Title: Lecture - Causal Inference, PHYS 777

Speakers: Robert Spekkens

Collection/Series: Causal Inference (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Quantum Foundations

Date: April 28, 2025 - 11:30 AM

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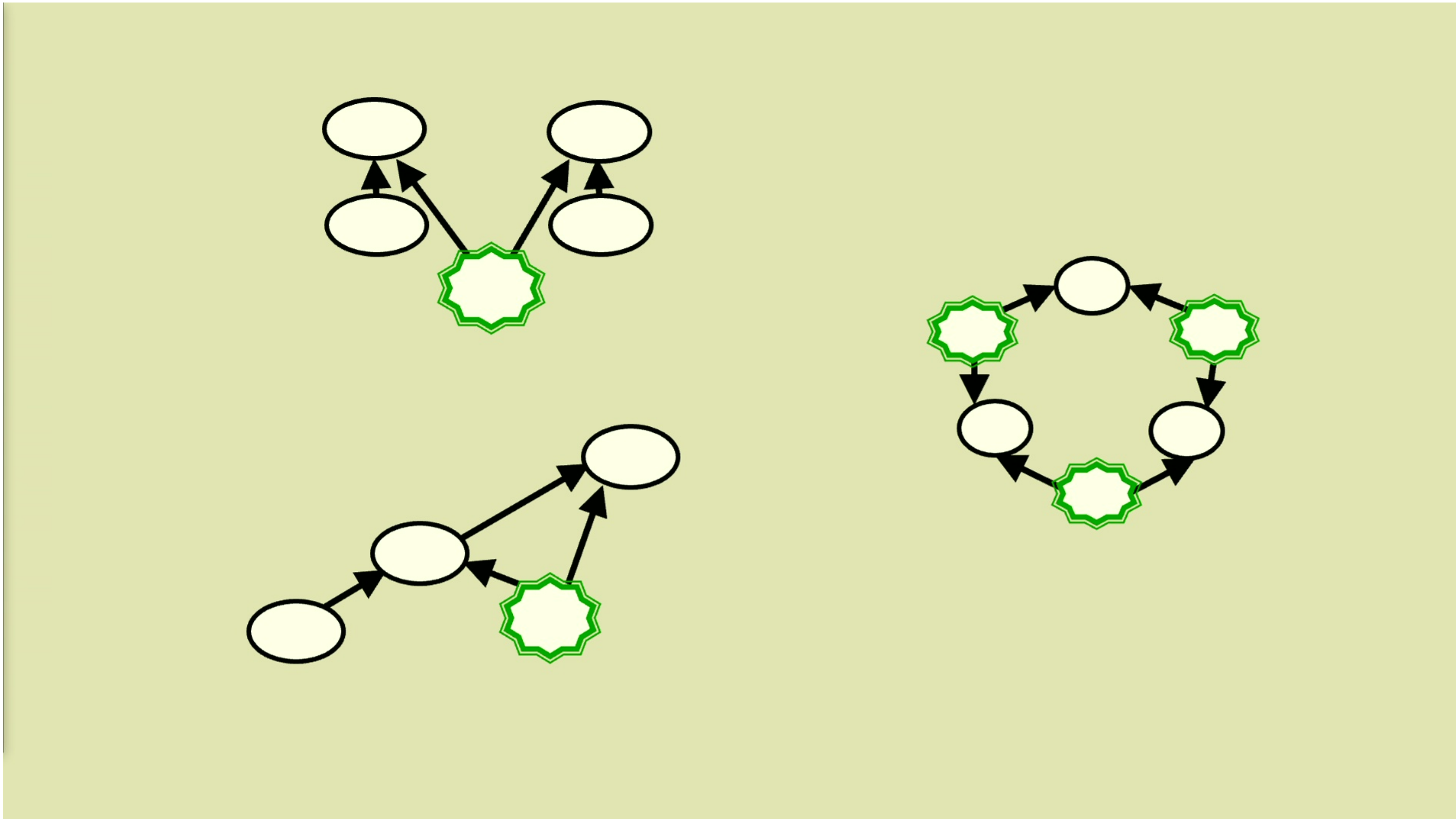
Causal compatibility in causal models with quantum latents



Conventional assumption:
dimension of latent quantum system is arbitrary

If Z is a complete common cause of X and Y, then

$$P_{XY|Z} = P_{X|Z}P_{Y|Z}$$

If C is a complete common cause of A and B, then

$$\rho_{AB|C} = \rho_{A|C}\rho_{B|C}$$

where

$$[\rho_{A|C}, \rho_{B|C}] = 0$$

If Z is a complete common cause of X_1,…,X_n, then

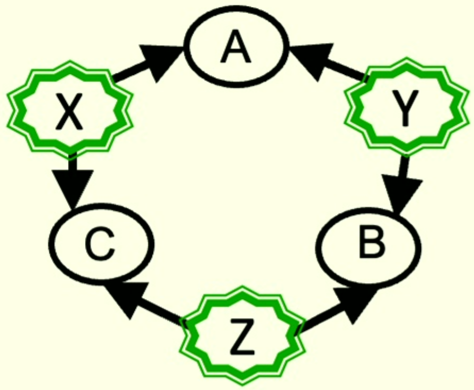
$$P_{X_1 \ldots X_n|Z} = \prod_i P_{X_i|Z}$$

If C is a complete common cause of A_1, …, A_n, then

$$\rho_{A_1 \ldots A_n|C} = \prod_i \rho_{A_i|C}$$

where

$$[\rho_{A_i|C}, \rho_{A_j|C}] = 0 \ \ \forall i, j$$



$\rho_{A|XY}$

$\rho_{B|YZ}$

$\rho_{C|XZ}$

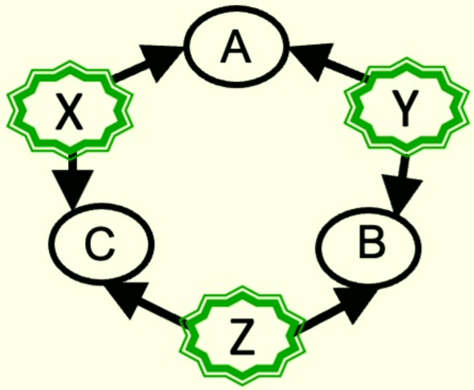
ρ_X

ρ_Y

ρ_Z

$$[\rho_{A|XY}, \rho_{B|YZ}] = 0$$
$$[\rho_{B|YZ}, \rho_{C|XZ}] = 0$$
$$[\rho_{A|XY}, \rho_{C|XZ}] = 0$$

$$P_{ABC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{B|YZ}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$



$\rho_{A|X_A Y_A}$

$\rho_{B|Y_B Z_B}$

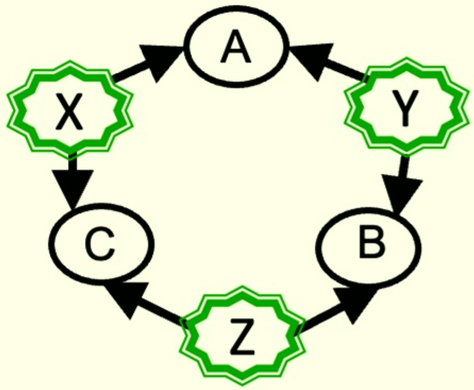
$\rho_{C|X_C Z_C}$

$\rho_{X_A X_C}$

$\rho_{Y_A Y_B}$

$\rho_{Z_B Z_C}$

$$P_{ABC} = \mathrm{Tr}_{X_A X_C Y_A Y_B Z_C Z_B}\left(\rho_{A|X_A Y_A}\rho_{B|Y_B Z_B}\rho_{C|X_C Z_C}\rho_{X_A X_C}\rho_{Y_A Y_B}\rho_{Z_B Z_C}\right)$$



$\{E_a^{X_A Y_A}\}_a$

$\{E_b^{Y_B Z_B}\}_b$

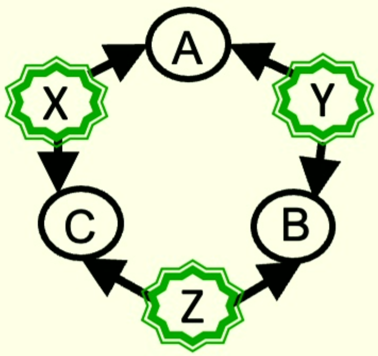
$\{E_c^{X_C Z_C}\}_c$

$\rho_{X_A X_C}$

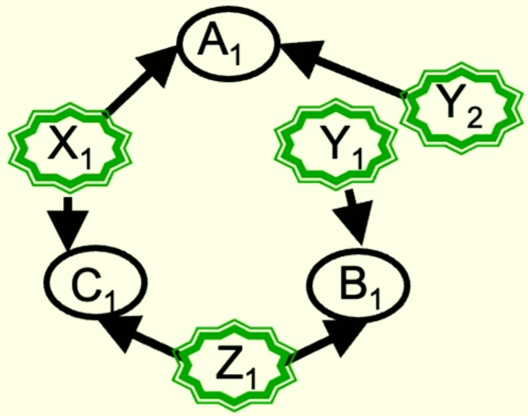
$\rho_{Y_A Y_B}$

$\rho_{Z_B Z_C}$

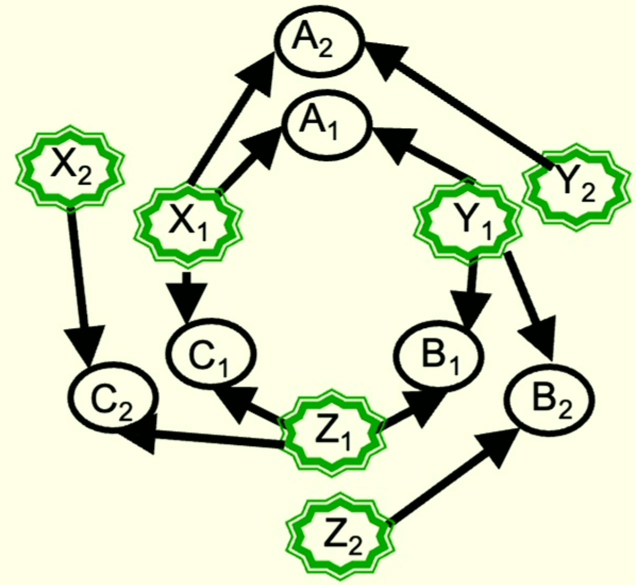
$$P_{ABC}(abc) = \mathrm{Tr}_{X_A X_C Y_A Y_B Z_A Z_C}\left(E_a^{X_A Y_A} E_b^{Y_B Z_B} E_c^{X_C Z_C} \rho_{X_A X_C} \rho_{Y_A Y_B} \rho_{Z_B Z_C}\right)$$



Triangle



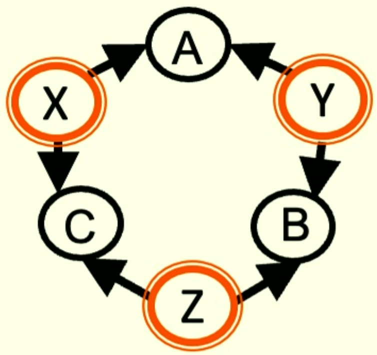
Cut inflation of Triangle



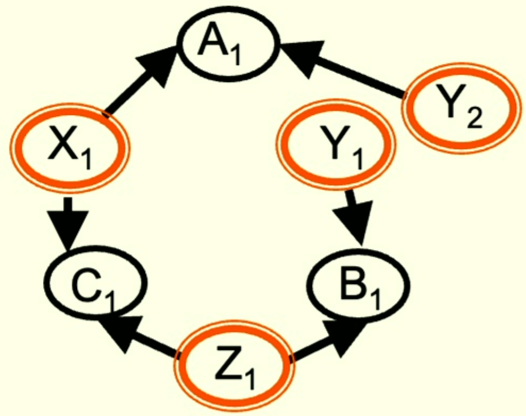
Spiral inflation of Triangle

Nonfanout

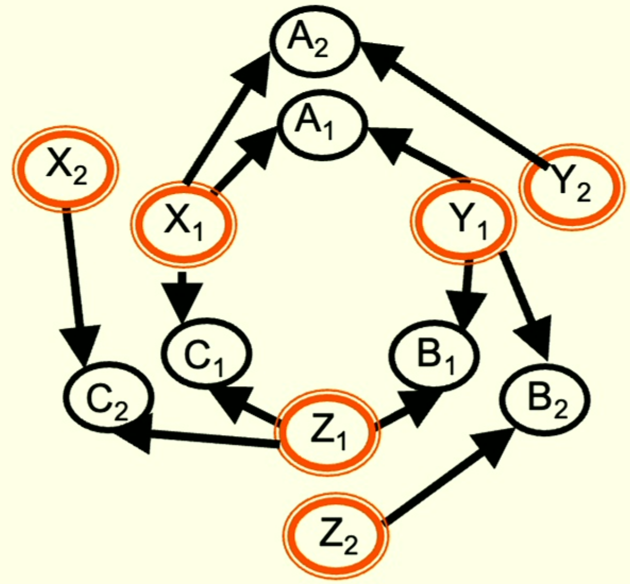
Fanout



Triangle



Cut inflation of Triangle



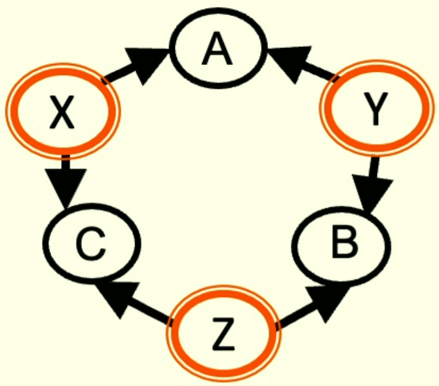
Spiral inflation of Triangle

Nonfanout

Fanout

Nonfanout inflations for classical latents

model M on DAG G



$P_{A|XY}$

$P_{B|YZ}$

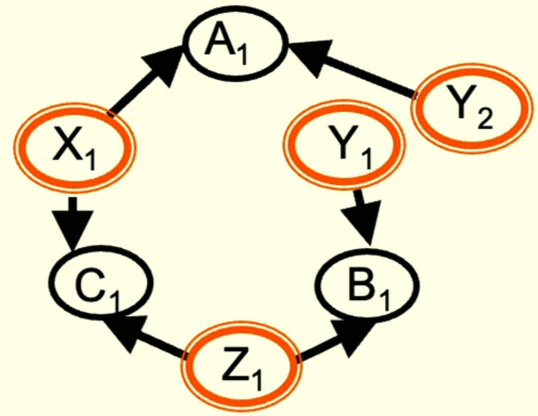
$P_{C|XZ}$

P_X

P_Y

P_Z

M' = G $\rightarrow$ G' Inflation of M



$P_{A_1|X_1Y_2}$

$P_{B_1|Y_1Z_1}$

$P_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

P_{Y_2}

P_{Z_1}

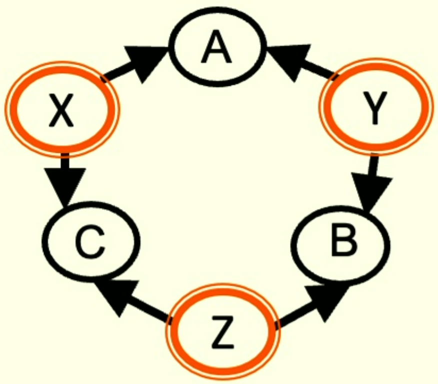
with symmetry constraint: $P_{Y_1} = P_{Y_2}$

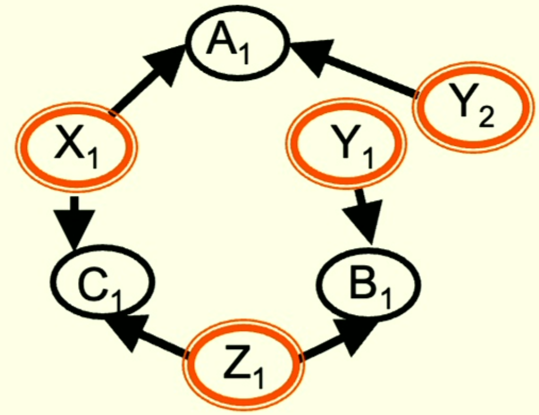
$\{A_1C_1\}$ is an injectable set

$$P_{A_1C_1} = \sum_{X_1Y_2Z_1} P_{A_1|X_1Y_2}P_{C_1|X_1Z_1}P_{X_1}P_{Y_2}P_{Z_1}$$

$$P_{AC} = \sum_{XYZ} P_{A|XY}P_{C|XZ}P_XP_YP_Z$$

$$P_{AC} \text{ compatible with } M \implies P_{A_1C_1} = P_{AC} \text{ compatible with } M'$$

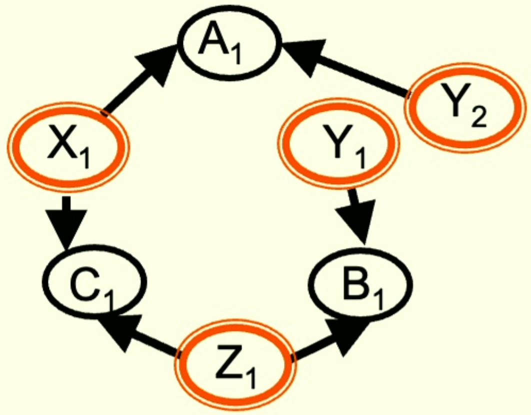




$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$
is a causal compatibility
inequality for M

$P_{A_1} + P_{B_1} + P_{C_1} - P_{A_1} P_{B_1} - P_{B_1 C_1} - P_{A_1 C_1} \leq 1$
is a causal compatibility
inequality for M'

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy
$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

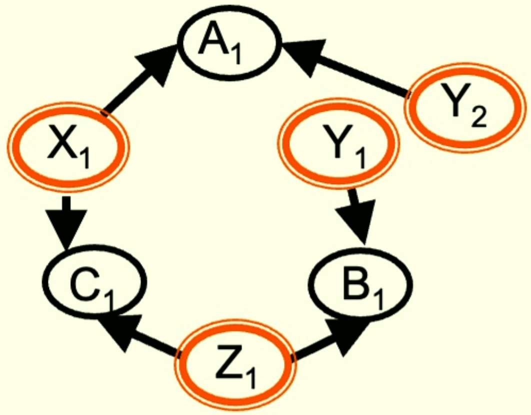


$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is compatible with M'

$$A_1 \perp_d B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is compatible with M' $\implies$ $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is a valid set of marginals $\Longrightarrow$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

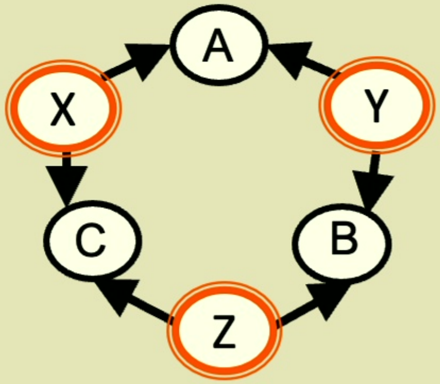


$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

$$A_1 \perp_d B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M' $\Longrightarrow$ $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

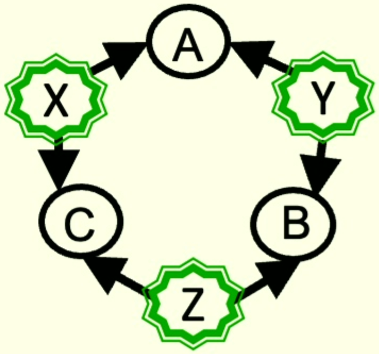
$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$ is a causal compatibility inequality for M'



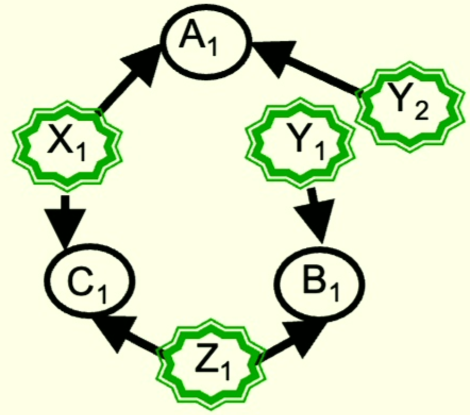
$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

rules out, for example:

$$P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$$



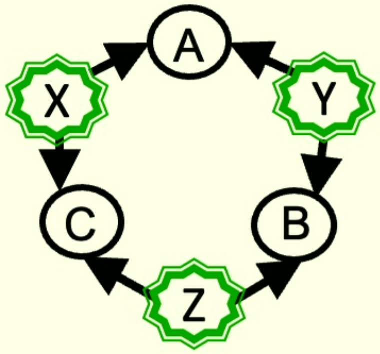
Triangle



Cut inflation of Triangle

Nonfanout

Quantum model M on DAG G



$\rho_{A|XY}$

$\rho_{B|YZ}$

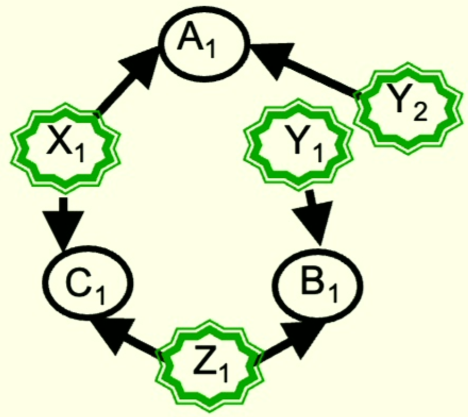
$\rho_{C|XZ}$

ρ_X

ρ_Y

ρ_Z

Quantum model M' = G → G' Inflation of M



$\rho_{A_1|X_1Y_2}$

$\rho_{B_1|Y_1Z_1}$

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ρ_{X_1}

ρ_{Y_1}

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with symmetry constraint:

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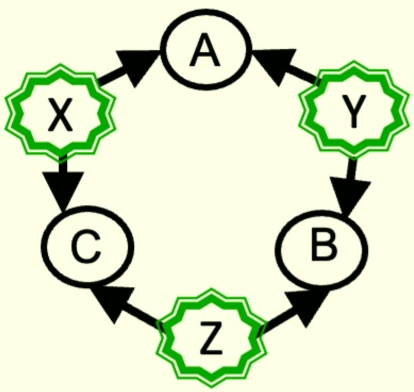
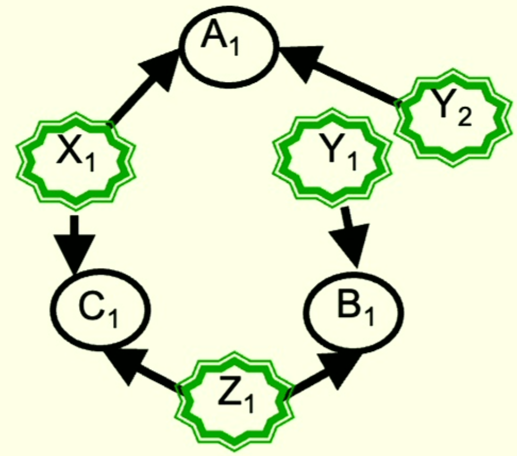
$\{A_1C_1\}$ is an injectable set

$$P_{A_1C_1} = \mathrm{Tr}_{X_1Y_2Z_1}\left(\rho_{A_1|X_1Y_2}\rho_{C_1|X_1Z_1}\rho_{X_1}\rho_{Y_2}\rho_{Z_1}\right)$$

$$P_{AC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$

$$P_{AC} \text{ compatible with } M \implies P_{A_1C_1} = P_{AC} \text{ compatible with } M'$$

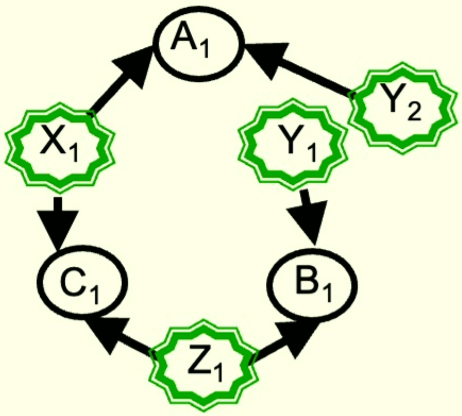




$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$
is a causal compatibility
inequality for M

$P_{A_1} + P_{B_1} + P_{C_1} - P_{A_1} P_{B_1} - P_{B_1 C_1} - P_{A_1 C_1} \leq 1$
is a causal compatibility
inequality for M'

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$



$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

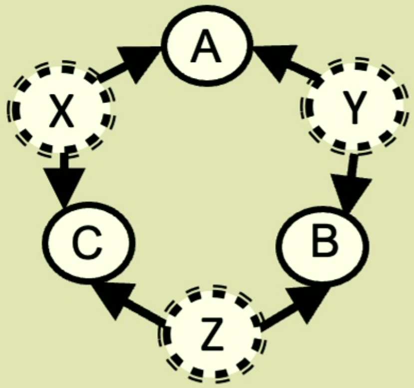
$$A_1 \perp_d B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$$

Implications from d-separation to conditional independences hold in quantum causal models

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M' $\implies$ $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$ is a causal compatibility inequality for M'

Causal model for an arbitrary generalized probabilistic theory (GPT)



$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

Inequalities derived from nonfanout inflations are causal compatibility constraints that hold *for all theories*

Violation of causal compatibility inequalities from **nonfanout** inflations

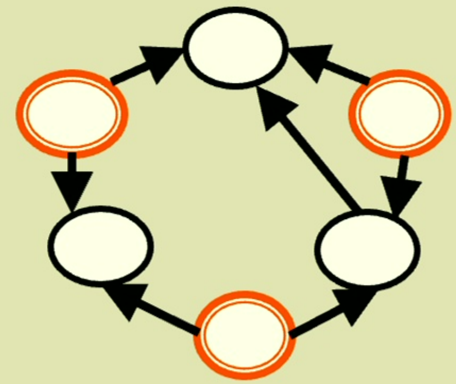
$$P_A+P_B+P_C-P_AP_B-P_{BC}-P_{AC} \leq 1$$

Witnessing quantumness ? No

Violation of causal compatibility inequalities from **nonfanout** inflations

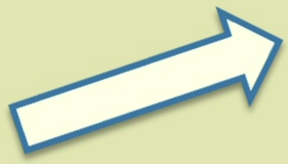
$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

Witnessing need for different structure

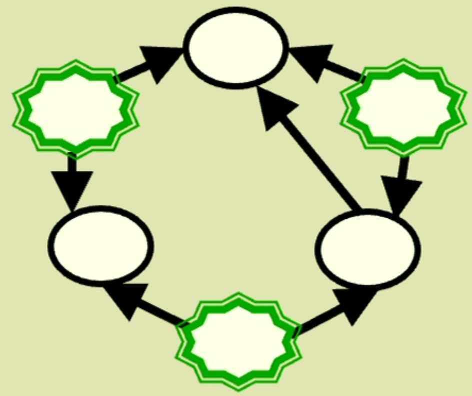


Violation of causal compatibility inequalities from **nonfanout** inflations

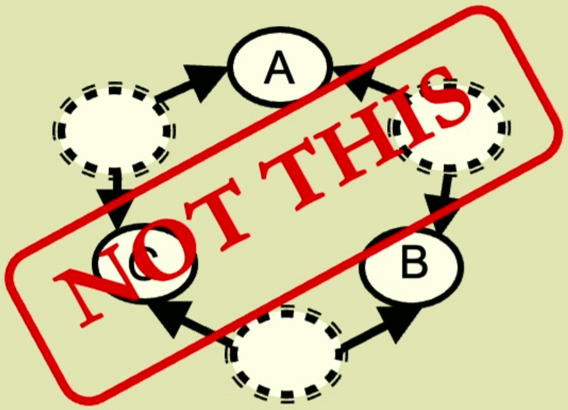
$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$



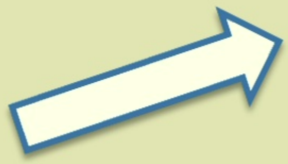
Witnessing need for different structure



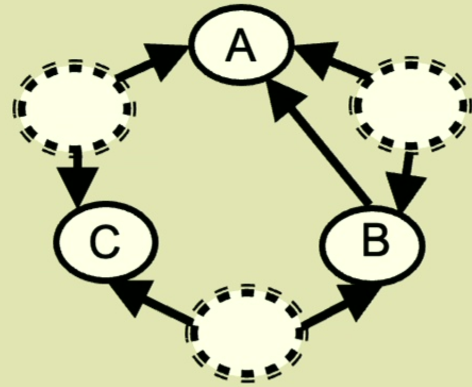
Violation of causal compatibility inequalities from **nonfanout** inflations



$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$



Witnessing need for different structure



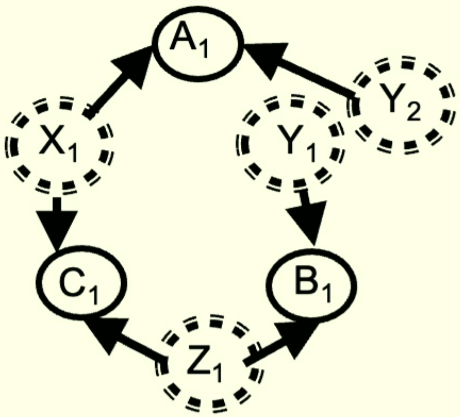
Inequalities derived from nonfanout inflations are causal compatibility constraints that hold *for all theories*

Consequently, the only inequality constraints that can serve to prove gaps between theories (e.g., quantum-classical gaps or postquantum-quantum gaps) are those derived from fanout inflations

Violation of causal compatibility inequalities from **nonfanout** inflations

$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$



$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

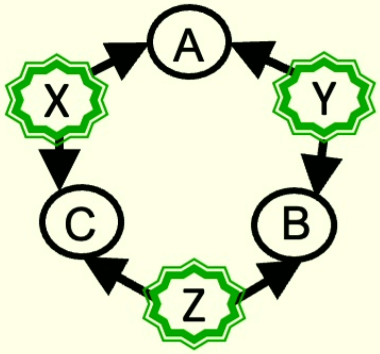
$$A_1 \perp_d B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$$

Implications from d-separation to conditional independences hold for causal models based on an arbitrary GPT

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M' $\implies$ $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$ is a causal compatibility inequality for M'

Quantum model M on DAG G



$\rho_{A|XY}$

$\rho_{B|YZ}$

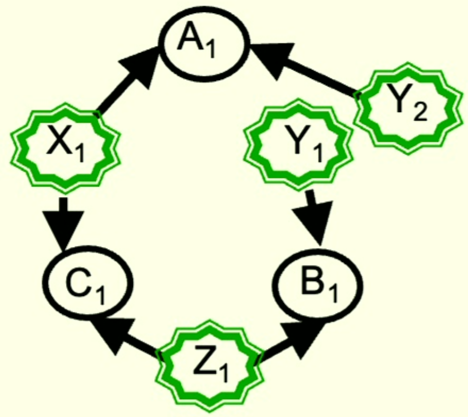
$\rho_{C|XZ}$

ρ_X

ρ_Y

ρ_Z

Quantum model M' = G → G' Inflation of M



$\rho_{A_1|X_1Y_2}$

$\rho_{B_1|Y_1Z_1}$

$\rho_{C_1|X_1Z_1}$

ρ_{X_1}

ρ_{Y_1}

ρ_{Y_2}

ρ_{Z_1}

with symmetry constraint:

$\rho_{Y_1} = \rho_{Y_2}$

Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_1C_1\}, \{B_1C_1\}$

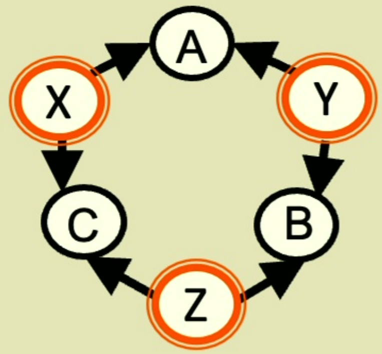
$(P_A, P_B, P_C, P_{AC}, P_{BC})$ is not compatible with M $\Longleftarrow$ $(P_{A_1}, P_{B_1}, P_{C_1}, P_{A_1C_1}, P_{B_1C_1})$

where $\mathrm{P}_{A_1} = \mathrm{P}_A$ $\quad \mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$

$\mathrm{P}_{B_1} = \mathrm{P}_B$ $\quad \mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$

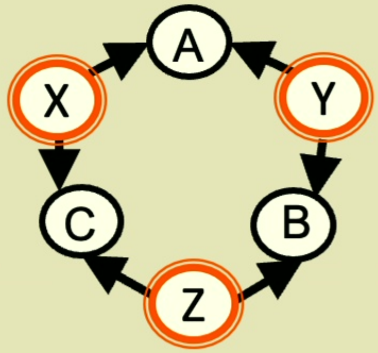
$\mathrm{P}_{C_1} = \mathrm{P}_C$

is not compatible with M'

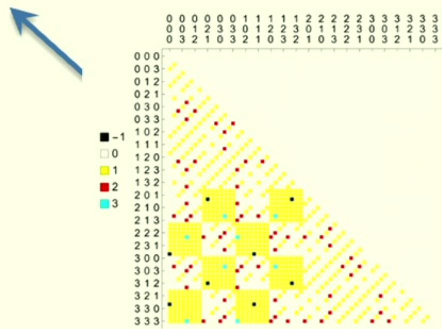


$$P_A(1)P_B(1)P_C(1) \leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) + P_{AC}(11)P_B(1) + P_{ABC}(000)$$

This inequality was obtained from the spiral inflation, which is fanout, and therefore might be quantumly violated



$$\sum_{a,b,c,a',b',c'} y_{abca'b'c'} P_{ABC}(abc) P_{ABC}(a'b'c') \geq 0$$

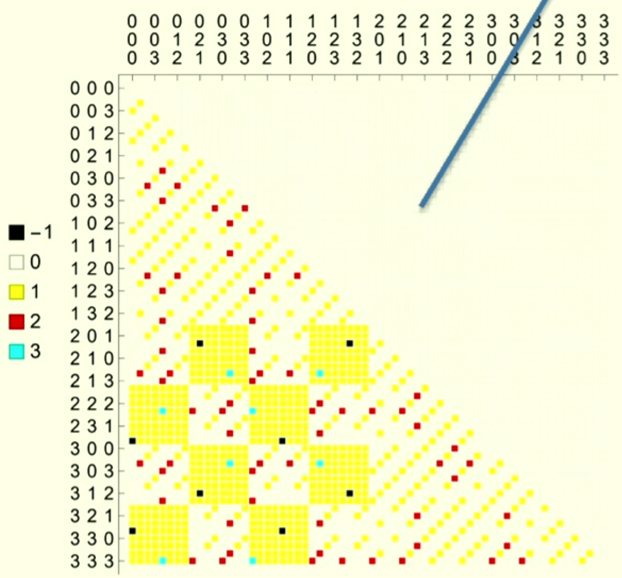


Polino et al., Nat. Comm. 14, 909 (2023)

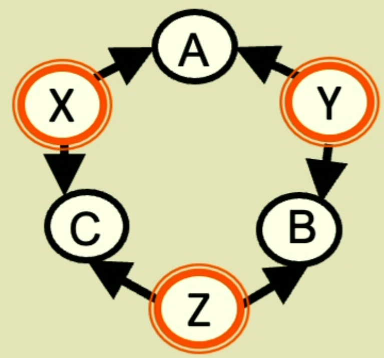
This inequality was obtained from the web inflation, which is fanout, and therefore might be quantumly violated

Data-seeded inflation technique

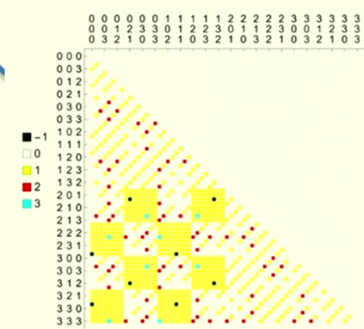
$$V := \sum_{a,b,c,a',b',c'} y_{abca'b'c'} P_{ABC}(abc) P_{ABC}(a'b'c') \geq 0$$



(b) **The coefficients of a quadratic inequality**



$$\sum_{a,b,c,a',b',c'} y_{abca'b'c'} P_{ABC}(abc) P_{ABC}(a'b'c') \geq 0$$


(b) The coefficients of a quadratic inequality

Polino et al., Nat. Comm. 14, 909 (2023)

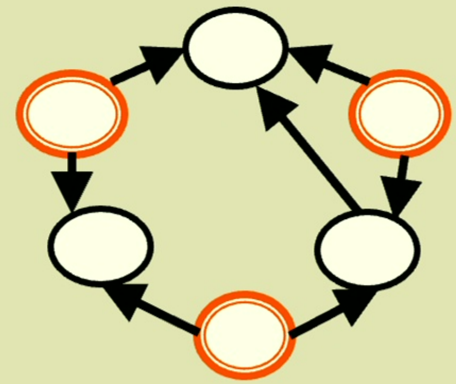
This inequality was obtained from the web inflation, which is fanout, and therefore might be quantumly violated

Violation of causal compatibility inequalities from **fanout** inflations

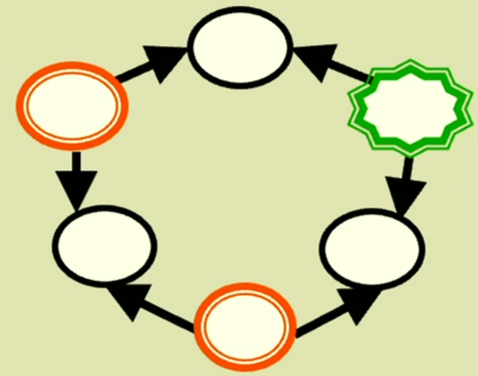
$$\sum_{a,b,c,a',b',c'} y_{abca'b'c'} P_{ABC}(abc) P_{ABC}(a'b'c') \geq 0$$

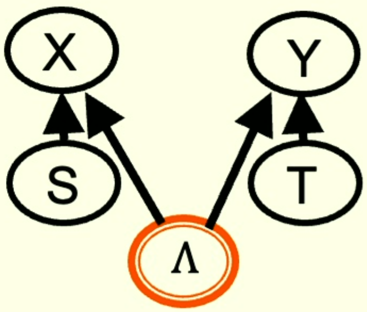
or

Witnessing need for different structure

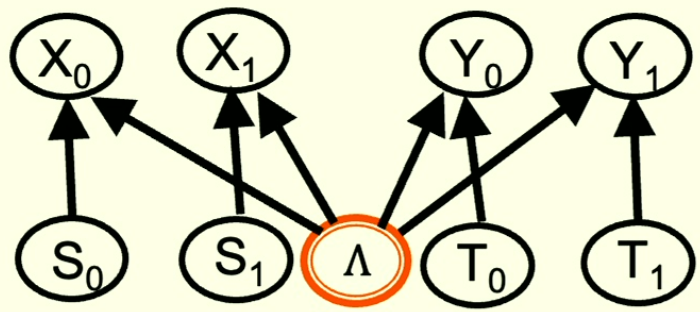


Witnessing quantumness



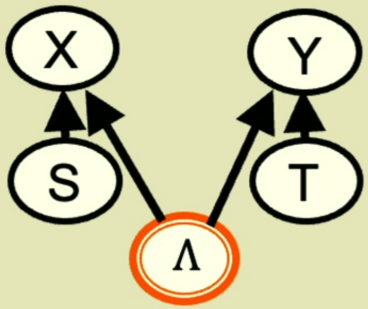


Bell



Standard inflation
of Bell

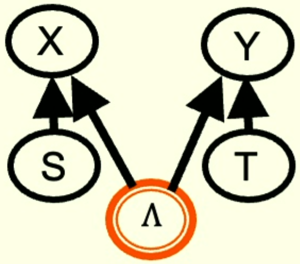
Fanout



$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

Note that the derivation of the CHSH inequalities from brute-force quantifier elimination did not provide much conceptual insight into their origin

Bell model M

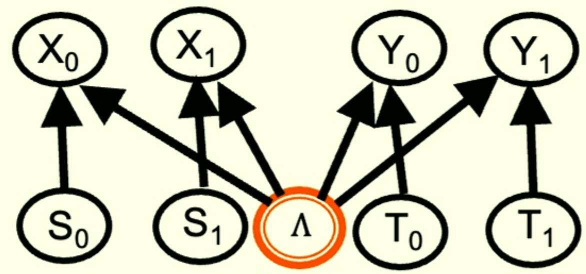


$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

$P_{XY|ST}$
compatible with M

Inflated model M'



$P_{X_0|S_0\Lambda}$
$P_{X_1|S_1\Lambda}$
$P_{Y_0|T_0\Lambda}$
$P_{Y_1|T_1\Lambda}$
P_{Λ}

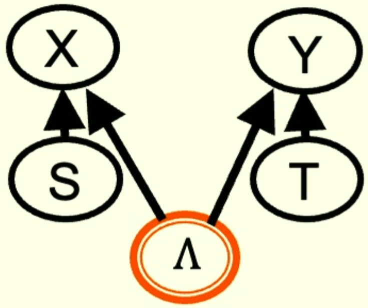
$P_{X_0|S_0\Lambda} = P_{X_1|S_1\Lambda}$
$P_{Y_0|T_0\Lambda} = P_{Y_1|T_1\Lambda}$

$$P_{X_0Y_0|S_0T_0} = \sum_{\Lambda} P_{X_0|S_0\Lambda} P_{Y_0|T_0\Lambda} P_{\Lambda}$$
$$P_{X_0Y_1|S_0T_1} = \sum_{\Lambda} P_{X_0|S_0\Lambda} P_{Y_1|T_1\Lambda} P_{\Lambda}$$
$$P_{X_1Y_0|S_1T_0} = \sum_{\Lambda} P_{X_1|S_1\Lambda} P_{Y_0|T_0\Lambda} P_{\Lambda}$$
$$P_{X_1Y_1|S_1T_1} = \sum_{\Lambda} P_{X_1|S_1\Lambda} P_{Y_1|T_1\Lambda} P_{\Lambda}$$

$\Rightarrow$

$P_{X_0Y_0|S_0T_0}, P_{X_0Y_1|S_0T_1}, P_{X_1Y_0|S_1T_0}, P_{X_1Y_1|S_1T_1}$
where $P_{X_iY_j|S_iT_j} = P_{XY|ST}$
compatible with M'

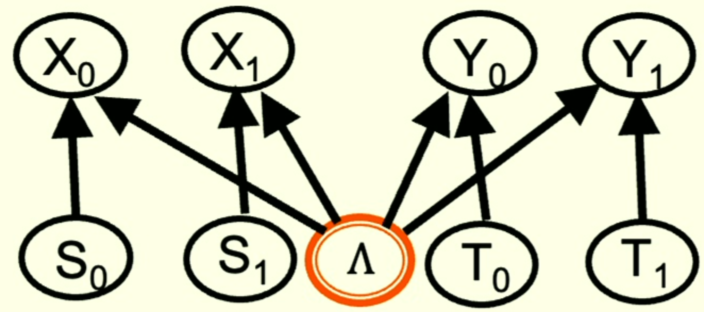
Bell model M



$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

Causal compatibility inequality in M

Inflated model M’



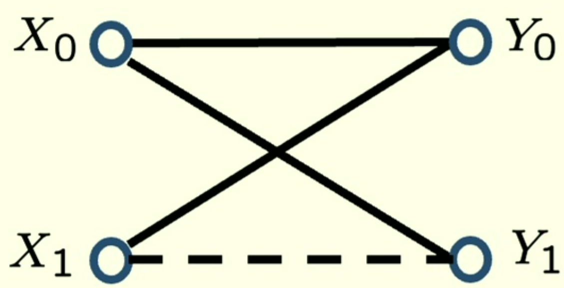
$$\frac{1}{4}\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \frac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \frac{3}{4}$$

Causal compatibility inequality in M’

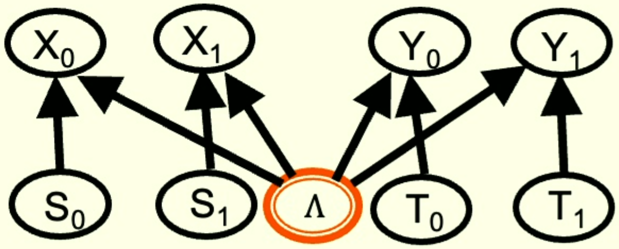
Consider any distribution over 4 variables $Q_{X_0 X_1 Y_0 Y_1}$

The marginals $Q_{X_0Y_0}, Q_{X_0Y_1}, Q_{X_1Y_0}, Q_{X_0Y_1}$ satisfy

$$\tfrac{1}{4}\sum_{x=y} Q_{X_0Y_0}(xy) + \tfrac{1}{4}\sum_{x=y} Q_{X_0Y_1}(xy)$$
$$\tfrac{1}{4}\sum_{x=y} Q_{X_1Y_0}(xy) + \tfrac{1}{4}\sum_{x\neq y} Q_{X_1Y_1}(xy) \leq \tfrac{3}{4}$$



Inflated model M'



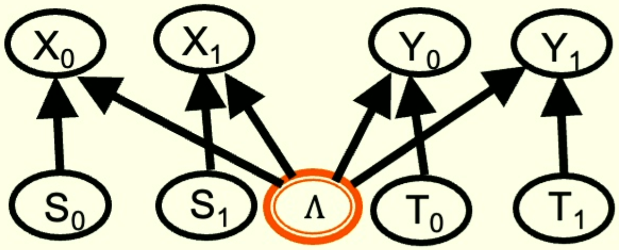
$$X_0Y_0 \perp S_1T_1|S_0T_0 \implies P_{X_0Y_0|S_0T_0S_1T_1} = P_{X_0Y_0|S_0T_0}$$

$$X_0Y_1 \perp S_1T_0|S_0T_1 \implies P_{X_0Y_1|S_0T_0S_1T_1} = P_{X_0Y_1|S_0T_1}$$

$$X_1Y_0 \perp S_0T_1|S_1T_0 \implies P_{X_1Y_0|S_0T_0S_1T_1} = P_{X_1Y_0|S_1T_0}$$

$$X_1Y_1 \perp S_0T_0|S_1T_1 \implies P_{X_1Y_1|S_0T_0S_1T_1} = P_{X_1Y_1|S_1T_1}$$

Inflated model M'



$$X_0Y_0 \perp S_1T_1|S_0T_0 \implies P_{X_0Y_0|S_0T_0S_1T_1} = P_{X_0Y_0|S_0T_0}$$

$$X_0Y_1 \perp S_1T_0|S_0T_1 \implies P_{X_0Y_1|S_0T_0S_1T_1} = P_{X_0Y_1|S_0T_1}$$

$$X_1Y_0 \perp S_0T_1|S_1T_0 \implies P_{X_1Y_0|S_0T_0S_1T_1} = P_{X_1Y_0|S_1T_0}$$

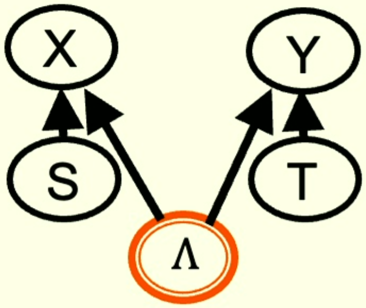
$$X_1Y_1 \perp S_0T_0|S_1T_1 \implies P_{X_1Y_1|S_0T_0S_1T_1} = P_{X_1Y_1|S_1T_1}$$

$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_0Y_0|S_0S_1T_0T_1}(xy|0101) + \tfrac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0S_1T_0T_1}(xy|0101)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_1Y_0|S_0S_1T_0T_1}(xy|0101) + \tfrac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_0S_1T_0T_1}(xy|0101) \leq \tfrac{3}{4}$$

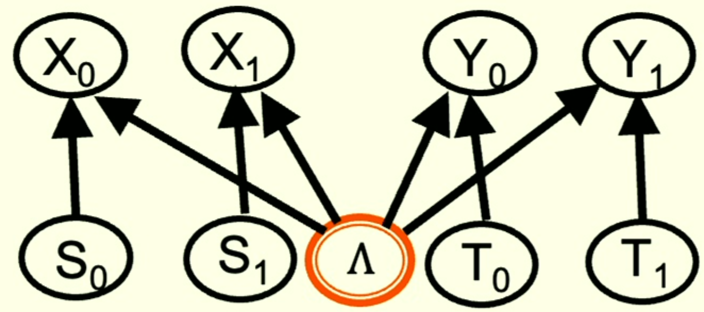
$\Rightarrow$

$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \tfrac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \tfrac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \tfrac{3}{4}$$

Bell model M



Inflated model M'



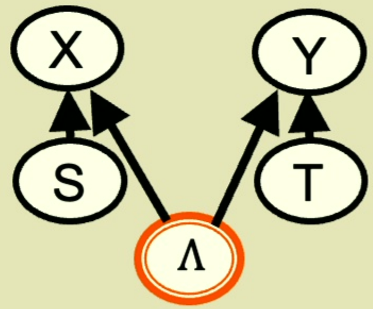
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{XY|ST}(xy|00) + \tfrac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{XY|ST}(xy|10) + \tfrac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \tfrac{3}{4}$$

Causal compatibility inequality in M

⇐

$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \tfrac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \tfrac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \tfrac{3}{4}$$

Causal compatibility inequality in M'



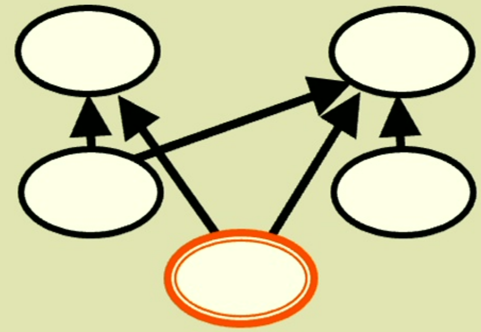
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

Violation of causal compatibility inequalities from **fanout** inflations

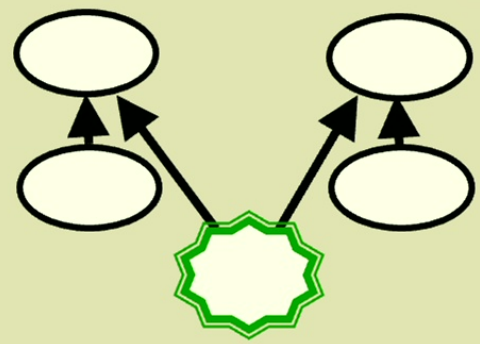
$$\tfrac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \tfrac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\tfrac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \tfrac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \tfrac{3}{4}$$

or

Witnessing need for different structure



Witnessing quantumness



Violation of causal compatibility inequalities from **nonfanout** inflations

?

No! All nonfanout inflations of the Bell DAG lead to trivial constraints

Consequently, there are no nontrivial bounds in the Bell scenario that hold for all theories.

Quantum inflation technique

Wolfe, Pozas-Kerstjens, Grinberg, Rosset, Acín, Navascués, Phys. Rev. X **11**, 021043 (2021)

Generalization of NPA semidefinite programming hierarchy for Bell scenario

Navascués, Pironio, and Acín, New J. Phys. 10, 073013 (2008)

Quantum inflation technique
Wolfe, Pozas-Kerstjens, Grinberg, Rosset, Acín, Navascués, Phys. Rev. X **11**, 021043 (2021)

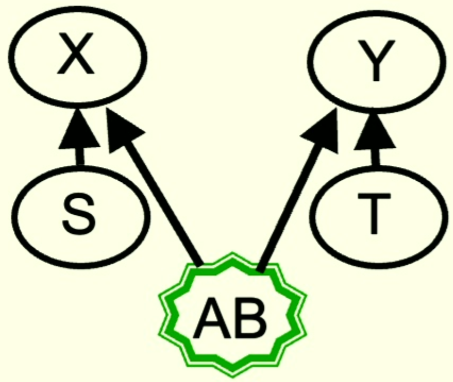
Generalization of NPA semidefinite programming hierarchy for Bell scenario
Navascués, Pironio, and Acín, New J. Phys. 10, 073013 (2008)

For convergence result, see:
Ligthart, Gachechiladze, and Gross, arXiv:2110.14659 (2021)

GPT-latent-permitting causal models (Generalized Probabilistic Theories)

Henson, Lal and Pusey, New J. Phys. 16, 113043 (2014)
Fritz, Comm. Math. Phys. 341, 391 (2016)

Quantum-latent Bell model



$\rho_{X|SA}$

$\rho_{Y|TB}$

ρ_{AB}

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

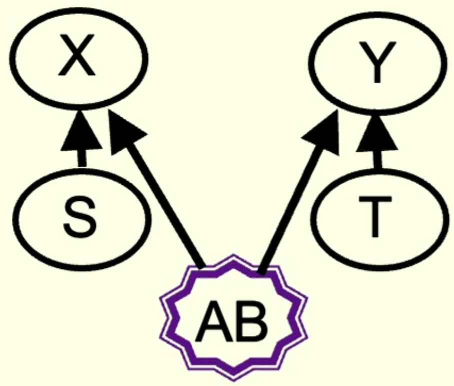
$X \perp T|S$

$Y \perp S|T$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq 0.85$$

Can violate Bell inequalities. Satisfies new inequalities.

Boxworld-latent Bell model



$$\mathbf{r}^A_{x|s} \in \mathbb{R}^{d_A}$$

$$\mathbf{r}^B_{y|t} \in \mathbb{R}^{d_B}$$

$$\mathbf{s}^{AB} \in \mathbb{R}^{d_A} \otimes \mathbb{R}^{d_B}$$

Boxworld

- Only product effects
- All states that are positive on product effects

$$P_{XY|ST}(xy|st) = (\mathbf{r}^A_{x|s} \otimes \mathbf{r}^B_{y|t}) \cdot \mathbf{s}^{AB}$$

$X \perp T|S$

$Y \perp S|T$

$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{XY|ST}(xy|00) + \tfrac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{XY|ST}(xy|10) + \tfrac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq 1$$

No nontrivial inequality constraints

J. Barrett, Phys. Rev. A 75, 032304 (2005)

Popescu-Rohrlich box

$$P^{\mathrm{PR}}_{XY|ST}(xy|st) = \begin{cases} \frac{1}{2}[00] + \frac{1}{2}[11] \text{ if } (s,t) \in \{(0,0),(0,1),(1,0)\} \\ \frac{1}{2}[01] + \frac{1}{2}[10] \text{ if } (s,t) = (1,1) \end{cases}$$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) = \underline{1}$$

compatible with boxworld Bell model

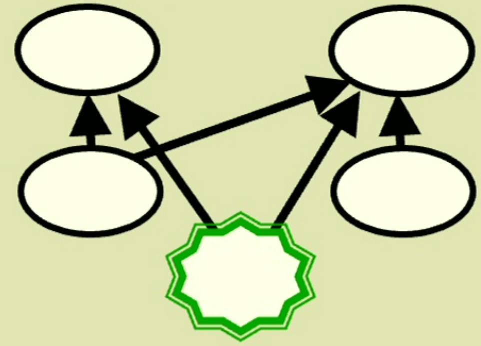
incompatible with quantum Bell model

Violation of causal compatibility inequalities from **nonfanout** inflations

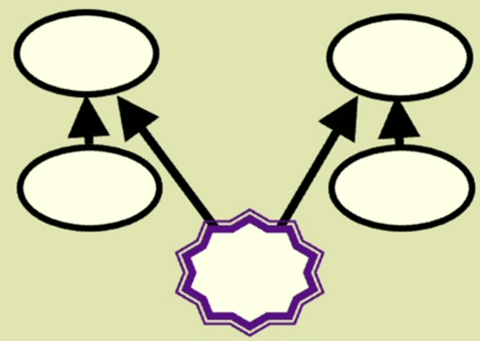
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

or

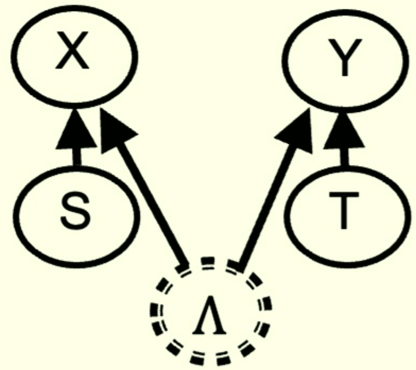
Witnessing need for different structure



Witnessing **post-quantumness**

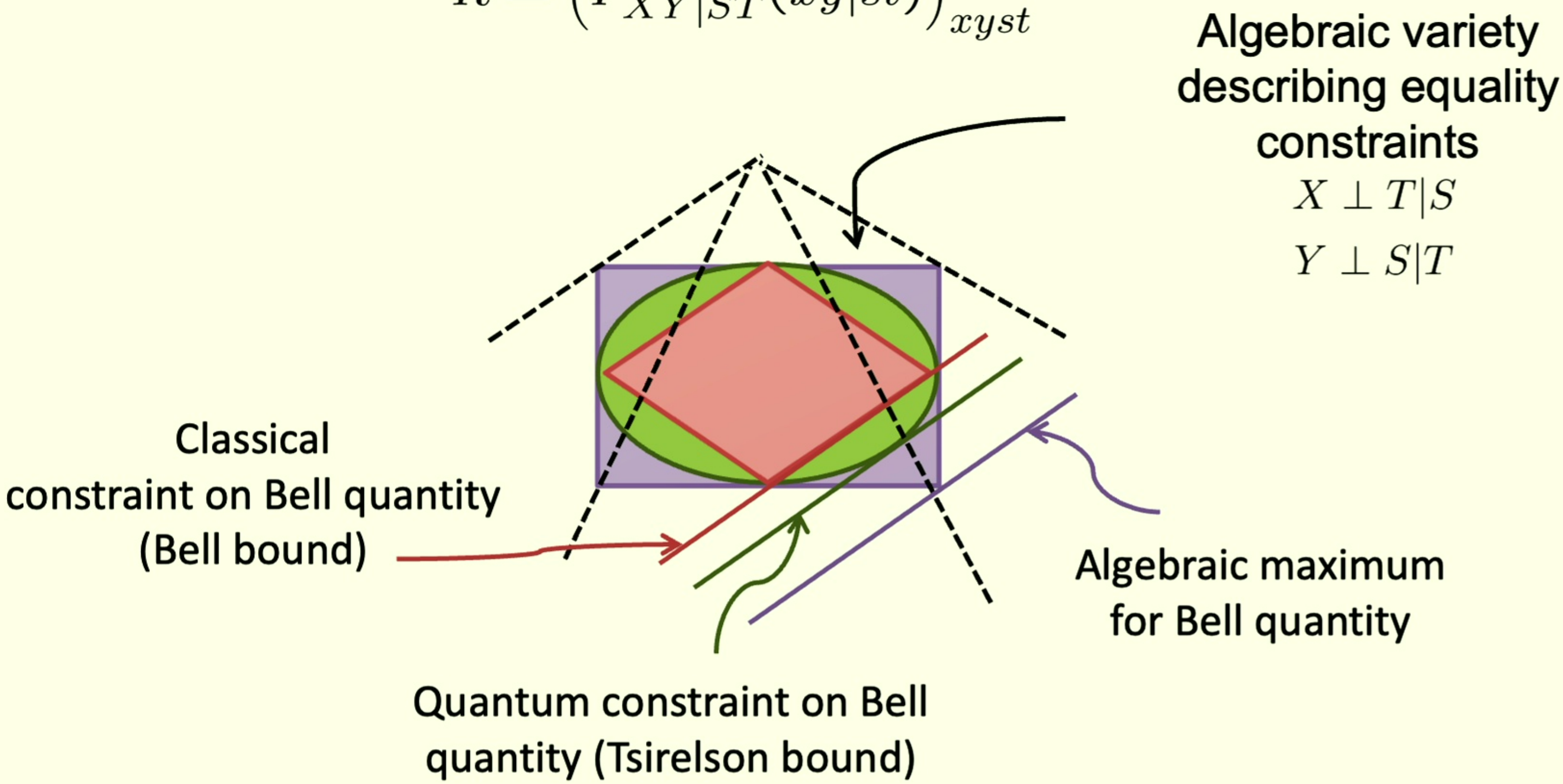


Bell model



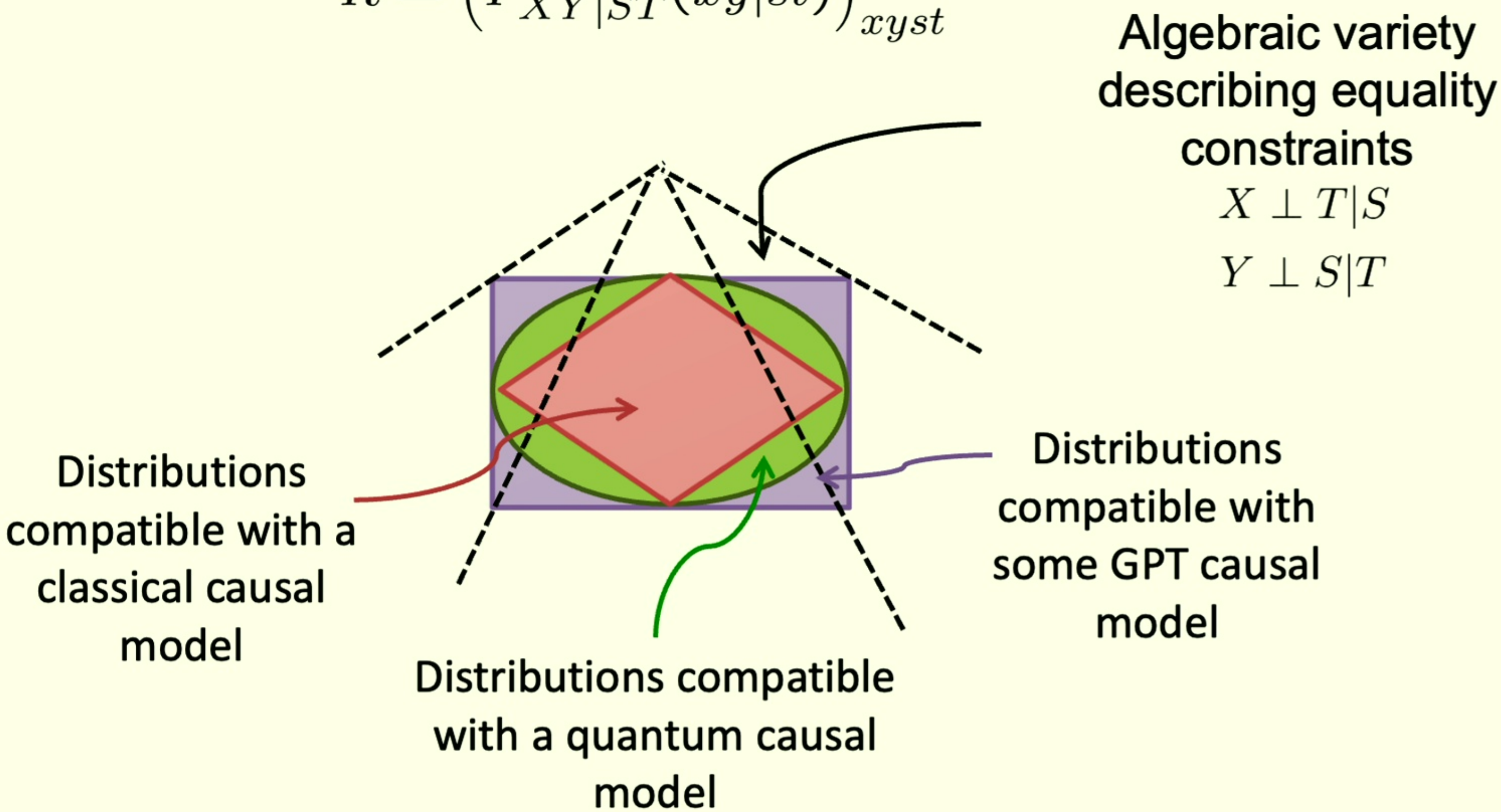
Space of compatible probability distributions

$$\vec{R} = \left(P_{XY|ST}(xy|st)\right)_{xyst}$$



Space of compatible probability distributions

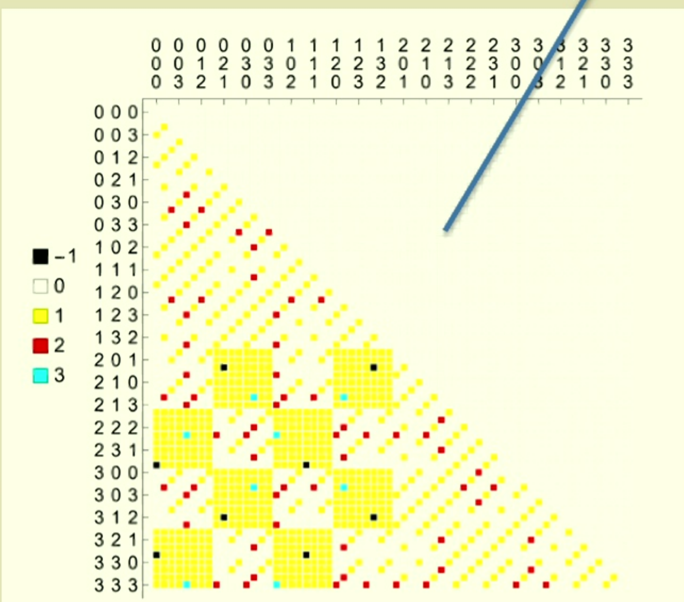
$$\vec{R} = \left(P_{XY|ST}(xy|st)\right)_{xyst}$$



Next: Causal compatibility in causal models with quantum latents and quantum visibles

Data-seeded inflation technique

$$V := \sum_{a,b,c,a',b',c'} y_{abca'b'c'} P_{ABC}(abc) P_{ABC}(a'b'c') \geq 0$$



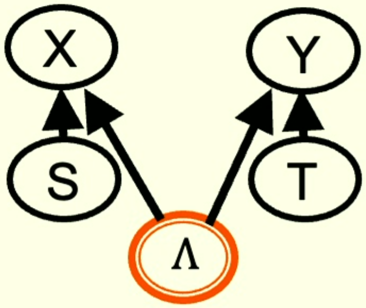
(b) **The coefficients of a quadratic inequality**

This causal compatibility inequality is violated by the data

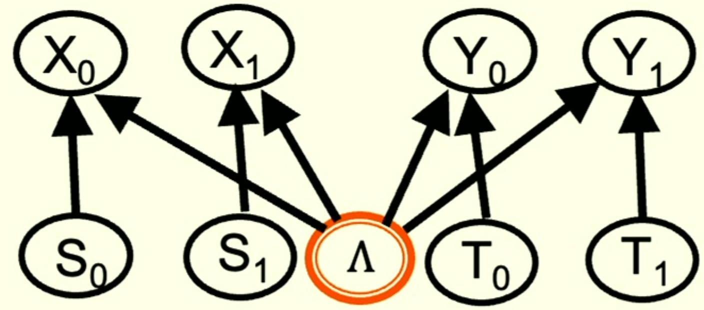
$$V_{\text{exp}} = -0.02436 \pm 0.00016$$

Polino et al., Nat. Comm. 14, 909 (2023)

Bell model M



Inflated model M'

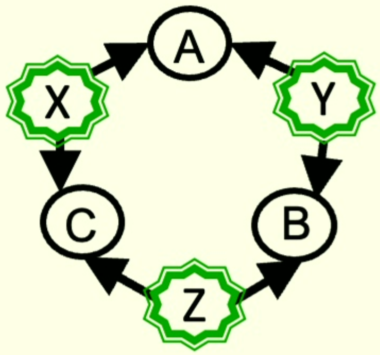


$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$
$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$

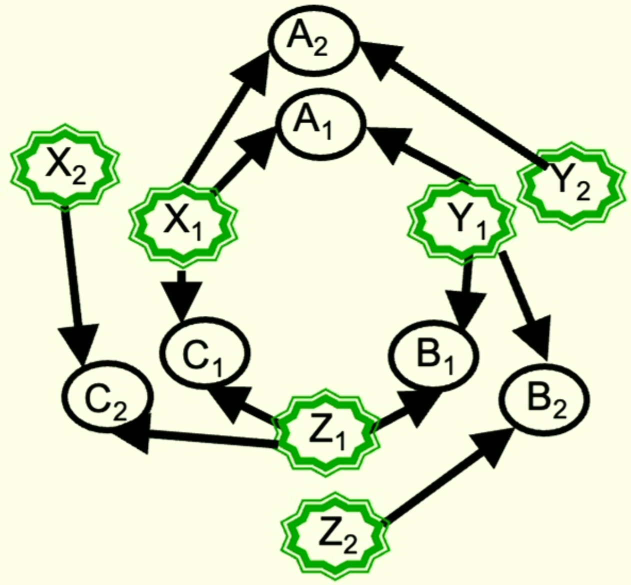
Causal compatibility inequality in M

$\frac{1}{4}\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \frac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$
$\frac{1}{4}\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \frac{3}{4}$

Causal compatibility inequality in M'



Triangle



Spiral inflation of Triangle

Fan-out