Title: Lecture - Causal Inference, PHYS 777

Speakers: Robert Spekkens

Collection/Series: Causal Inference (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Quantum Foundations

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Quantum causal models, part 2

Joint states for systems that are common-cause connected

	Classical	Quantum
State of knowledge	P_A	ρ_A
Normalization	$\sum_A P_A = 1$	$\mathrm{Tr}_A(\rho_A) = 1$
Joint state	P_{AB}	ρ_{AB}
Marginalization	$P_B = \sum_A P_{AB}$	$\rho_B = \mathrm{Tr}_A(\rho_{AB})$

Conditional states for systems that are cause-effect connected

Classical belief propagation

$$P_B = \Gamma_{B|A}[P_A]$$

Stochastic map preserving positivity and normalization

In terms of a conditional

$$P_B = \sum_A P_{B|A} P_A$$
$$\sum_B P_{B|A} = 1$$
$$P_{B|A} \geq 0$$

Quantum belief propagation

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

Completely positive trace-preserving map

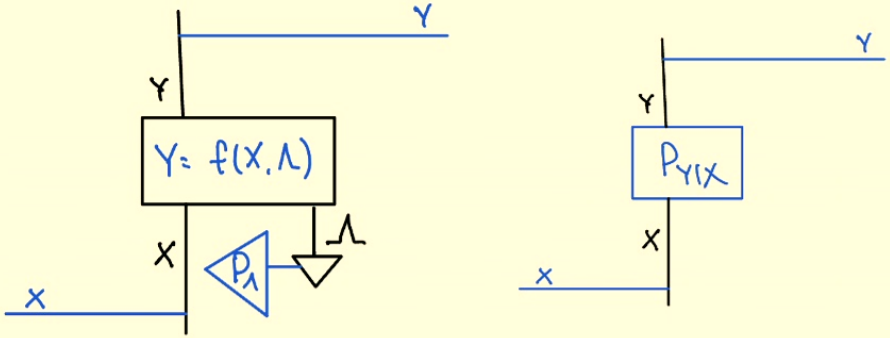
In terms of a conditional

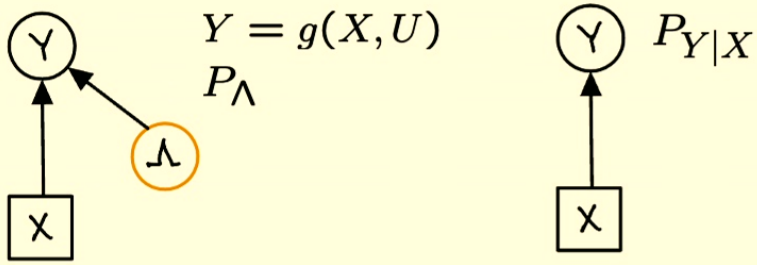
$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$
$$\mathrm{Tr}_B(\rho_{B|A}) = I_A$$
$$\rho_{B|A}^{T_A} \geq 0$$

The quantum analogue of the difference between structural equation models and Bayesian causal models

structural equation model

Bayesian causal model





$Y = g(X, U)$

P_Λ

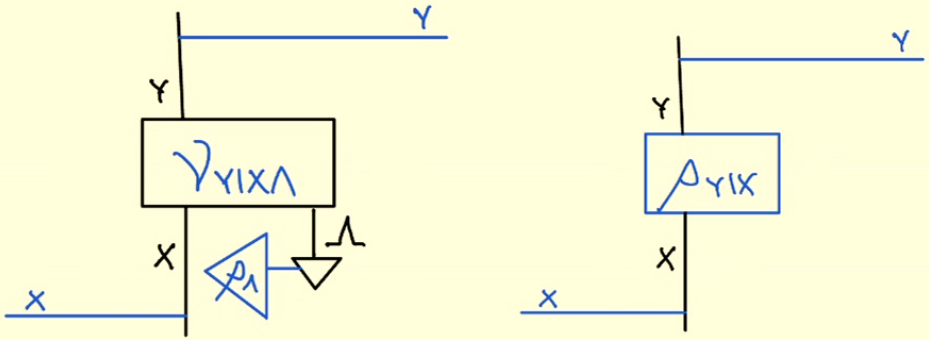
$P_{Y|X}$

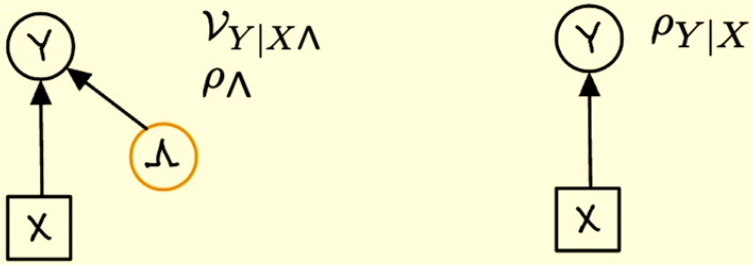
$$P_{Y|X} = \sum_\Lambda P^{\text{det}}_{Y|X\Lambda} P_\Lambda$$

where $P^{\text{det}}_{Y|X\Lambda} = \delta_{Y, f(X,\Lambda)}$

structural equation model

Bayesian causal model





$\mathcal{V}_{Y|X\Lambda}$

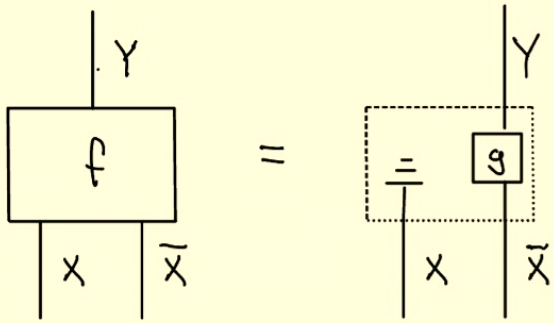
ρ_Λ

$\rho_{Y|X}$

$$\rho_{Y|X} = \text{Tr}_\Lambda \left(\rho^{\text{det}}_{Y|X\Lambda} \rho_\Lambda \right)$$

where $\rho^{\text{det}}_{Y|X\Lambda}$ is CJ-isomorphic to $\mathcal{V}_{Y|X\Lambda}$

variable X has no influence on variable Y

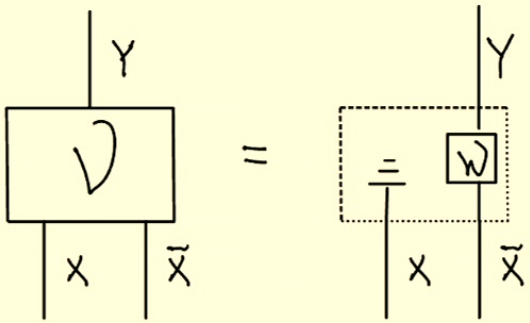


$$f(X,\bar{X}) = g(\bar{X})$$

$$P^{\text{det}}_{Y|X\bar{X}} = P^{\text{det}}_{Y|\bar{X}}$$

$$Y \perp X|\bar{X}$$

system X has no influence on system Y



$$\mathcal{V}_{Y|X\bar{X}} = \mathrm{Tr}_X \otimes \mathcal{V}_{Y|\bar{X}}$$

$$\rho^{\text{det}}_{Y|X\bar{X}} = \rho^{\text{det}}_{Y|\bar{X}}$$

$$Y \perp X|\bar{X}$$

One system a complete common cause of two others

B and C are conditionally independent given A

$$\rho_{BC|A} = \rho_{B|A}\rho_{C|A}$$

Denote this $(B \perp C|A)$

$$I(B:C|A) = 0$$

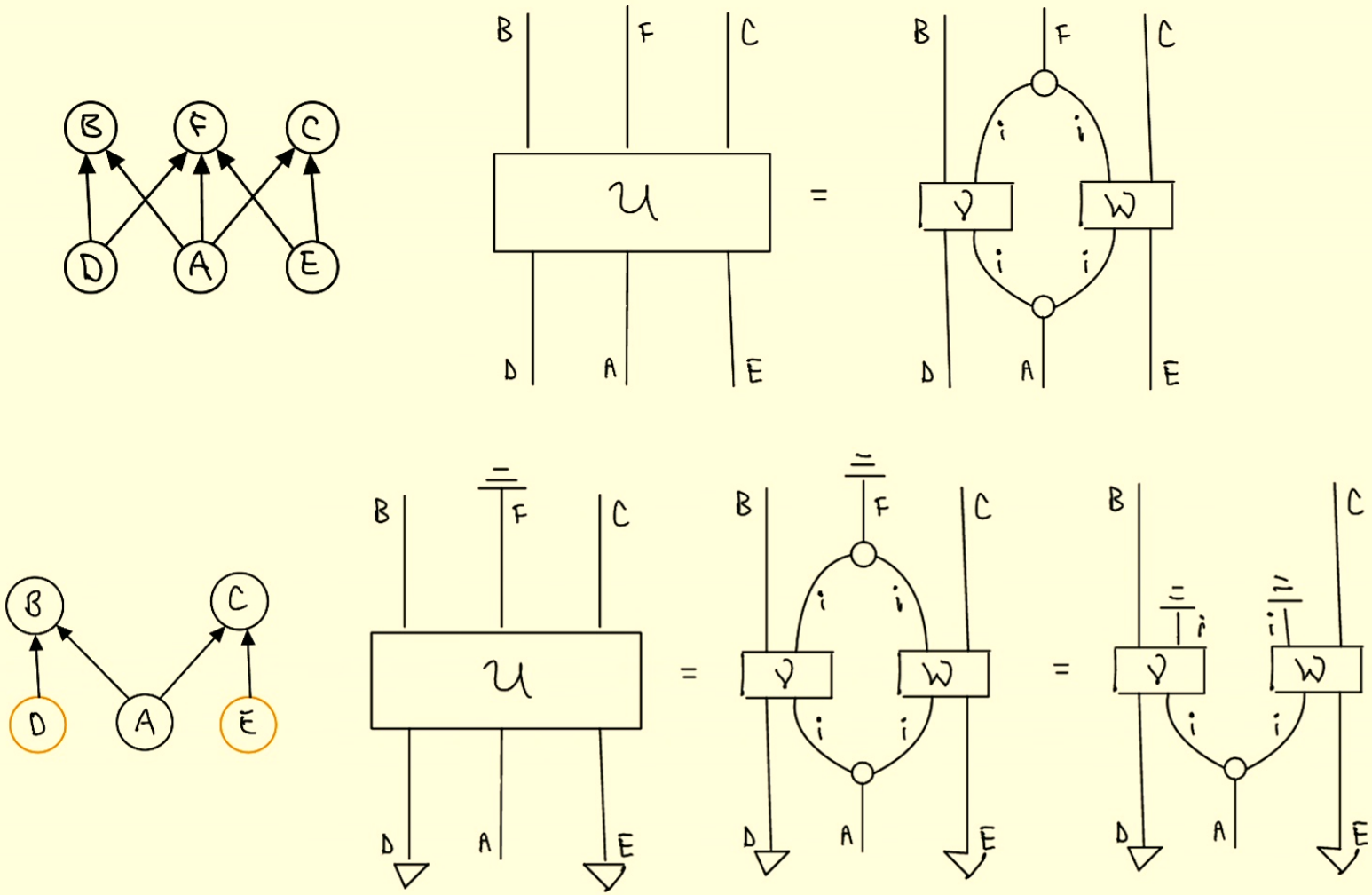
where

$$I(B:C|A) := S(AC) + S(AB) - S(ABC) - S(A)$$

for $$\rho_{ABC} = \rho_{BC|A}^{T_A}\frac{1}{d_A}$$

$$= \mathrm{Tr}_{A'}\left[\rho_{BC|A'}(|\Phi^+\rangle_{AA'}\langle\Phi^+|)\right]$$

Why should a rational agent organize their beliefs so that A being a complete common cause of B and C implies that B and C are conditionally independent given A?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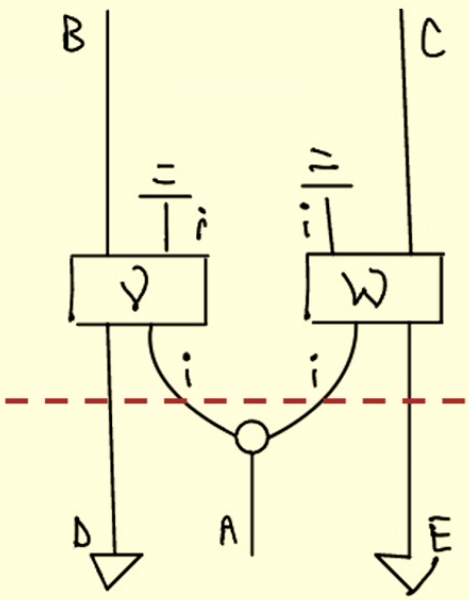


$\mathcal{V}$ is block-diagonal across the *i* sectors and is nontrivial only on $\mathcal{H}_{A_i^L}$

$$\mathcal{H}_D \otimes \left(\bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R} \right) \otimes \mathcal{H}_E$$

$$\rho^{\text{det}}_{BC|ADE} := (\mathcal{U}_{BC|A'D'E'} \otimes \text{id}_{ADE})(\textstyle\sum_{j,k} |j\rangle\langle k|_{A'D'E'} \otimes |k\rangle\langle j|_{ADE})$$

$$\rho^{\text{det}}_{BC|ADE} = \rho^{\text{det}}_{B|AD} \rho^{\text{det}}_{C|AE}$$

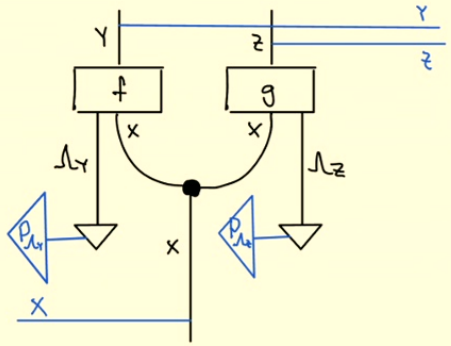
$$\rho^{\text{det}}_{B|AD} = \textstyle\sum_i \rho^{\text{det}}_{B|DA_i^L} \otimes I_{A_i^R}$$

$$\rho^{\text{det}}_{C|AE} = \textstyle\sum_i I_{A_i^R} \otimes \rho^{\text{det}}_{C|A_i^R E}$$

$$P_{YZ|X} = \sum_{\lambda_Y \lambda_Z} P^{\text{det}}_{YZ|X\lambda_Y\lambda_Z} P_{\lambda_Y\lambda_Z}$$

$$P^{\text{det}}_{YZ|X\lambda_Y\lambda_Z} = P^{\text{det}}_{Y|X\lambda_Y} P^{\text{det}}_{Z|X\lambda_Z}$$

$$P_{\lambda_Y\lambda_Z} = P_{\lambda_Y} P_{\lambda_Z}$$



$$P_{YZ|X} = \sum_{\lambda_Y,\lambda_Z} P^{\text{det}}_{Y|X\lambda_Y} P^{\text{det}}_{Z|X\lambda_Z} P_{\lambda_Y} P_{\lambda_Z}$$

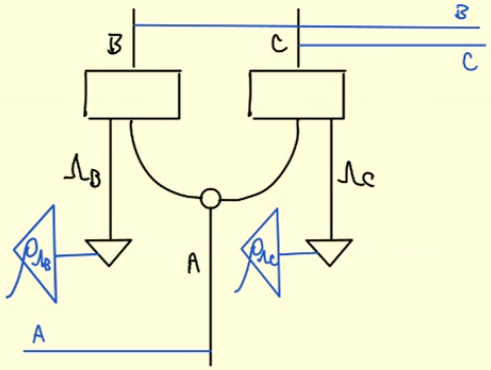
$$= \left(\sum_{\lambda_Y} P^{\text{det}}_{Y|X\lambda_Y} P_{\lambda_Y}\right)\left(\sum_{\lambda_Z} P^{\text{det}}_{Z|X\lambda_Z} P_{\lambda_Z}\right)$$

$$= P_{Y|X} P_{Z|X}$$

$$\rho_{BC|A} = \text{Tr}_{\lambda_B\lambda_C}\left(\rho^{\text{det}}_{BC|A\lambda_B\lambda_C}\rho_{\lambda_B\lambda_C}\right)$$

$$\rho^{\text{det}}_{BC|A\lambda_B\lambda_C} = \rho^{\text{det}}_{B|A\lambda_B}\rho^{\text{det}}_{C|A\lambda_C}$$

$$\rho_{\lambda_B\lambda_C} = \rho_{\lambda_B} \otimes \rho_{\lambda_C}$$

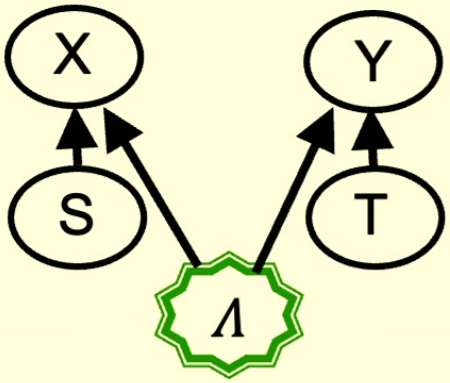


$$\rho_{BC|A} = \text{Tr}_{\lambda_B\lambda_C}\left(\rho^{\text{det}}_{B|A\lambda_B}\rho^{\text{det}}_{C|A\lambda_C}(\rho_{\lambda_B} \otimes \rho_{\lambda_C})\right)$$

$$= \text{Tr}_{\lambda_B}(\rho^{\text{det}}_{B|A\lambda_B}\rho_{\lambda_B})\text{Tr}_{\lambda_C}(\rho^{\text{det}}_{C|A\lambda_C}\rho_{\lambda_C})$$

$$= \rho_{B|A}\rho_{C|A}$$

Quantum causal model



$\rho_{X|S\Lambda}$

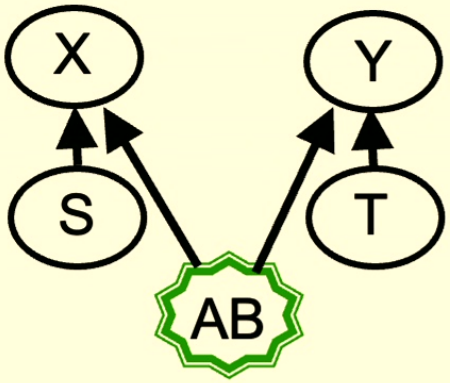
$\rho_{Y|T\Lambda}$

ρ_{Λ}

$$[\rho_{X|S\Lambda}, \rho_{Y|T\Lambda}] = 0$$

$$P_{XY|ST} = \mathrm{Tr}_{\Lambda}(\rho_{X|S\Lambda}\rho_{Y|T\Lambda}\rho_{\Lambda})$$

Quantum causal model



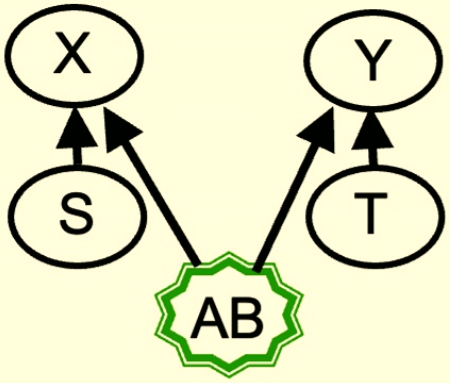
$\{E^A_{x|s}\}_x$ for each s

$\{E^B_{y|t}\}_y$ for each t

ρ_{AB}

$$P_{XY|ST}(xy|st) = \mathrm{Tr}_{AB}((E^A_{x|s} \otimes E^B_{y|t})\rho_{AB})$$

Quantum causal model



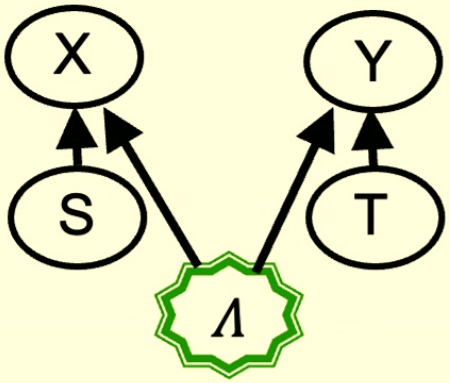
$\rho_{X|SA}$

$\rho_{Y|TB}$

ρ_{AB}

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Quantum causal model



$\rho_{X|S\Lambda}$

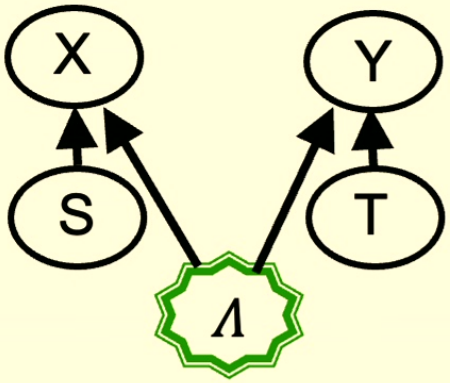
$\rho_{Y|T\Lambda}$

ρ_{Λ}

$$[\rho_{X|S\Lambda}, \rho_{Y|T\Lambda}] = 0$$

$$P_{XY|ST} = \mathrm{Tr}_{\Lambda}(\rho_{X|S\Lambda}\rho_{Y|T\Lambda}\rho_{\Lambda})$$

Quantum causal model



$\rho_{X|S\Lambda}$

$\rho_{Y|T\Lambda}$

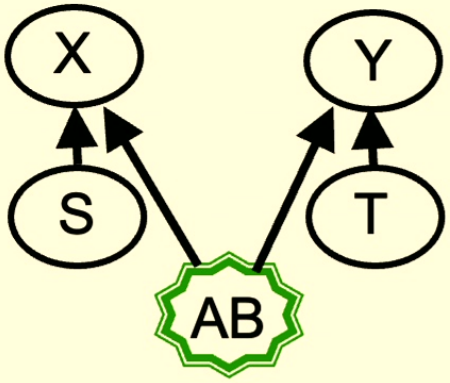
ρ_{Λ}

$$[\rho_{X|S\Lambda}, \rho_{Y|T\Lambda}] = 0$$

$$P_{XY|ST} = \mathrm{Tr}_{\Lambda}(\rho_{X|S\Lambda}\rho_{Y|T\Lambda}\rho_{\Lambda})$$

Leifer and RWS, PRA 88, 052130 (2013)
Henson, Lal & Pusey NJP 16, 113043 (2014)
Costa, Shrapnel NJP 18(6) (2016)
Allen, Barrett, Horsman, Lee & RWS, PRX 7, 031021 (2017)
Barrett, Lorenz, Oreshkov, arXiv:1906.10726

Quantum causal model



$\rho_{X|SA}$

$\rho_{Y|TB}$

ρ_{AB}

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Causal compatibility constraints:

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

Tsirelson, Lett. Math. Phys. 4, 93 (1980)

Remaining questions about the inferential theory:

What is a joint state over systems that are cause-effect related?

What is a conditional state between systems that are common-cause related?

Relation of conditional to joint	$P_{B\|A} = \frac{P_{AB}}{P_A}$	$\rho_{B\|A} = \rho_A^{-1/2} \rho_{AB} \rho_A^{-1/2}$ $\rho_{B\|A} \geq 0 \quad \text{(positive)}$
	$P_{AB} = P_{B\|A} P_A$	$\rho_{AB} = \rho_A^{1/2} \rho_{B\|A} \rho_A^{1/2}$
Normalization condition	$\sum_B P_{B\|A} = 1$	$\mathrm{Tr}_B(\rho_{B\|A}) = I_A$
Belief propagation	$P_B = \sum_A P_{B\|A} P_A$ $P_B = \Gamma_{B\|A}[P_A]$	$\rho_B = \mathrm{Tr}_A(\rho_{B\|A} \rho_A)$ $\rho_B = \mathcal{E}_{B\|A}(\rho_A)$ $\mathcal{E}_{B\|A} \circ T_A \in \mathrm{CP} \quad \text{(co-CP)}$
Bayesian inversion	$P_{B\|A} = \frac{P_{A\|B} P_B}{P_A}$	$\rho_{B\|A} = \rho_A^{-1/2} \rho_B^{1/2} \rho_{A\|B} \rho_B^{1/2} \rho_A^{-1/2}$

See: Leifer, PRA 74, 042310 (2006), Leifer & RWS, PRA A 88, 052130 (2013)

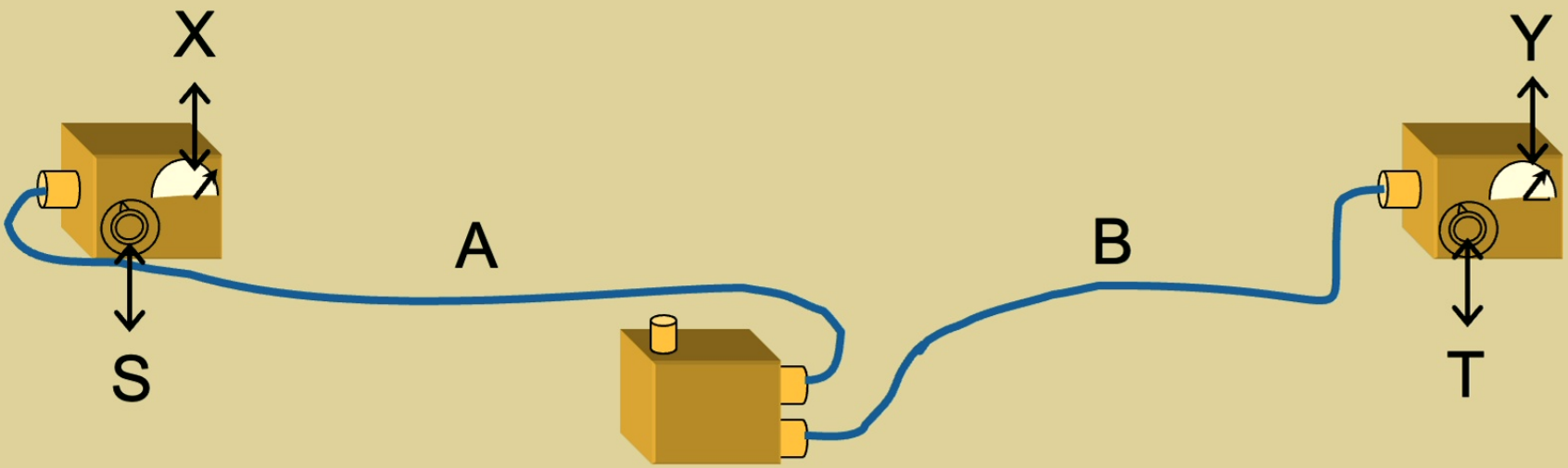
Joint states for systems that are cause-effect connected

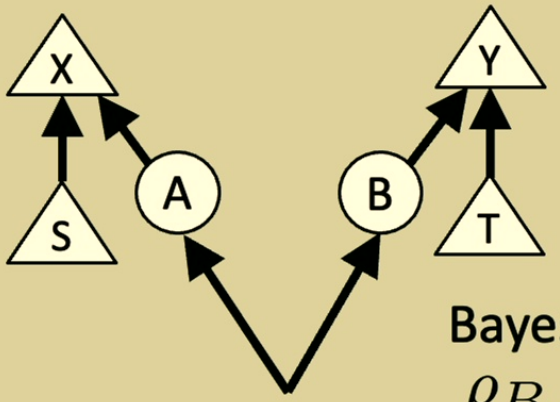
“Quantum states over time”

Still unclear how to do this in general

Problems with various proposals are highlighted here:
Horsman et al., Proc. R. Soc. A **473**, 20170395 (2017)

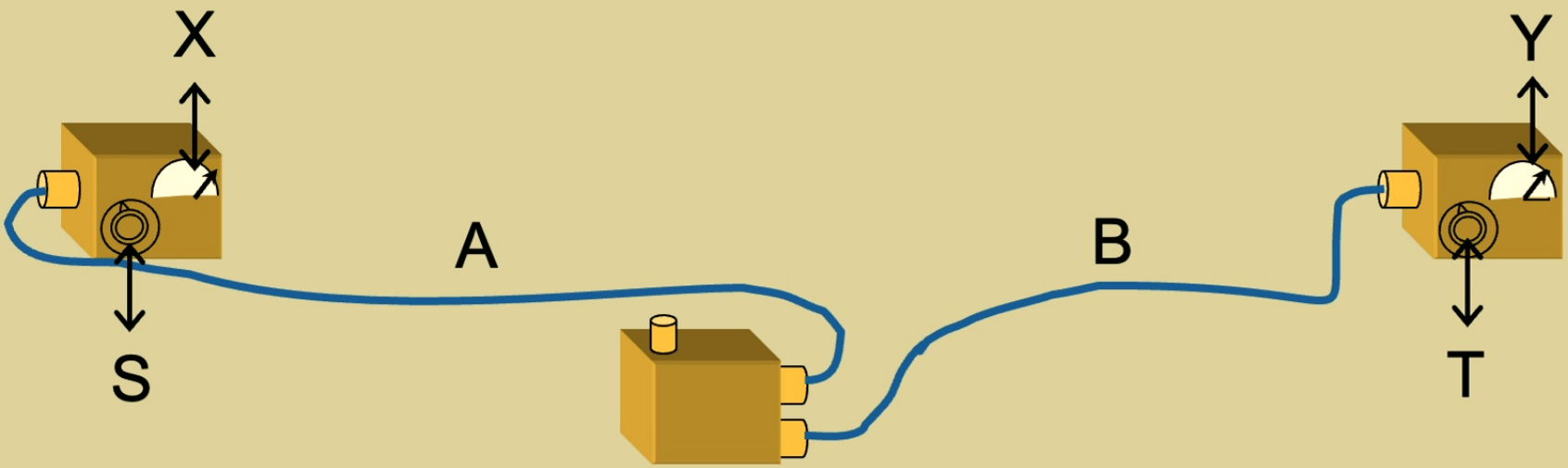
The objective is to find a solution that can be part of a full quantum theory of inference

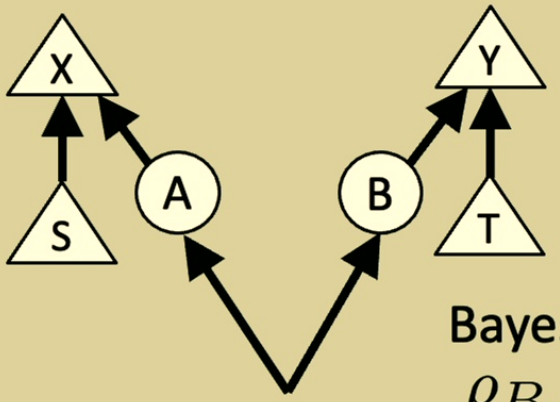




Bayesian updating

$$\rho_B \to \rho_{B|SX}$$





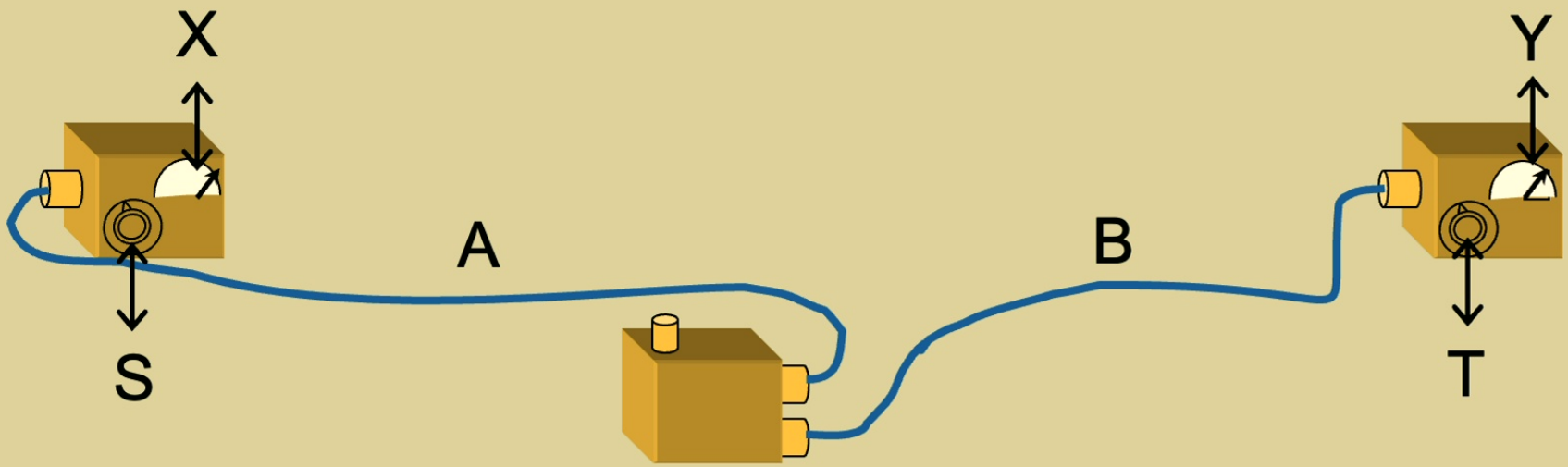
Given:

ρ_{AB}

$\rho_{X|SA}$

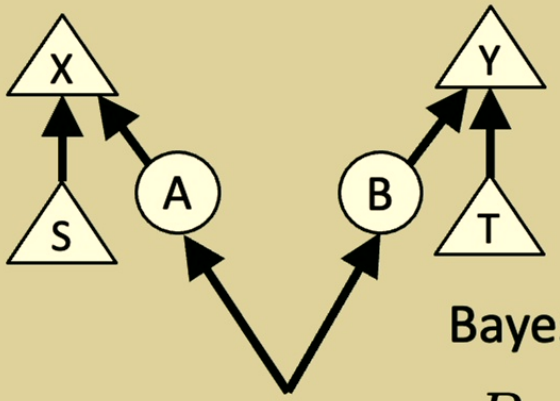
Bayesian updating

$\rho_B \to \rho_{B|SX}$



Bayesian inversion

$$P_{A|SX} = \frac{P_{X|AS} P_A}{P_{X|S}}$$



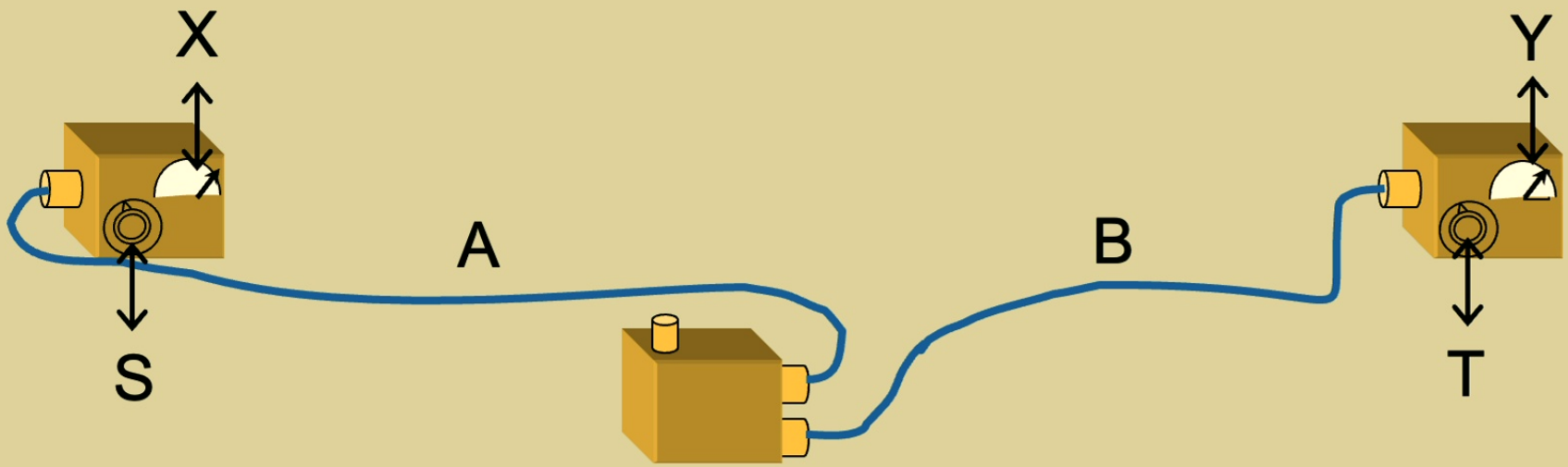
Given:

P_{AB}

$P_{X|AS}$

Bayesian updating

$$P_B \to P_{B|SX}$$

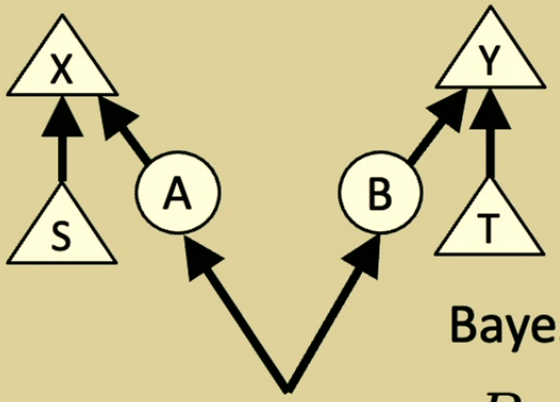


Bayesian inversion

$$P_{A|SX} = \frac{P_{X|AS} P_A}{P_{X|S}}$$

Conditional from joint

$$P_{B|A} = \frac{P_{AB}}{P_A}$$



Given:

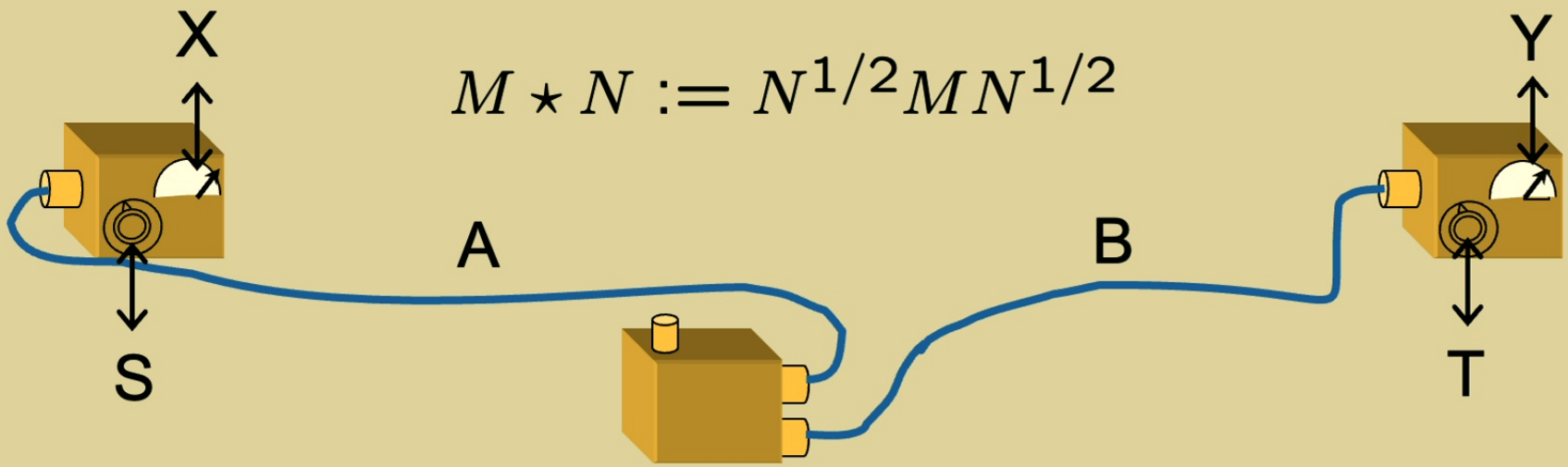
P_{AB}

$P_{X|AS}$

Bayesian updating

$$P_B \to P_{B|SX}$$

$$M \star N := N^{1/2} M N^{1/2}$$



Bayesian inversion

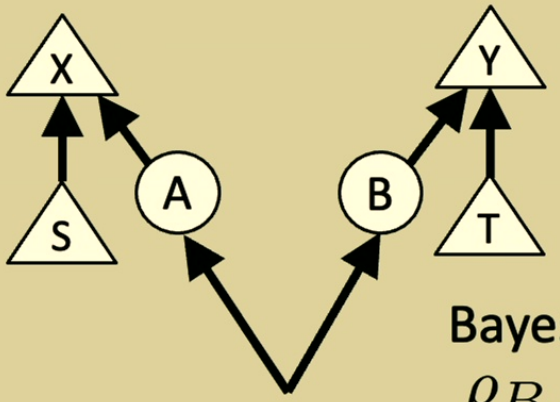
$$\rho_{A|XS} = \rho_{X|AS} \star \rho_A \rho_{X|S}^{-1}$$

Conditional from joint

$$\rho_{B|A} = \rho_{AB} \star \rho_A^{-1}$$

Belief propagation

$$\rho_{B|SX} = \mathrm{tr}_A(\rho_{B|A}\rho_{A|SX})$$



Given:

ρ_{AB}

$\rho_{X|SA}$

Bayesian updating

$$\rho_B \to \rho_{B|SX}$$

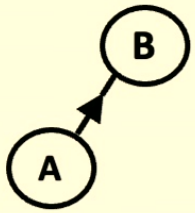
An application of quantum-classical differences in the inferential theory

$$P_{B|A} = P_{AB} P_A^{-1}$$

$$P_B = \sum_A P_{B|A} P_A$$

$$P_B = \Gamma_{B|A}[P_A]$$

$\Gamma_{B|A} \in$ Stochastic maps



$\rho_{B|A} =$ image of $\mathcal{E}_{B|A}$ under CJ-isomorphism

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$\rho_{B|A}^{T_A} \geq 0$ (conditional is PPT)

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

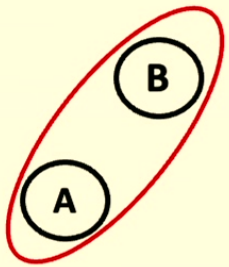
$\mathcal{E}_{B|A} \in$ CP (map is CP)

$$P_{B|A} = P_{AB} P_A^{-1}$$

$$P_B = \sum_A P_{B|A} P_A$$

$$P_B = \Gamma_{B|A}[P_A]$$

$\Gamma_{B|A} \in$ Stochastic maps



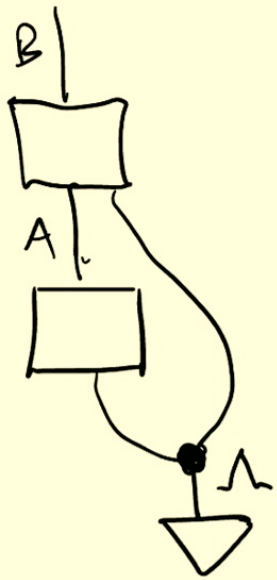
$$\rho_{B|A} = \rho_A^{-1/2} \rho_{AB} \rho_A^{-1/2}$$

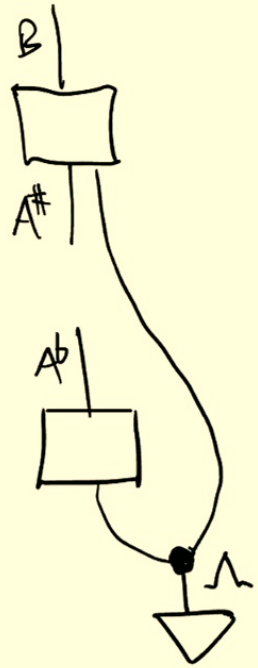
$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$\rho_{B|A} \geq 0$ (conditional is positive)

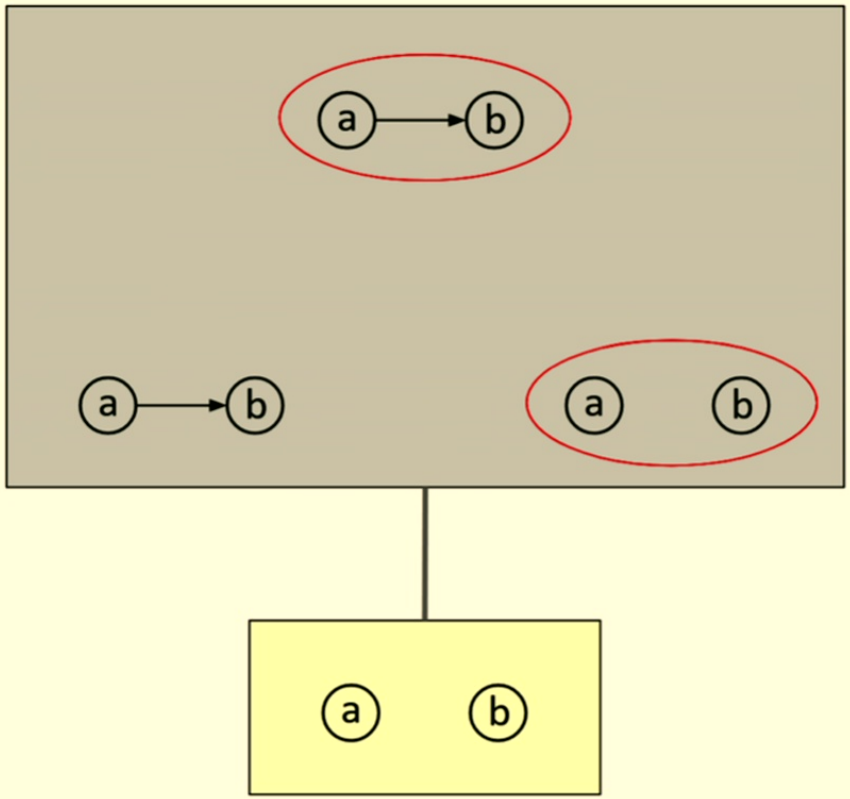
$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

$\mathcal{E}_{B|A} \circ T_A \in$ CP (map is co-CP)


B
A


B
A#
Ab

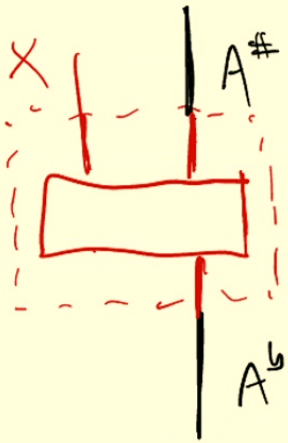
Observational order of 2-node mDAGs



Meaningless to ask if this holds quantumly because there is no quantum analogue of passive observation

Therefore, we consider a different type of probing scheme which, like passive observation, cannot discriminate cause-effect from common-cause classically but which does have a quantum analogue

Informationally symmetric probing schemes



$$P_{X A^\sharp | A^\flat}$$

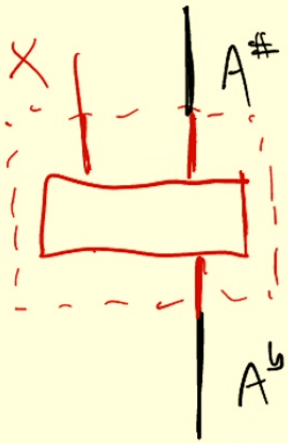
s.t., given no prior knowledge of $A^\flat$, we have

$$P_{A^\sharp | X} = P_{A^\flat | X}$$

Therefore, we consider a different type of probing scheme which, like passive observation, cannot discriminate cause-effect from common-cause classically but which does have a quantum analogue

In the dominance order induced by this probing scheme, we find that there is a difference between the classical and quantum cases

Informationally symmetric probing schemes



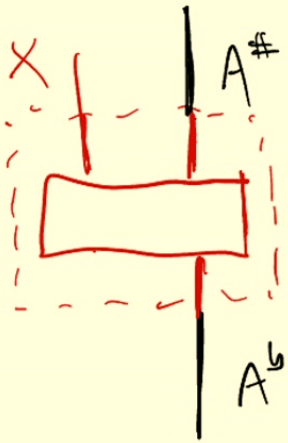
$$P_{X A^\sharp | A^\flat}$$

s.t., given no prior knowledge of $A^\flat$, we have

$$P_{A^\sharp | X} = P_{A^\flat | X}$$

Passive observation is of this type

Informationally symmetric probing schemes



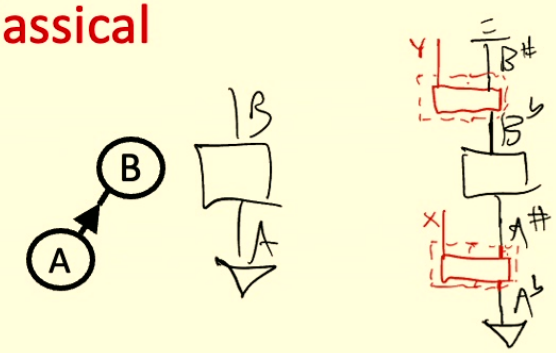
$$\rho_{X A^\sharp | A^\flat}$$

s.t., given no prior knowledge of $A^\flat$, we have

$$\rho_{A^\sharp | X} = \rho_{A^\flat | X}$$

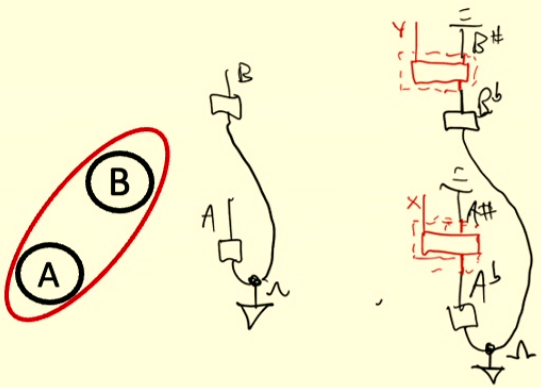
Rank-1 projector-valued measure with von-Neumann update rule is of this type

Classical



$$P_{Y|X} = \sum_{A^\#, B^\flat} P_{Y|B^\flat} P_{B^\flat|A^\#} P_{A^\#|X}$$

where $P_{A^\#|X} = \sum_{A^\flat} \frac{P_{XA^\#|A^\flat} P_{A^\flat}}{P_X}$



$$P_{Y|X} = \sum_{A^\flat, B^\flat} P_{Y|B^\flat} P_{B^\flat|A^\flat} P_{A^\flat|X}$$

where $P_{A^\flat|X} = \sum_{A^\#} \frac{P_{XA^\#|A^\flat} P_{A^\flat}}{P_X}$

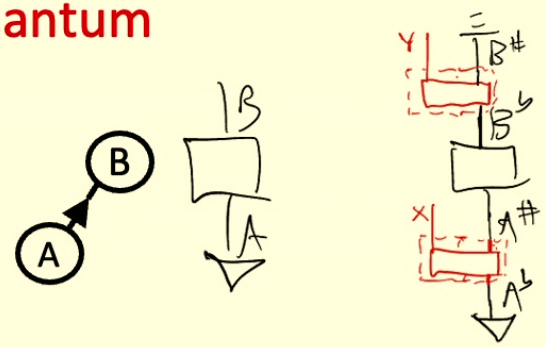
Same scope of possibilities for conditionals

$$\{P_{B|A^\#}\} = \{P_{B|A^\flat}\}$$

For probing schemes with informational symmetry

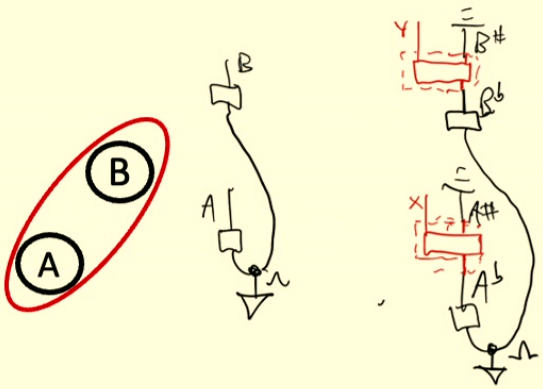
$$P_{A^\#|X} = P_{A^\flat|X}$$

Quantum



$$\rho_{Y|X} = \mathrm{Tr}_{A^\# B^\flat}\left(\rho_{Y|B^\flat}\,\rho_{B^\flat|A^\#}\,\rho_{A^\#|X}\right)$$

where $\rho_{A^\#|X} = \mathrm{Tr}_{A^\flat}\left(\rho_{XA^\#|A^\flat}\,\rho_{A^\flat}\,\rho_X^{-1}\right)$



$$\rho_{Y|X} = \mathrm{Tr}_{A^\flat B^\flat}\left(\rho_{Y|B^\flat}\,\rho_{B^\flat|A^\flat}\,\rho_{A^\flat|X}\right)$$

where $\rho_{A^\flat|X} = \mathrm{Tr}_{A^\#}\left(\rho_{XA^\#|A^\flat}\,\rho_{A^\flat}\,\rho_X^{-1}\right)$

Different scope of possibilities for conditionals

$$\{\rho_{B|A^\#}\} \neq \{\rho_{B|A^\flat}\}$$

For probing schemes with informational symmetry

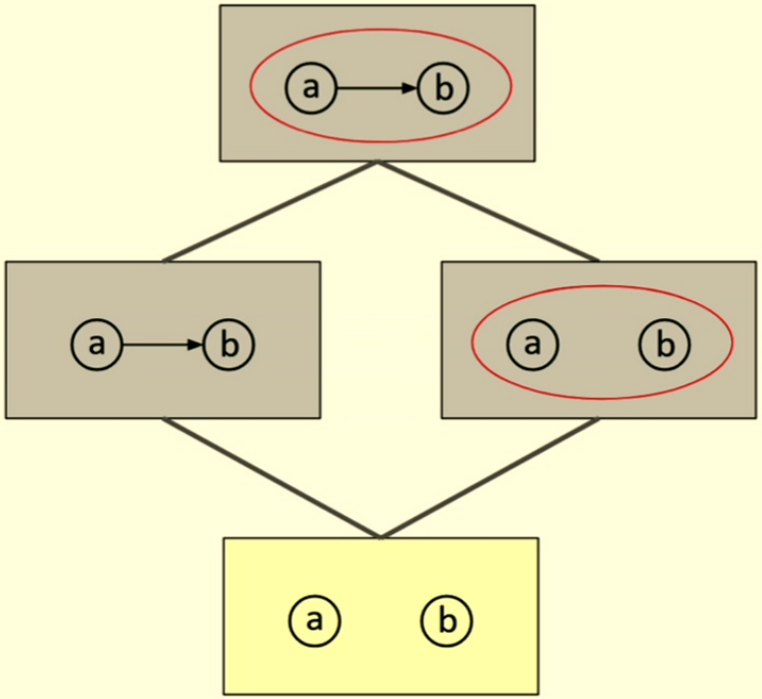
$$\rho_{A^\#|X} = \rho_{A^\flat|X}$$

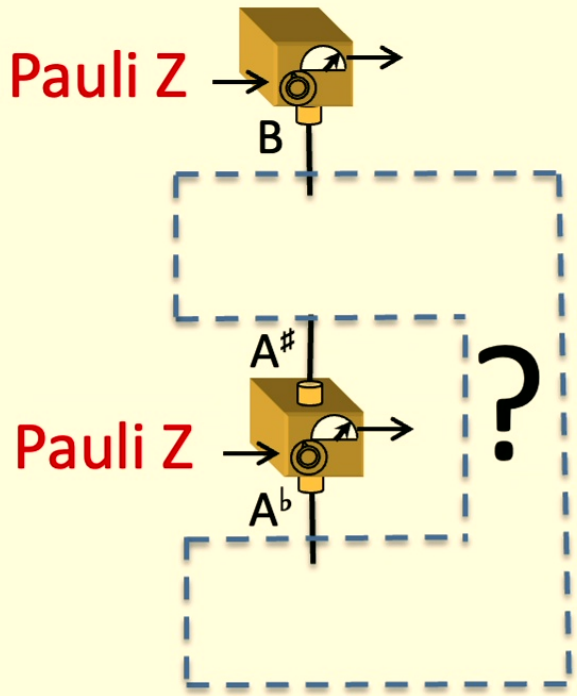
$\Rightarrow$

Different scope of possibilities for correlations

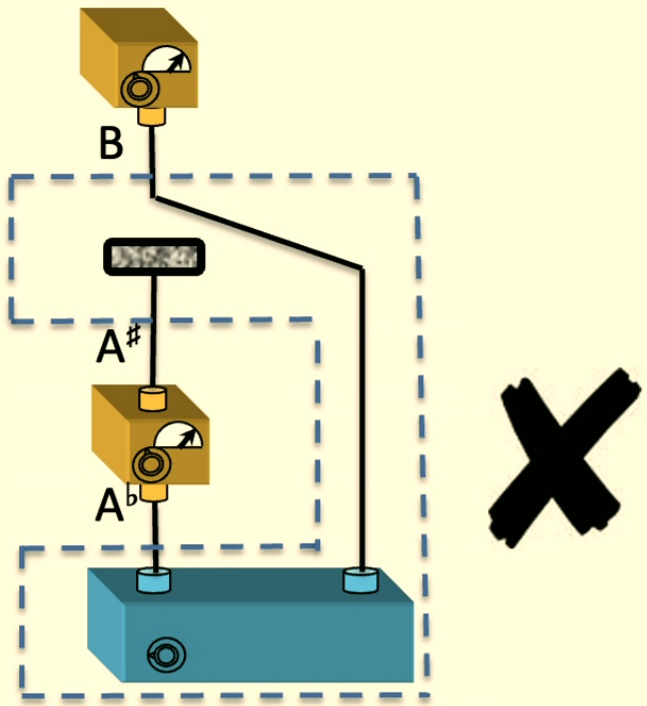
$$\{\rho^{\mathrm{ce}}_{Y|X}\} \neq \{\rho^{\mathrm{cc}}_{Y|X}\}$$

Order of 2-node mDAGs
relative to informationally symmetric probing scheme
Quantumly



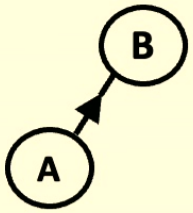


Outcomes perfectly correlated


B
A#
Ab

Pattern of correlations			Cause-effect explanation?	Common-cause explanation?
C_{XX}	C_{YY}	C_{ZZ}		
$+1$	$+1$	$+1$	$\mathcal{E}(\cdot) = I(\cdot)I$	No
$+1$	-1	-1	$\mathcal{E}(\cdot) = X(\cdot)X$	No
-1	$+1$	-1	$\mathcal{E}(\cdot) = Y(\cdot)Y$	No
-1	-1	$+1$	$\mathcal{E}(\cdot) = Z(\cdot)Z$	No
-1	-1	-1	No	$\rho = \lvert\Psi^-\rangle\langle\Psi^-\rvert$
-1	$+1$	$+1$	No	$\rho = \lvert\Phi^-\rangle\langle\Phi^-\rvert$
$+1$	-1	$+1$	No	$\rho = \lvert\Phi^+\rangle\langle\Phi^+\rvert$
$+1$	$+1$	-1	No	$\rho = \lvert\Psi^+\rangle\langle\Psi^+\rvert$

$$\rho_{B|A} = \text{image of } \mathcal{E}_{B|A} \text{ under CJ-isomorphism}$$



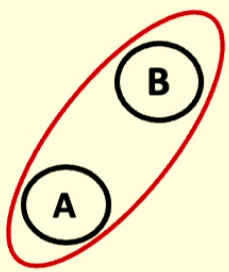
$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$$\rho_{B|A}^{T_A} \geq 0 \quad \text{(conditional is PPT)}$$

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

$$\mathcal{E}_{B|A} \in \mathrm{CP} \quad \text{(map is CP)}$$

$$\rho_{B|A} = \rho_A^{-1/2}\rho_{AB}\rho_A^{-1/2}$$



$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$$\rho_{B|A} \geq 0 \quad \text{(conditional is positive)}$$

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

$$\mathcal{E}_{B|A} \circ T_A \in \mathrm{CP} \text{ (map is co-CP)}$$

Cases in the intersection of the two sets

$$\rho_{B|A} \in \mathrm{PPT} \cap \text{positive}$$

$$\mathcal{E}_{B|A} \in \mathrm{CP} \cap \text{co-CP}$$

For two qubits

$$\rho_{B|A} \in \text{separable}$$

$$\mathcal{E}_{B|A} \in \text{entanglement-breaking}$$

More generally

$$\rho_{B|A} \in \text{separable} \cup \text{PPT entangled}$$

$$\mathcal{E}_{B|A} \in \text{PPT channels}$$

This sort of advantage arises also in classical *physics*, when we consider the restriction to Hamiltonian dynamics

In this sense, it is similar to other quantum features that have an analogue in classical Hamiltonian theories, such as no-cloning,

A quantum advantage for inferring causal structure
Ried , Agnew, Vermeyden, Janzing, RWS, Resch, Nature Physics 11, 414 (2015)