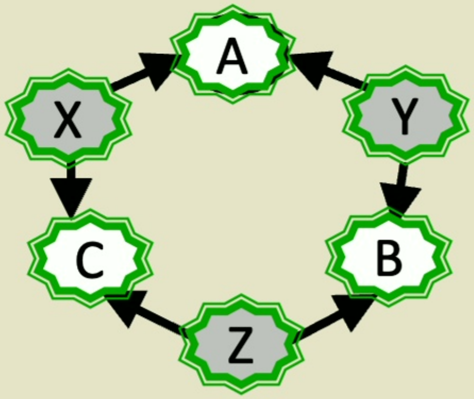
$\rho_{C|X_C Z_C}$

$\rho_{X_A X_C}$

$\rho_{Y_A Y_B}$

$\rho_{Z_B Z_C}$

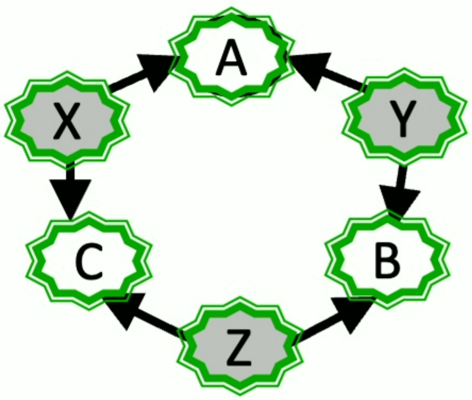
$$\rho_{ABC} = \mathrm{Tr}_{X_A X_C Y_A Y_B Z_C Z_B} \left(\rho_{A|X_A Y_A} \rho_{B|Y_B Z_B} \rho_{C|X_C Z_C} \rho_{X_A X_C} \rho_{Y_A Y_B} \rho_{Z_B Z_C}\right)$$

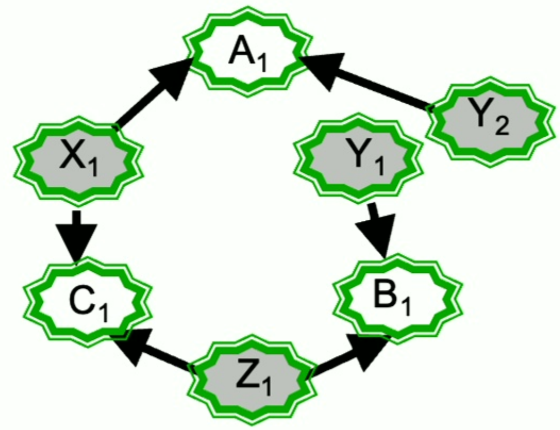


A, B and C qubits

$$I - \rho_A - \rho_B - \rho_C + \rho_A \otimes \rho_B + \rho_{BC} + \rho_{AC} \geq 0$$

where identity operators are implicit





$$I-\rho_A-\rho_B-\rho_C+\rho_A\otimes\rho_B+\rho_{BC}+\rho_{AC}\geq 0$$

is a causal compatibility inequality for M

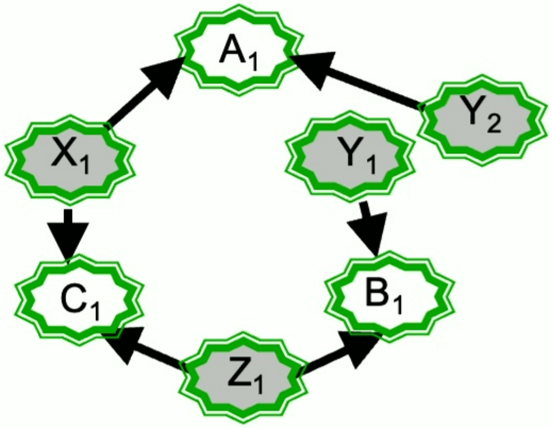
⇐

$$I-\rho_{A_1}-\rho_{B_1}-\rho_{C_1}+\rho_{A_1}\otimes\rho_{B_1}+\rho_{B_1C_1}+\rho_{A_1C_1}\geq 0$$

is a causal compatibility inequality for M'

$(\rho_{A_1C_1}, \rho_{B_1C_1}, \rho_{A_1B_1})$
is a valid set of marginals $\Longrightarrow$ $(\rho_{A_1C_1}, \rho_{B_1C_1}, \rho_{A_1B_1})$ satisfy
$$I - \rho_{A_1} - \rho_{B_1} - \rho_{C_1} + \rho_{A_1B_1} + \rho_{B_1C_1} + \rho_{A_1C_1} \geq 0$$

Butterley, Sudbery, Szulc, Found. Phys. 36, 83 (2006).



$(\rho_{A_1C_1}, \rho_{B_1C_1}, \rho_{A_1B_1})$
is compatible with M'

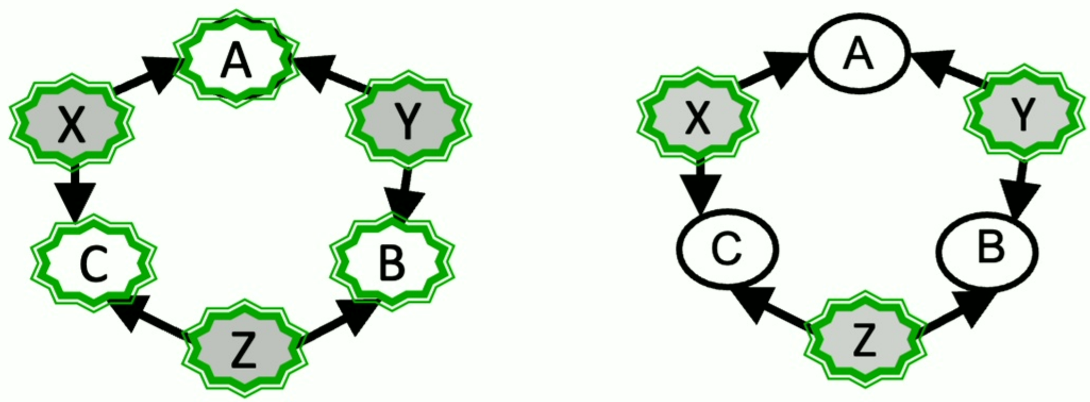
$$\Longrightarrow \rho_{A_1B_1} = \rho_{A_1} \otimes \rho_{B_1}$$

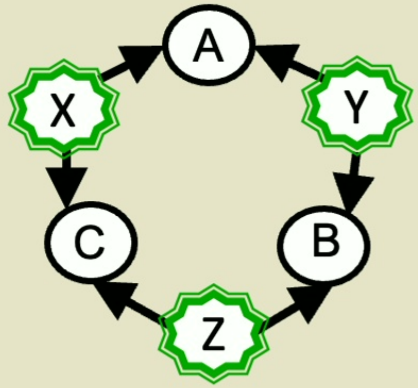
$(\rho_{A_1C_1}, \rho_{B_1C_1}, \rho_{A_1B_1})$
is compatible with M' $\Longrightarrow$
$$I - \rho_{A_1} - \rho_{B_1} - \rho_{C_1} + \rho_{A_1} \otimes \rho_{B_1} + \rho_{B_1C_1} + \rho_{A_1C_1} \geq 0$$

This is a causal compatibility inequality for M'

For quantum states that encode a classical distribution

A, B and C become effectively classical variables





$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

Derived from the cut inflation, which is nonfanout

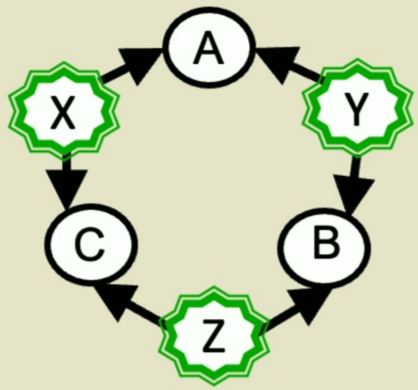
rules out

$$P_{ABC}^{(\mathrm{GHZ})} = \tfrac{1}{2}[000] + \tfrac{1}{2}[111]$$

$$\text{For } (A=0, B=0, C=1),\ LHS = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{4} - 0 - 0 = \tfrac{5}{4} > 1$$

The problem of characterizing the compatible states
Subsumes the problem of characterizing the compatible distributions

The latter problem is highly nontrivial, therefore so is the former



$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

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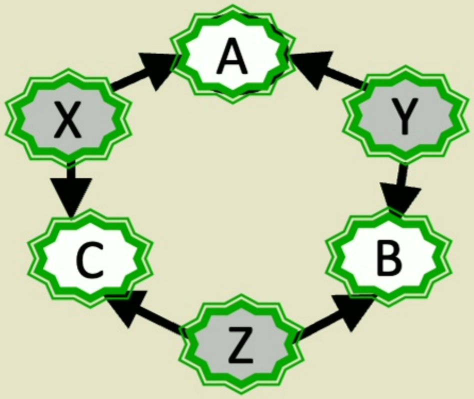
For $(A=0, B=0, C=1)$, $LHS = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} - 0 - 0 = \frac{5}{4} > 1$



A, B and C qubits

$$I-\rho_A-\rho_B-\rho_C+\rho_A\otimes\rho_B+\rho_{BC}+\rho_{AC}\geq 0$$

rules out, for example:

$$\rho_{ABC}=|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| \text{ where } |\mathrm{GHZ}\rangle=\tfrac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$$

$$\rho_{ABC}=|\mathrm{W}\rangle\langle\mathrm{W}| \text{ where } |\mathrm{W}\rangle=\tfrac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle)$$

Linear inequality constraints from marginal compatibility (from linear quantifier elimination)

Polynomial inequality constraints for causal compatibility with the original DAG

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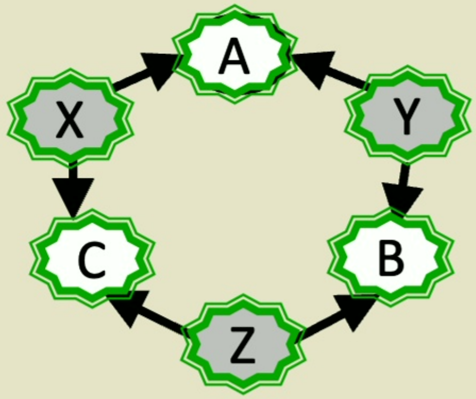
Polynomial equality constraints from causal compatibility with the inflated DAG (e.g., from d-separation relations)



A, B and C qubits

$$I - \rho_A - \rho_B - \rho_C + \rho_A \otimes \rho_B + \rho_{BC} + \rho_{AC} \geq 0$$

Factorization across a bipartition AB|C, AC|B or BC|A is clearly *sufficient* for compatibility. For three-qubit pure states, it is also *necessary*

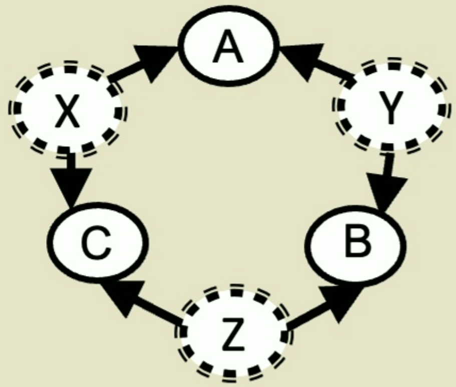
Fully quantum inflation provides a means of using results concerning the quantum marginal problem for the purpose of causal discovery

The problem of characterizing the compatible states
Subsumes the problem of characterizing the compatible distributions

The latter problem is highly nontrivial, therefore so is the former

Additional comments on the inflation technique

Causal compatibility constraints in terms of entropies and correlators



{+.-}-valued A, B and C

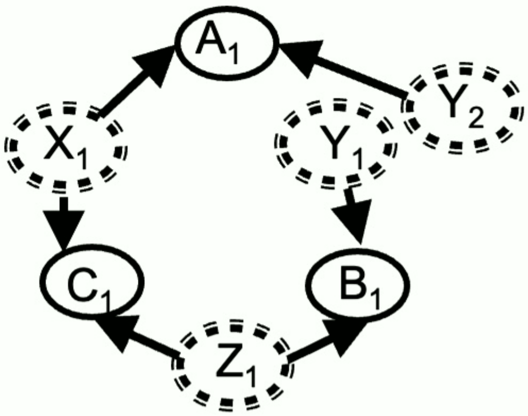
$$\langle AC\rangle + \langle BC\rangle \leq 1 + \langle A\rangle\langle B\rangle$$

This inequality is obtained from the cut inflation, which is nonfanout, and therefore is valid for all theories

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is a valid set of marginals

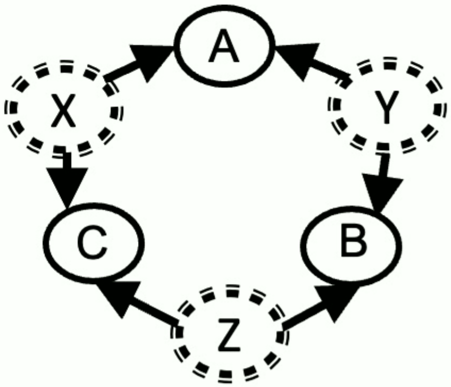
$$\Longrightarrow$$

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1}) \text{ satisfy}$$
$$\langle A_1C_1\rangle + \langle B_1C_1\rangle - \langle A_1B_1\rangle \leq 1$$



$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is compatible with M'

$$\Longrightarrow A_1 \perp B_1 \Longrightarrow \langle A_1B_1\rangle = \langle A_1\rangle\langle B_1\rangle$$



is incompatible with

$$P_{ABC} = \frac{1}{2}[+++] + \frac{1}{2}[---]$$

Test causal compatibility inequality:

$$\langle AC \rangle + \langle BC \rangle \leq 1 + \langle A \rangle \langle B \rangle$$

$$LHS = (+1) + (+1) = 2 \qquad RHS = 1 + 0 = 1$$

Shannon entropy

$$H(X) := -\sum_x P_X(x) \log P_X(x)$$

Conditional entropy

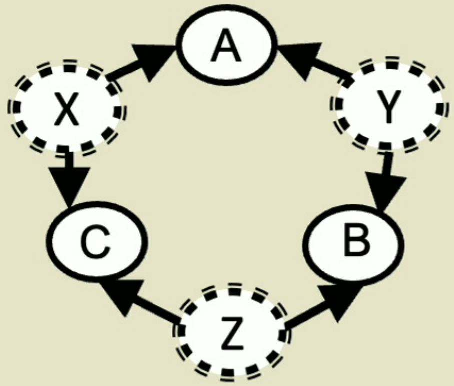
$$H(X|Y) := H(XY) - H(Y)$$

Mutual information

$$I(X:Y) = H(X) + H(Y) - H(XY)$$

Conditional mutual information

$$I(X:Y|Z) = H(XZ) + H(YZ) - H(XYZ) - H(Z)$$



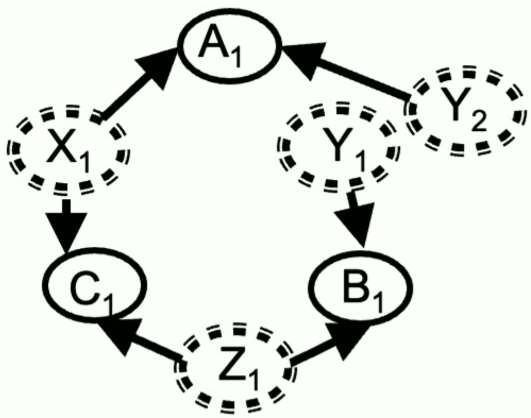
{+.-}-valued A, B and C

$$I(A:C) + I(C:B) \leq H(C)$$

This inequality is obtained from the cut inflation, which is nonfanout, and therefore is valid for all theories

Entropic inequalities are valid for arbitrary cardinalities of observed variables

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy $I(A_1 : C_1) + I(C_1 : B_1) - I(A_1 : B_1) \leq H(C_1)$

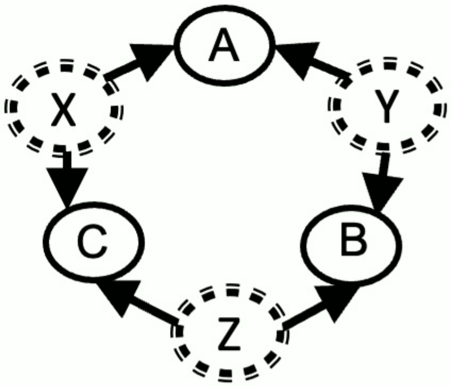


$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

$$\implies A_1 \perp B_1 \implies I(A_1 : B_1) = 0$$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M' $\implies$ $I(A_1 : C_1) + I(C_1 : B_1) \leq H(C_1)$

$I(A_1 : C_1) + I(C_1 : B_1) \leq H(C_1)$ is a causal compatibility inequality for M'



is incompatible with $P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$

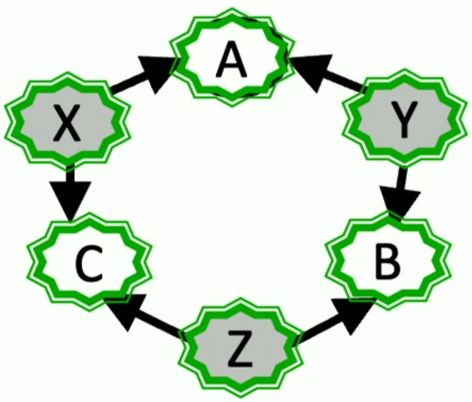
Test entropic causal compatibility inequality

$$I(A:C) + I(C:B) \leq H(C)$$

$LHS = (+1) + (+1) = 2$ $\qquad$ $RHS = 1$

$2 \not\leq 1$ violated!

One can also use nonfanout inflations to find entropic inequalities that serve as causal compatibility constraints for causal models with quantum latents and quantum visibles



$$I(A:B) = S(A) + S(B) - S(AB)$$
$$S(X) = -\mathrm{Tr}(\rho_X \log \rho_X)$$

$$\rho_{AB} = \rho_A \otimes \rho_B$$
$$I(A:B) = 0$$

Nonfanout inflations

Other Techniques for Causal Discovery

Covariance matrix techniques

A. Kela, K. von Prillwitz, J. Aberg, R. Chaves, and D. Gross, arXiv:μ70μ.00652 (20μ7).

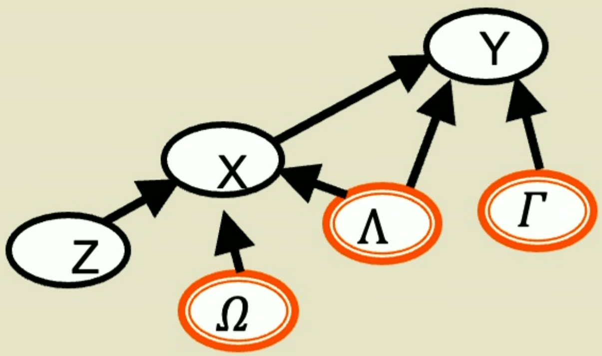
Using nonShannon-type entropic inequalities in the entropy cone technique

M. Weilenmann and R. Colbeck, Quantum 2, 57 (20μ8)

Using algorithmic independence of autonomous causal mechanisms

J. Lemeire and D. Janzing, Minds and Machines 23, 227 (20μ3)

Variations on the problem of causal compatibility



$X = f(\Lambda, \Gamma, Z)$
$Y = g(\Lambda, \Omega, X)$
P_Λ
P_Γ
P_Ω

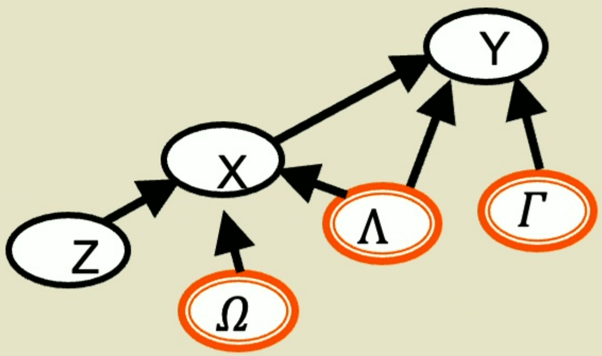
$$P_{XY|Z} = \sum_{\Lambda,\Gamma,\Omega} \delta_{X,f(\Lambda,\Omega,Z)} \delta_{Y,g(\Lambda,\Gamma,X)} P_\Lambda$$

Restricted functional dependences

- Linear dependences
- Monotonic functions
- Symplectic functions
- Local noise is additive
- symmetries

Restricted distributions of latents

- Only Gaussian distributions
- Restricted cardinalities
- symmetries



$X = f(\Lambda, \Omega, Z)$ $\quad f \in \mathcal{F}$

$Y = g(\Lambda, \Gamma, X)$ $\quad g \in \mathcal{G}$

P_Λ $\quad P_\Lambda \in \mathcal{P}_\Lambda$

P_Γ $\quad P_\Gamma \in \mathcal{P}_\Gamma$

P_Ω $\quad P_\Omega \in \mathcal{P}_\Omega$

$$P_{XY|Z} = \sum_{\Lambda,\Gamma,\Omega} \delta_{X,f(\Lambda,\Omega,Z)} \delta_{Y,g(\Lambda,\Gamma,X)} P_\Lambda \quad \Longrightarrow \quad P_{XY|Z} \in \mathcal{P}_{XY|Z}$$

Strength of causal conclusions $\propto$ Strength of causal assumptions

Contrast:

Causal explanations of the infinite-run statistics predicted by an operational theory

vs.

Causal explanations of the finite-run statistics accumulated in a real-world experiment or observation

Example of contrast:

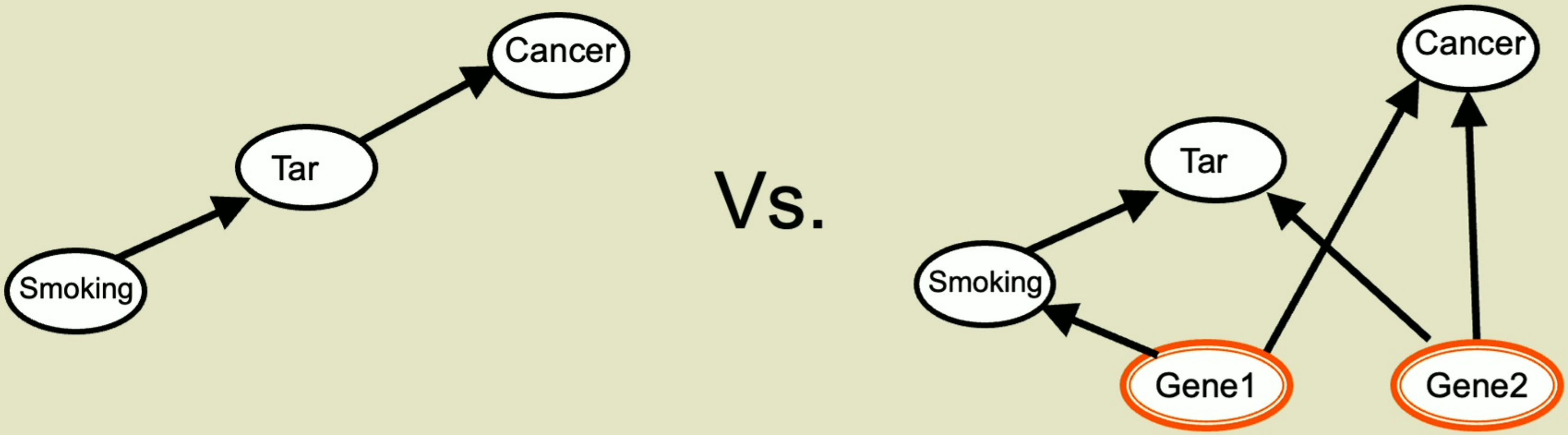
No-go theorem establishing that the idealized statistics predicted by operational quantum theory are incompatible with a classical causal model having the causal structure of the Bell DAG.

E.g., Bell's 1964 argument which appealed to perfect correlations

Hardy's 1993 argument which appealed to events with probability 0

vs.

An analysis technique for finite-run experimental data that can rule out with high confidence the possibility of a classical causal model having the causal structure of the Bell DAG



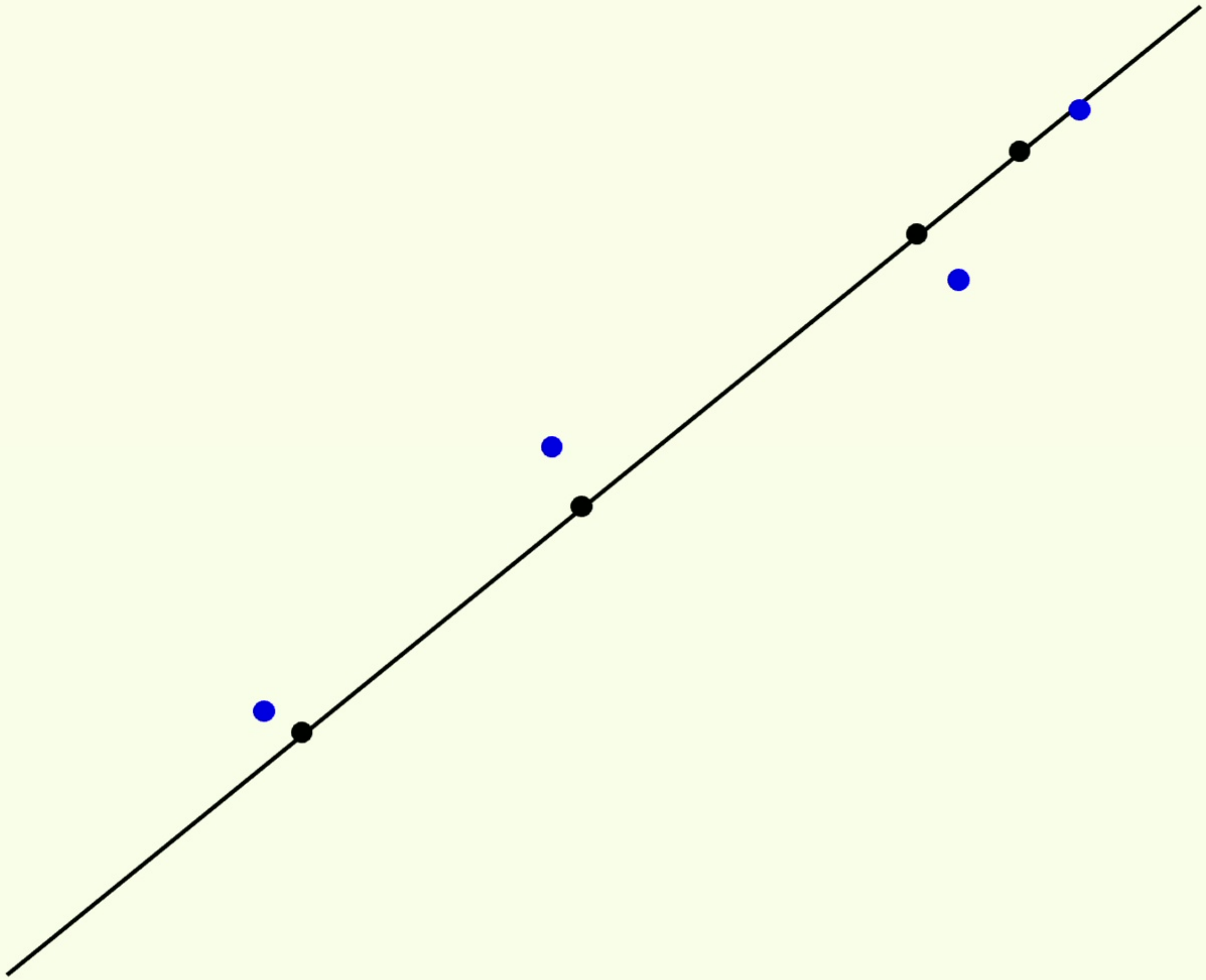
Observe P_{STC} such that

$$S \perp C | T$$

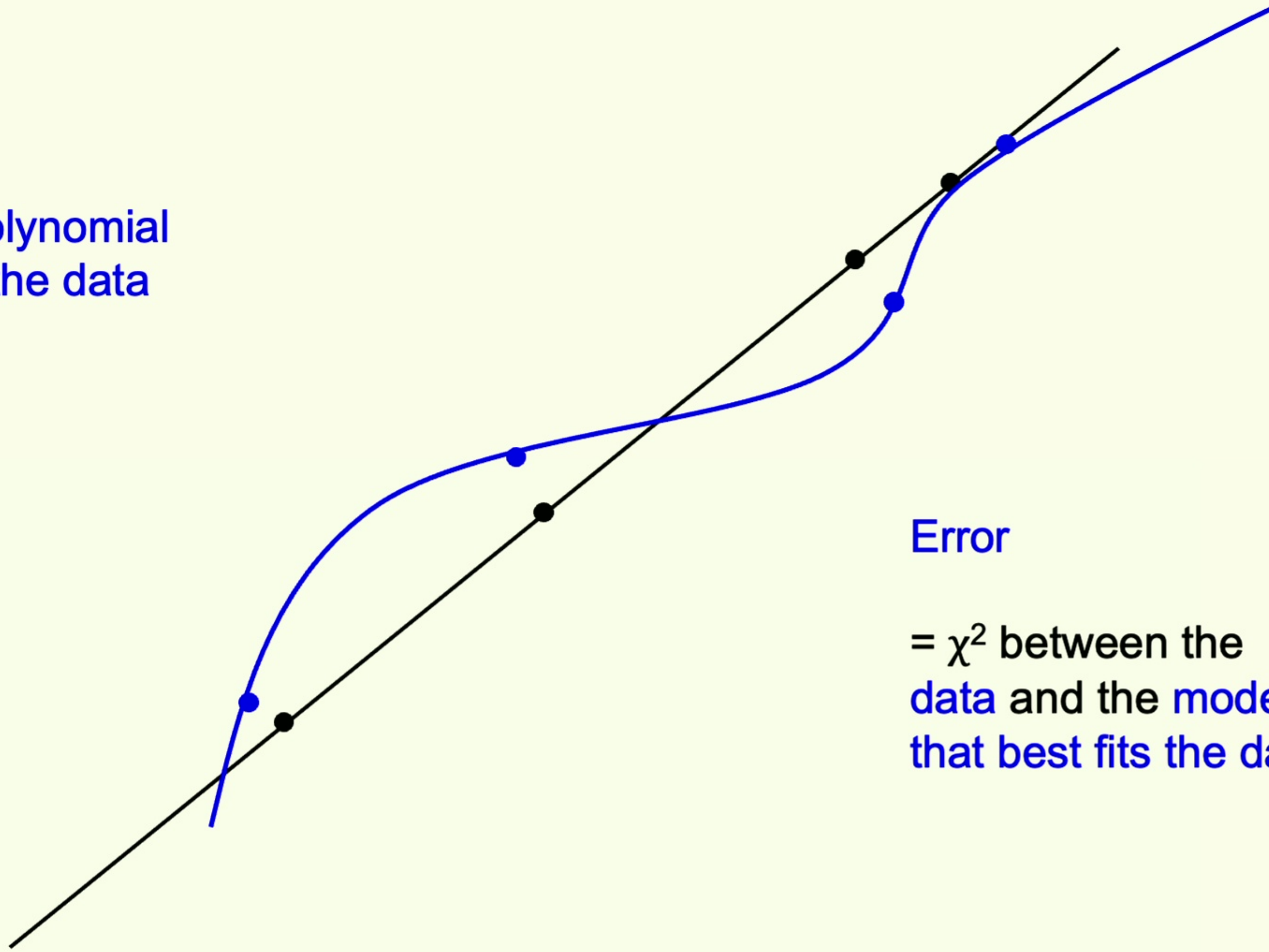
Principle of faithfulness/no fine-tuning:

Prefer those causal models for which the conditional independence relations are a consequence of the causal structure rather than the values of the parameters

Data



Data
&
Best-fit polynomial
model of the data



Error

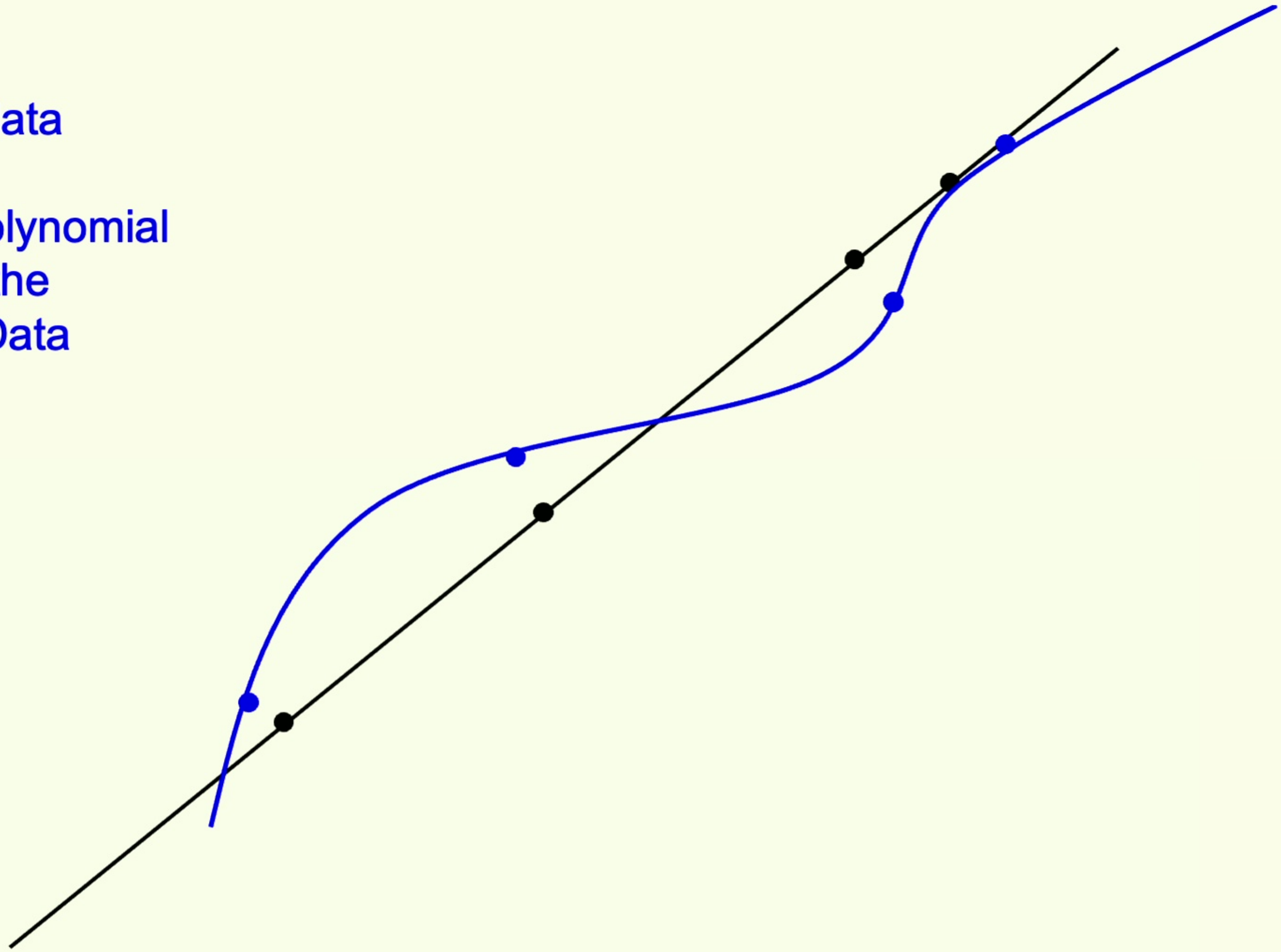
= χ^2 between the
data and the model
that best fits the data

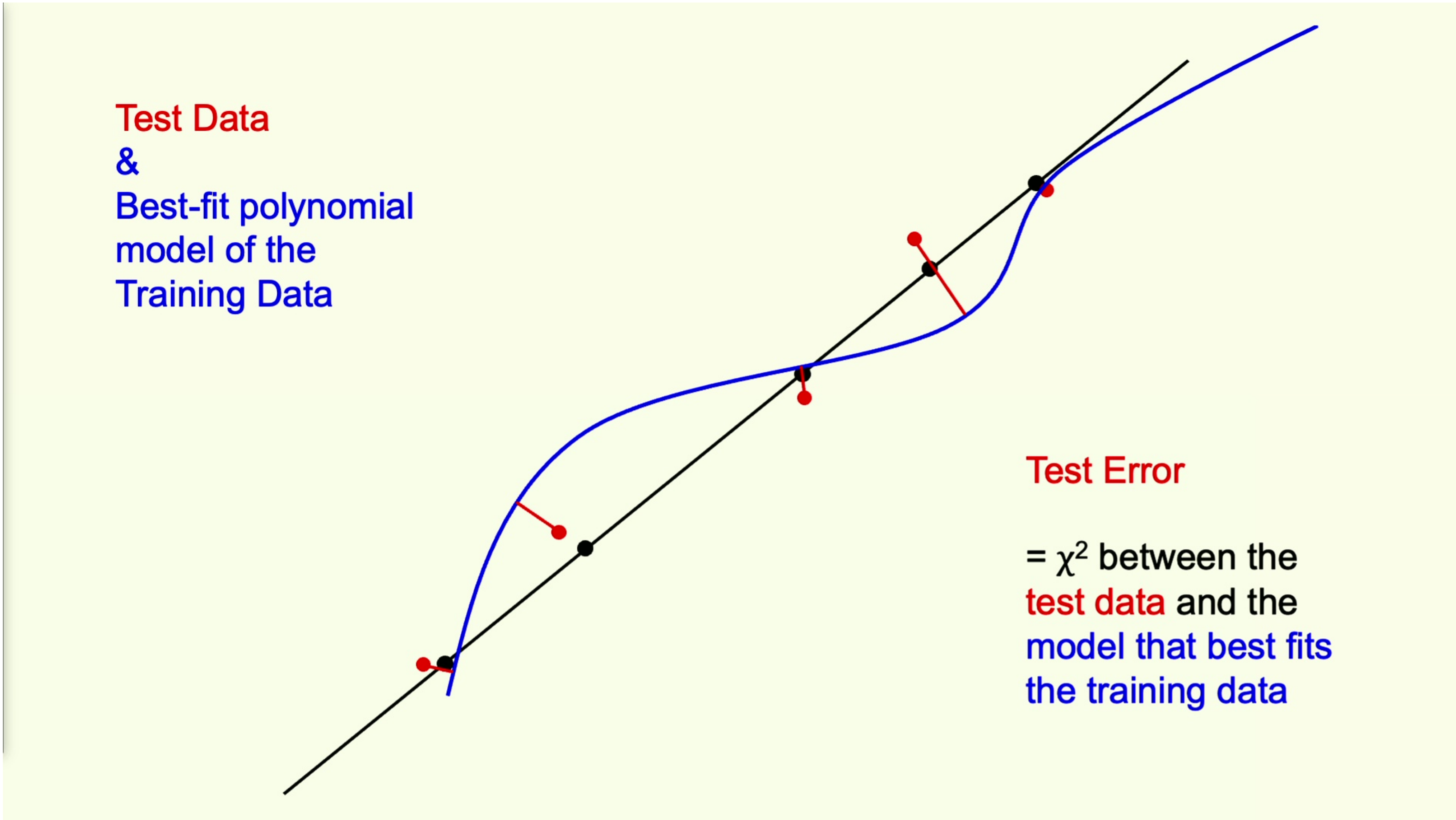
The high bar:

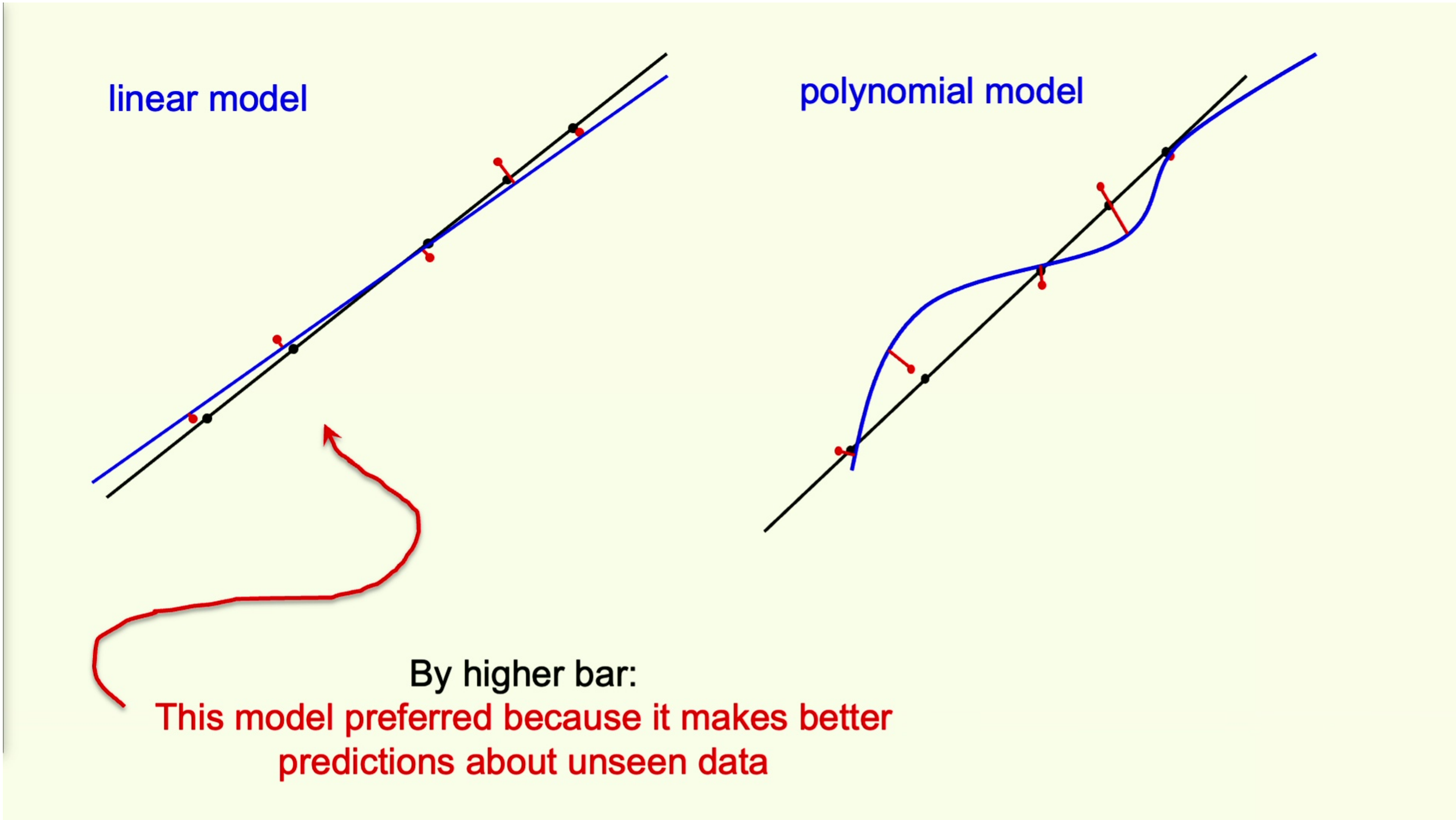
Predictive power

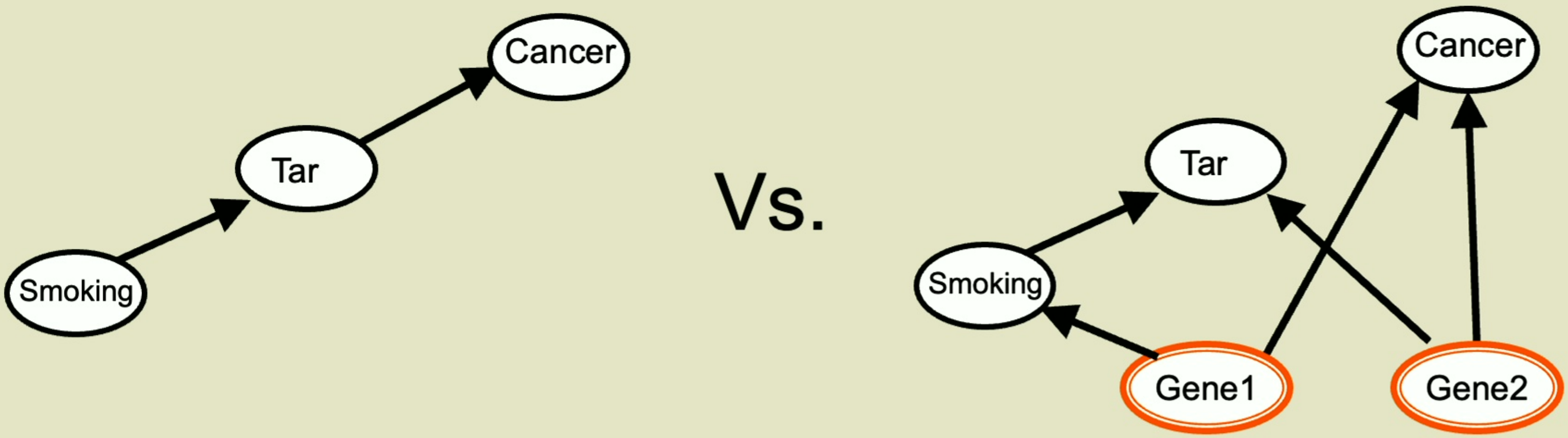
not underfitting the data
and also
not **overfitting** the data

Training data
&
Best-fit polynomial
model of the
Training Data




Test Data
&
Best-fit polynomial model of the Training Data
Test Error
= χ^2 between the test data and the model that best fits the training data

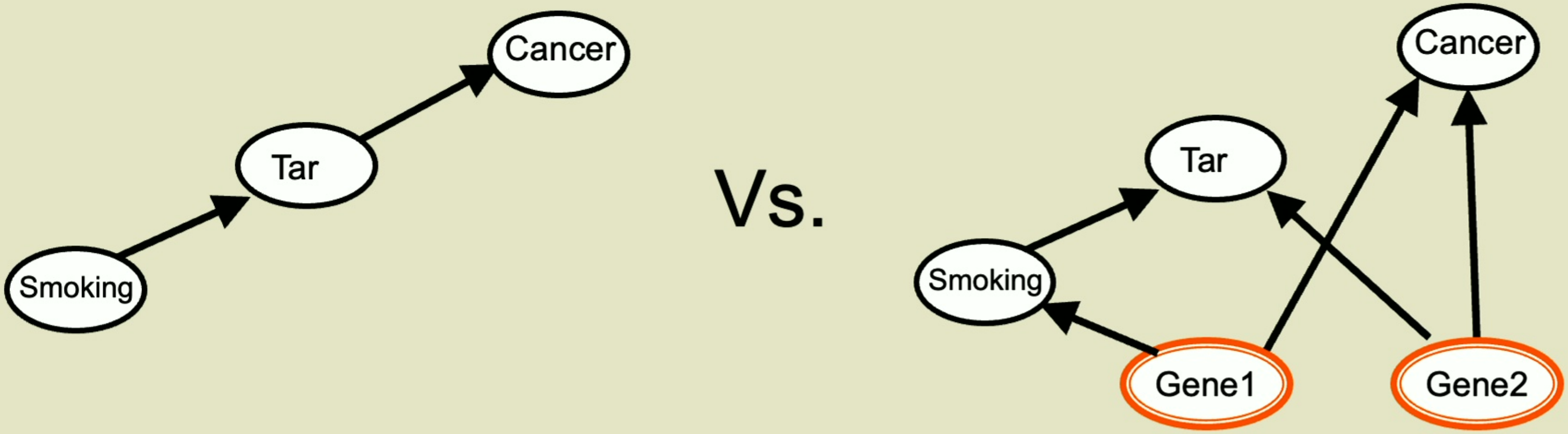

linear model
polynomial model
By higher bar:
This model preferred because it makes better predictions about unseen data



Observe P_{STC} such that **to good approximation**

$$S \perp C | T$$

If the deviations are a statistical fluctuations, the model on the right will tend to overfit the data by mistaking these for real features



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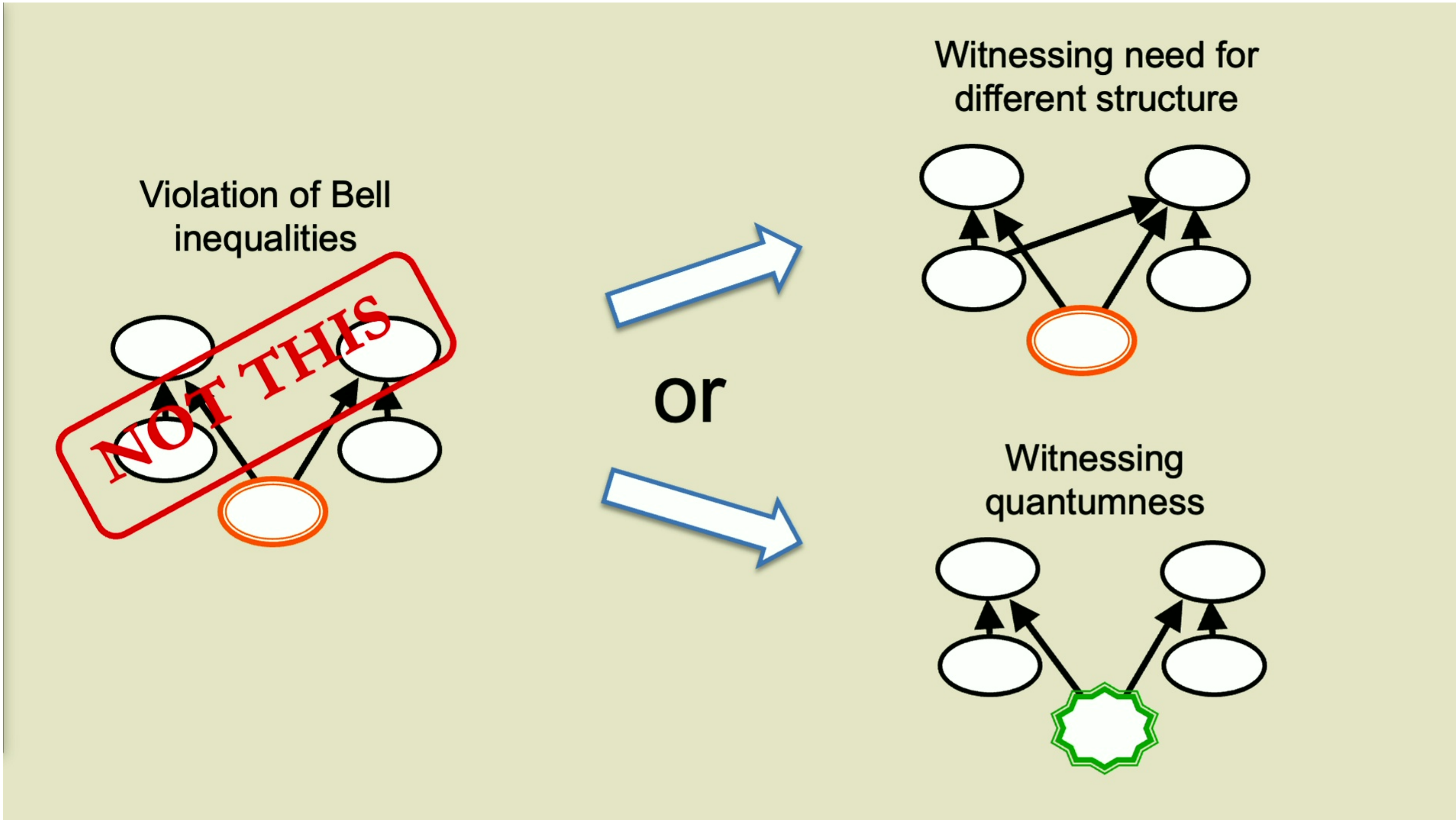
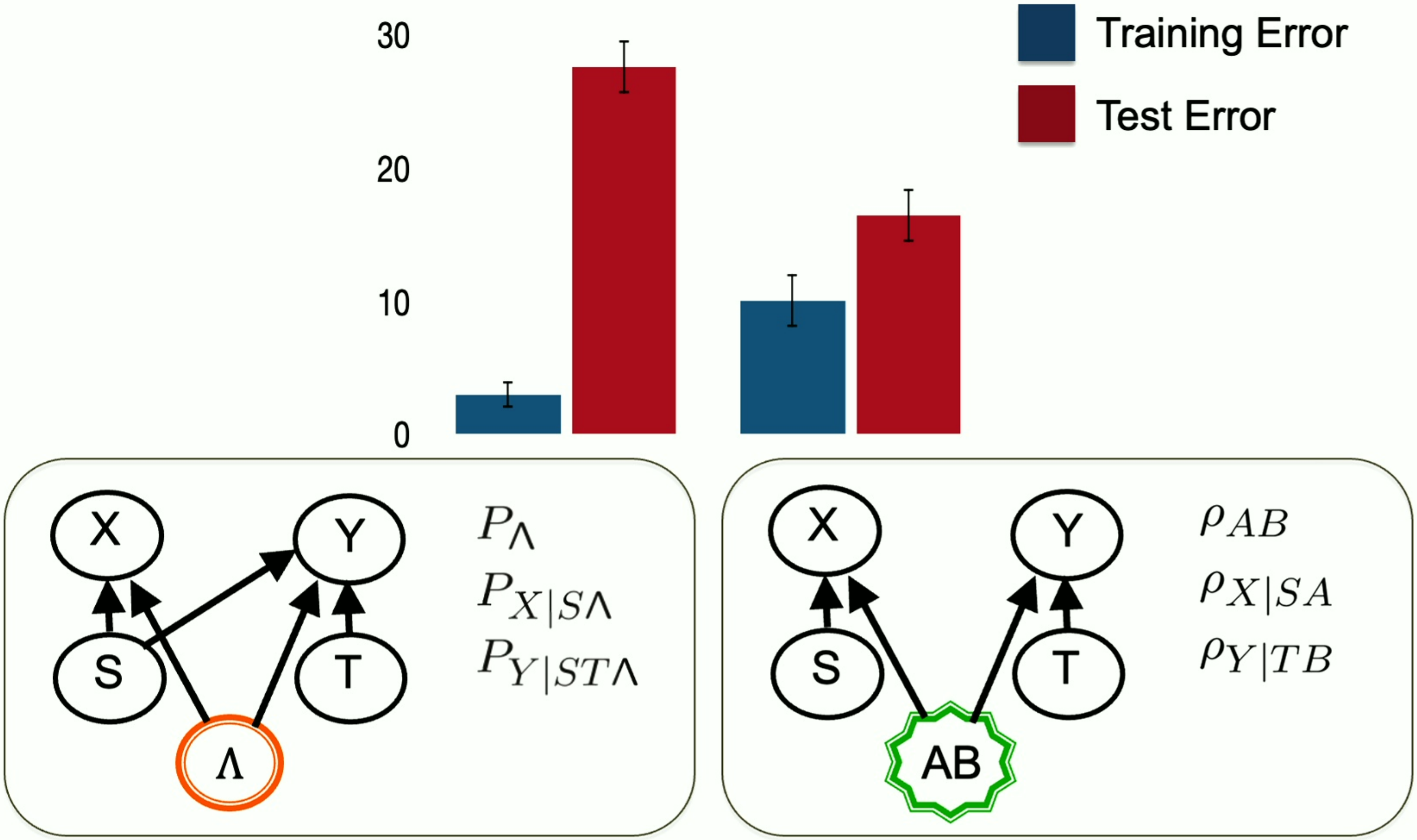
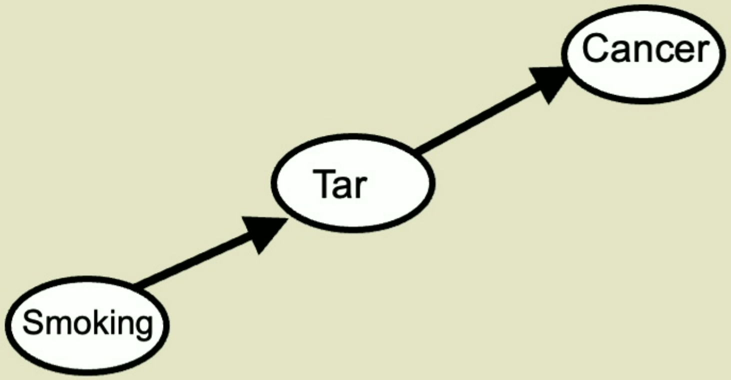
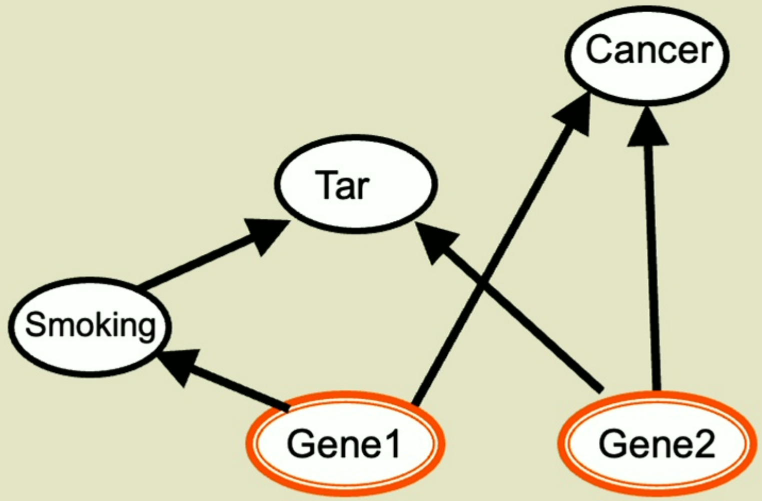


Violation of Bell inequalities
NOT THIS
or
Witnessing need for different structure
Witnessing quantumness


30
20
10
0
Training Error
Test Error
X
Y
S
T
Λ
P_Λ
$P_{X|S\Lambda}$
$P_{Y|ST\Lambda}$
X
Y
S
T
AB
ρ_{AB}
$\rho_{X|SA}$
$\rho_{Y|TB}$



Vs.



Requires fine-tuning

Observe P_{STC} such that

$$S \perp C|T$$