Title: Lecture - Causal Inference, PHYS 777

Speakers: Robert Spekkens

Collection/Series: Causal Inference (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Quantum Foundations

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Quantum causal models

"[...] our present Quantum Mechanical formalism [...] is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble."

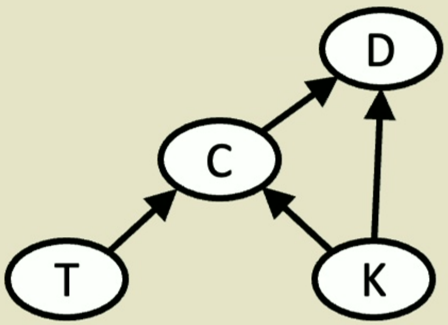
E.T. Jaynes, 1989

Unsatisfactory proposal for how to define "A causes B" classically

$$P_{B|\mathrm{do}A} \neq P_B$$

Reason to reject it:

Vernam cypher



D = (C + K)mod2

C = (T + K)mod2

$P_K = ½[0]_K + ½[1]_K$

$P_{C|\mathrm{do}\,T} = ½[0]_C + ½[1]_C = P_C$

Yet T is a cause of C

Unsatisfactory roposal for how to define "A causes B" quantumly

$$\mathcal{E}_{B|A}(\cdot) \neq \rho_B\ \mathrm{Tr}_A(\cdot)$$

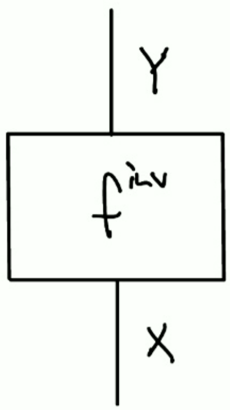
Reason to reject it:

Classical is a special case of quantum

Focus on deterministic and reversible evolution in closed systems:

Unitary evolution

invertible function



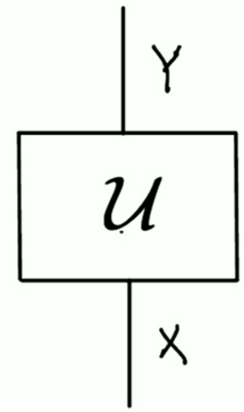
For $|X| = |Y|$

$f^{\mathsf{inv}} : X \to Y$

$:: x \mapsto f^{\mathsf{inv}}(x)$

where f^{inv} is an invertible function

Unitary channel



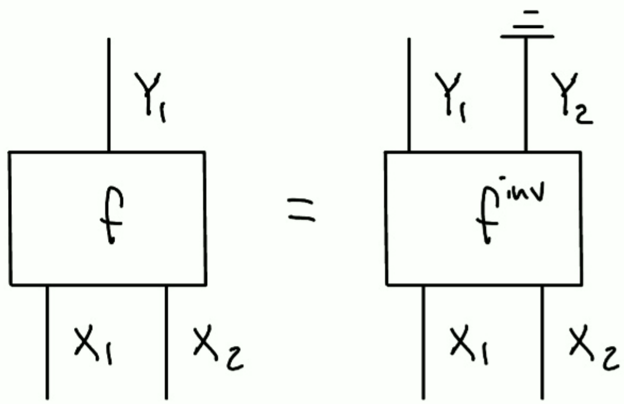
For $\dim(\mathcal{H}_X) = \dim(\mathcal{H}_Y)$

$\mathcal{U} : \mathcal{L}(\mathcal{H}_X) \to \mathcal{L}(\mathcal{H}_Y)$

$:: A \mapsto \mathcal{U}(A) := UAU^{\dagger}$

where U is a unitary operator

general function



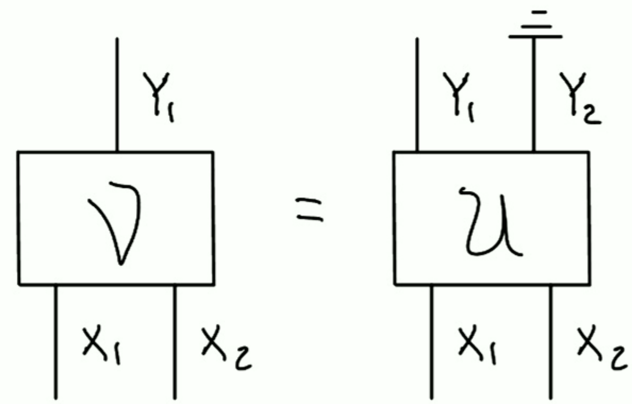
For $|X_1||X_2| = |Y_1||Y_2|$

$$f : X_1 \times X_2 \to Y_1$$

$$f := f^{\text{inv}}|_{Y_1}$$

where f^{inv} is an invertible function

Reduced unitary



For $\dim(\mathcal{H}_{X_1} \otimes \mathcal{H}_{X_2}) = \dim(\mathcal{H}_{Y_1} \otimes \mathcal{H}_{Y_2})$

$$\mathcal{V} : \mathcal{L}(\mathcal{H}_{X_1} \otimes \mathcal{H}_{X_2}) \to \mathcal{L}(\mathcal{H}_{Y_1})$$

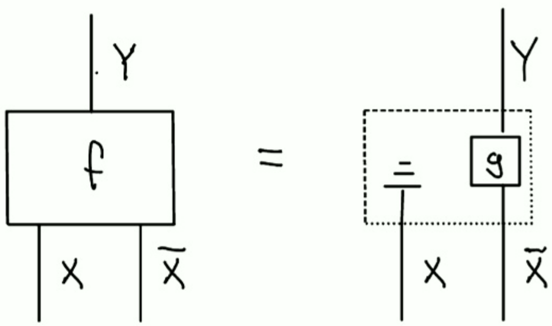
$$\mathcal{V} := \mathrm{Tr}_{Y_2} \circ \mathcal{U}$$

where $\mathcal{U}$ is a unitary channel

Classical

variable X has no influence on variable Y if Y has a trivial dependence on X

for a general function f

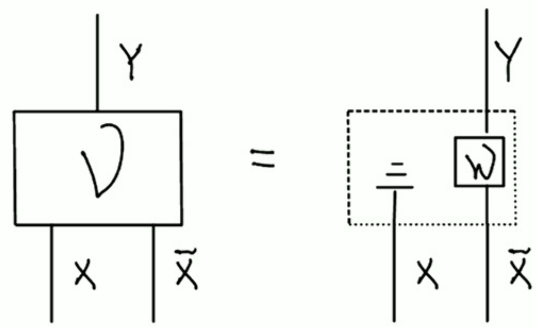


$$f(X, \bar{X}) = g(\bar{X})$$

Quantum

system X has no influence on system Y if Y has a trivial dependence on X

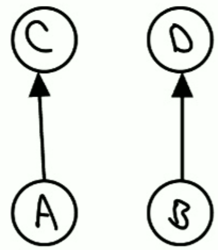
for a reduced unitary channel V



$$\mathcal{V}_{Y|X\bar{X}} = \mathrm{Tr}_X \otimes \mathcal{W}_{Y|\bar{X}}$$

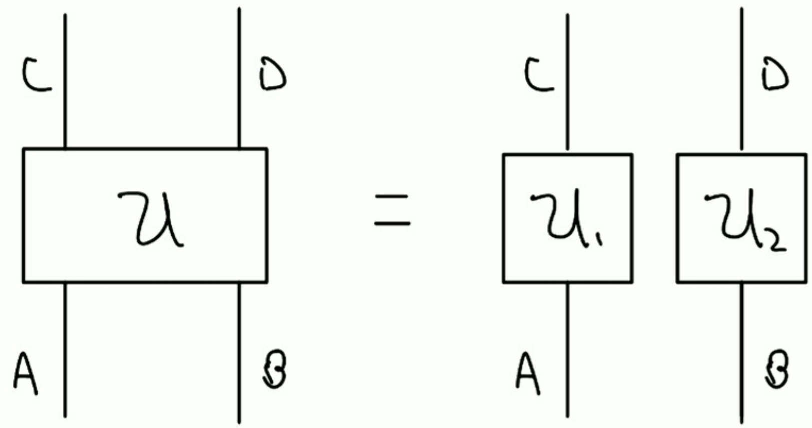
Note: this definition of no-influence satisfies "causal atomicity"

A only influences C
B only influences D

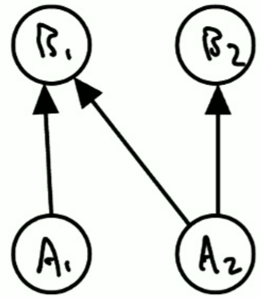


Quantumly:

$\exists\, \mathcal{U}_1, \mathcal{U}_2$:

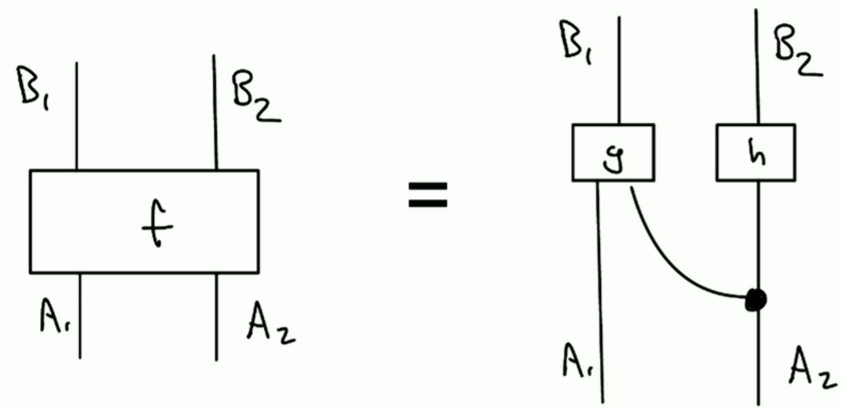


A_1 only influences B_1
A_2 influences B_1 and B_2



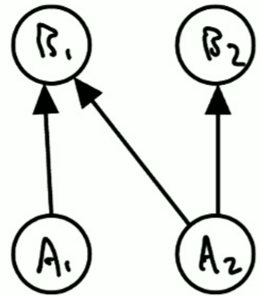
Classically

$\exists\, g, h :$



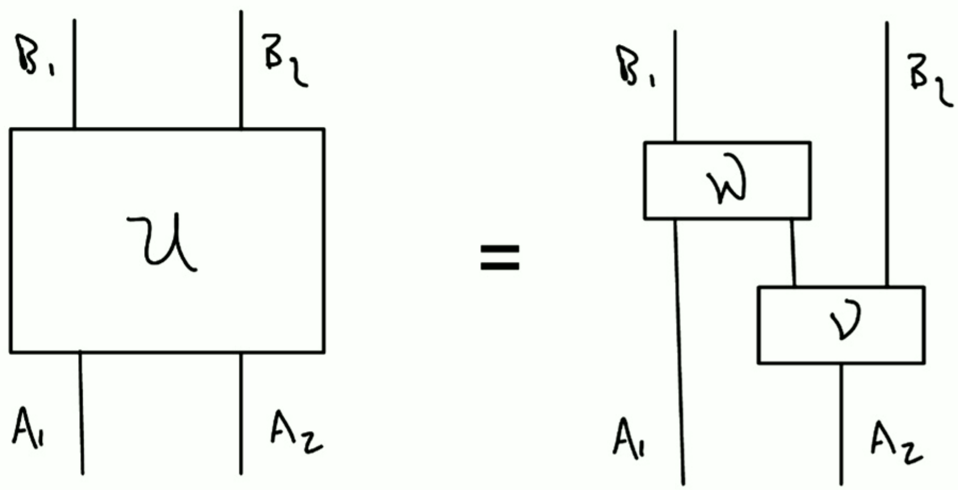
But there is no cloning in quantum theory

A_1 only influences B_1
A_2 influences B_1 and B_2

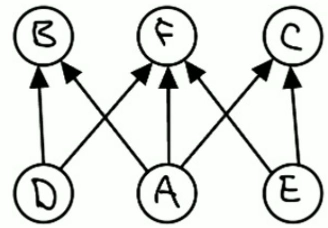


Quantumly:

$\exists\, \mathcal{V}, \mathcal{W}$:

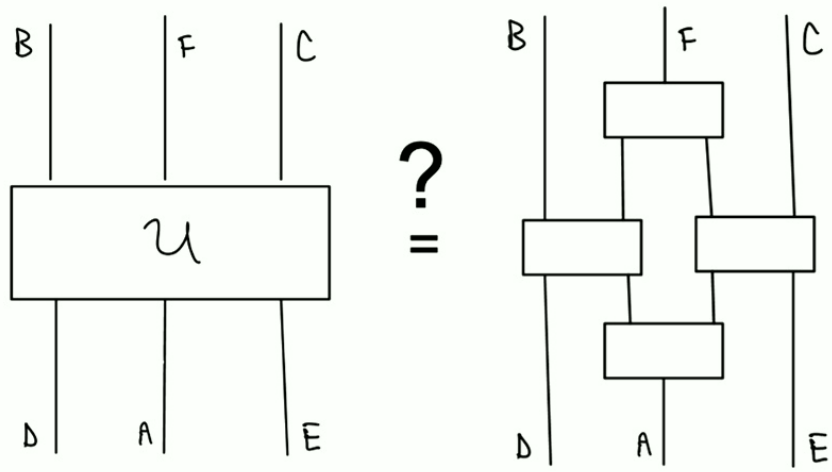


A is the complete common cause of B and C



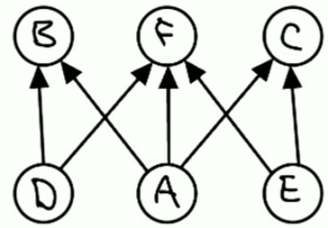
Quantumly

Is it always the case that we can find a decomposition

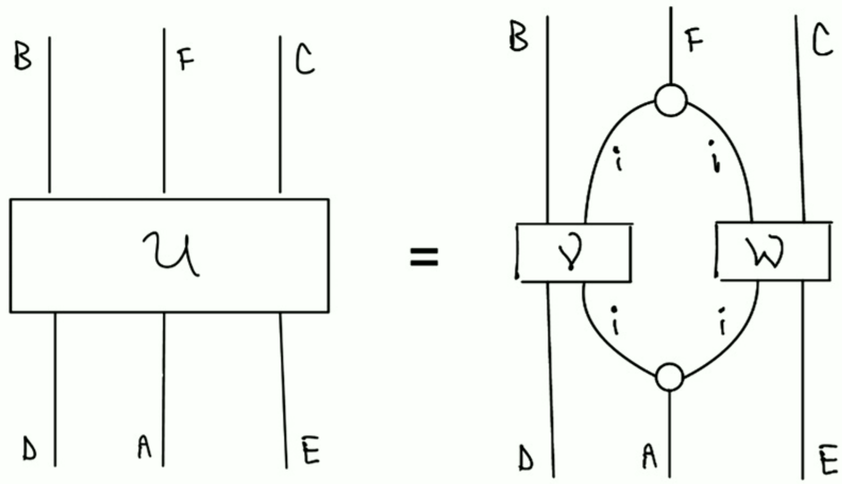


No!

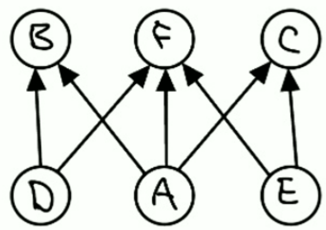
A is the complete common cause of B and C

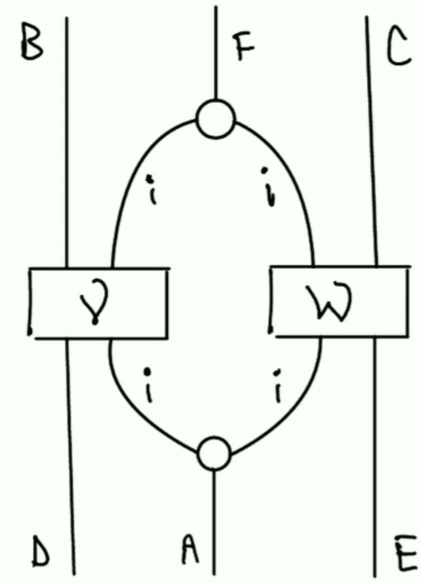


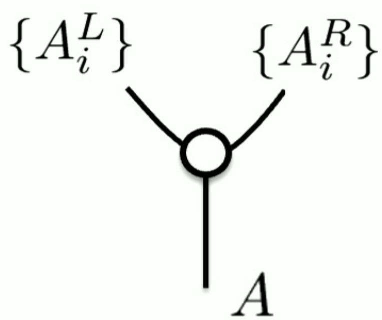
But we *can* always find a decomposition



See: Allen et al., Phys. Rev. X 7, 031021 (2017)







The dot describes a "factorization within subspaces" for A

$$\mathcal{H}_A = \bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R}$$

See: Hayden et al., Comm. Math. Phys. 246, 359 (2004)

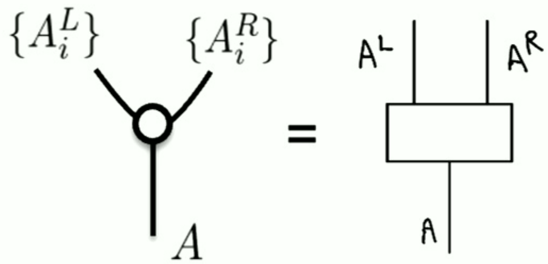
Special cases of
"factorization within subspaces"

$$\mathcal{H}_A = \bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R}$$

Pure factorization

$$\mathcal{H}_A = \mathcal{H}_{A^L} \otimes \mathcal{H}_{A^R}$$

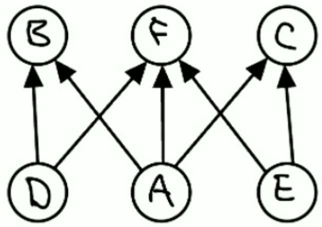
$$\dim(\mathcal{H}_{A^L}) \times \dim(\mathcal{H}_{A^R}) = \dim(\mathcal{H}_A)$$

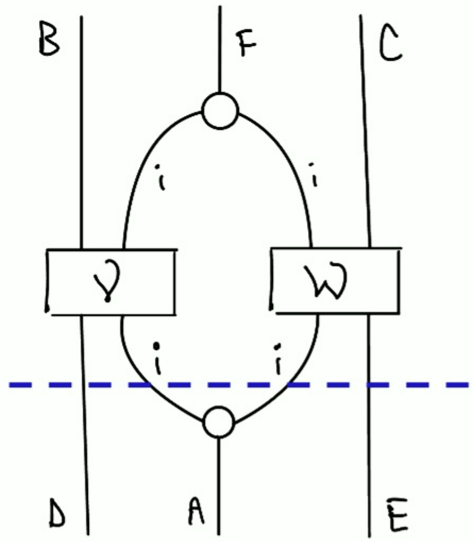


Coherent copy

$$\mathcal{H}_A = \bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R}$$

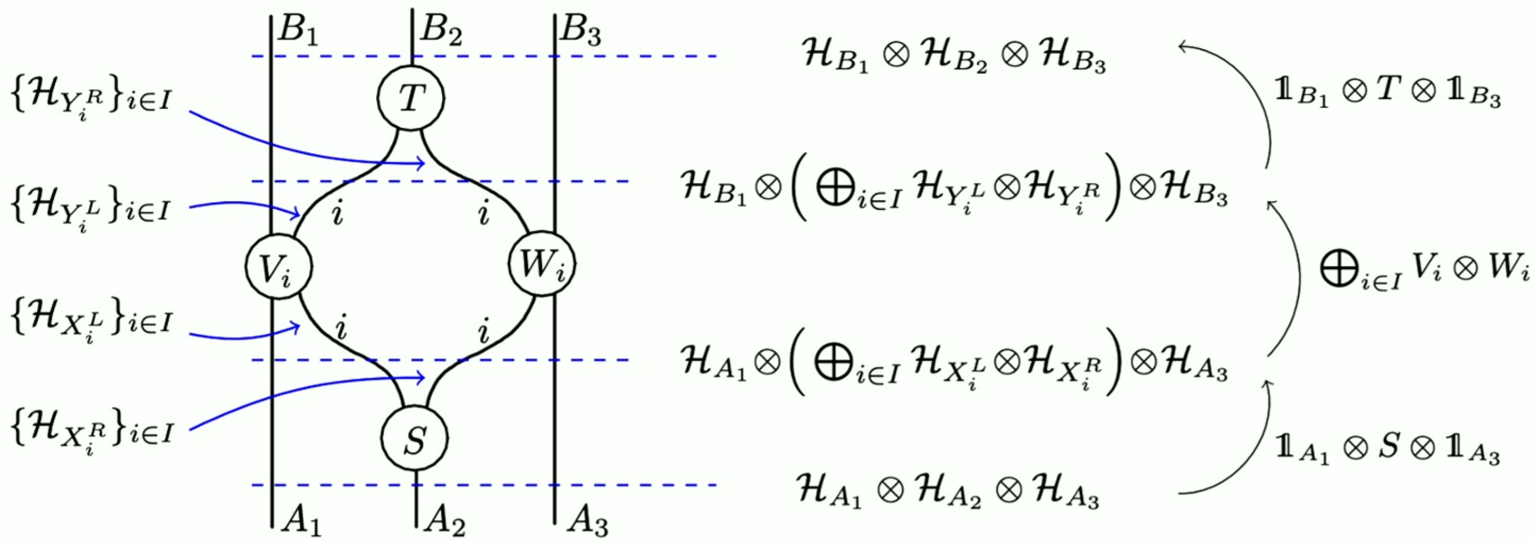
$$\dim(\mathcal{H}_{A_i^L}) = \dim(\mathcal{H}_{A_i^R}) = 1 \ \ \forall i$$





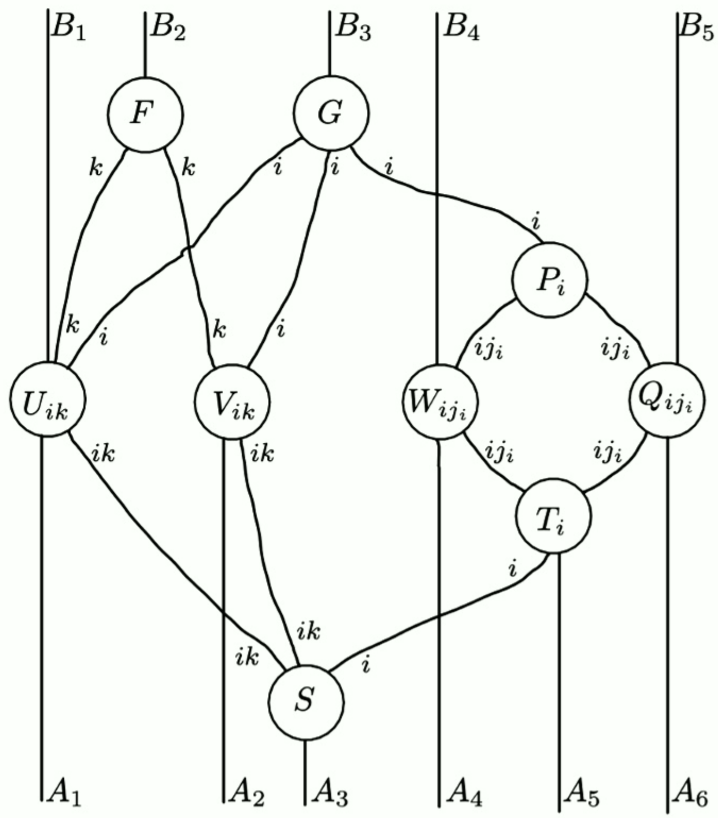
$$\mathcal{H}_D \otimes \left(\bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R} \right) \otimes \mathcal{H}_E$$

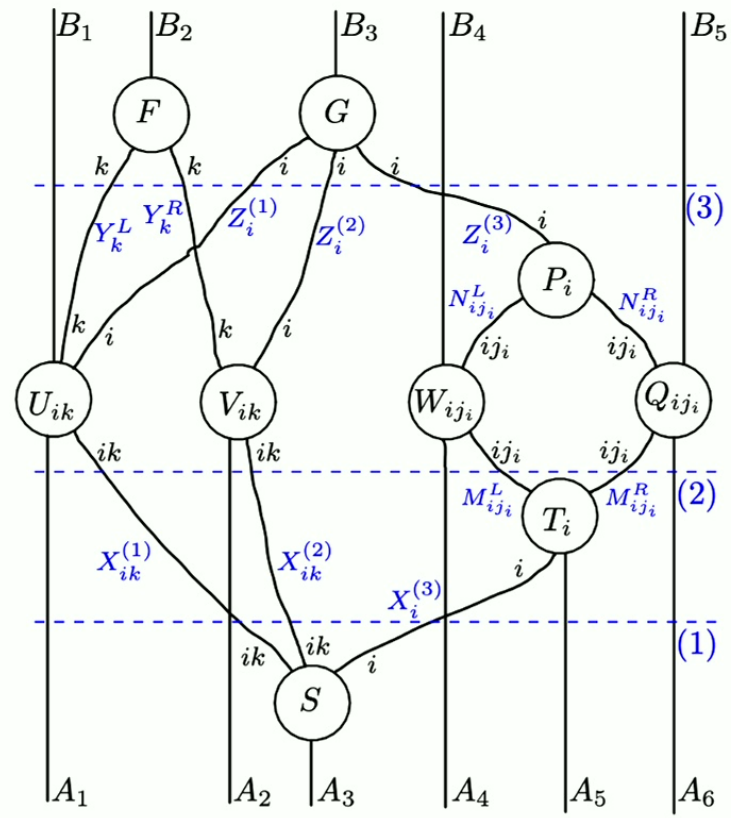
$\mathcal{V}$ is block-diagonal across the *i* sectors and is nontrivial only on $\{\mathcal{H}_{A_i^L}\}_i$



Lorenz and Barrett, Quantum 5, 511 (2021)

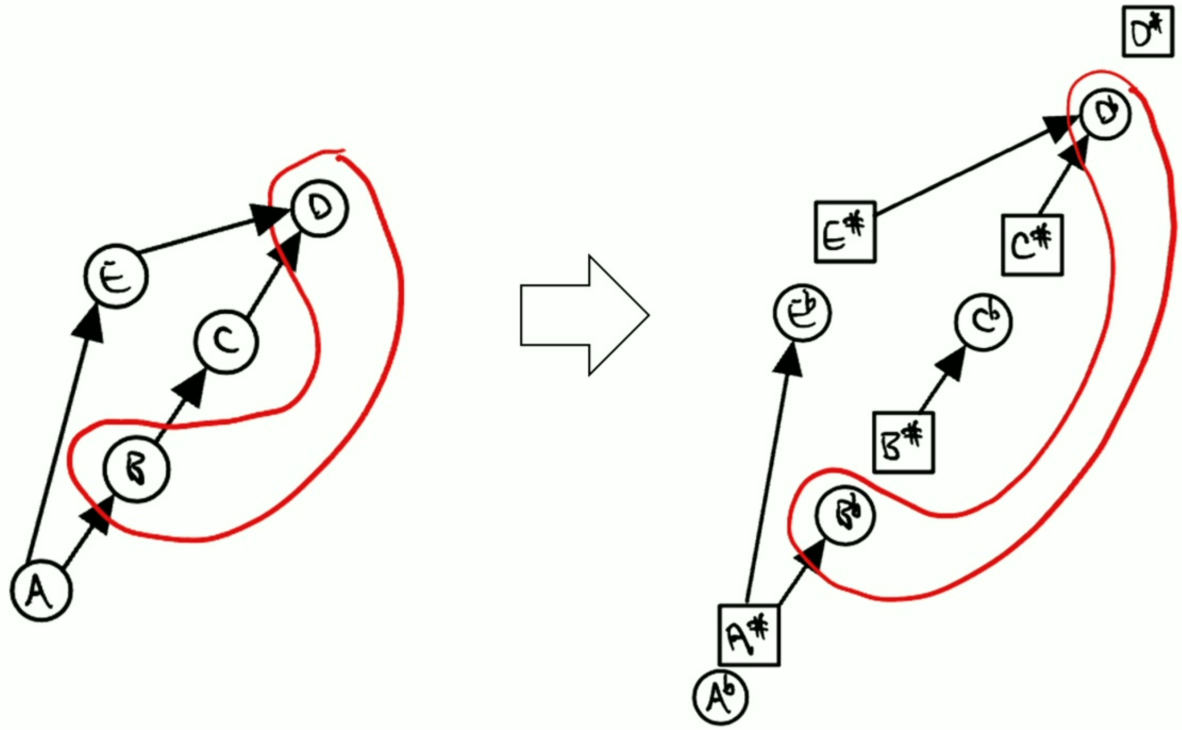
More complicated case



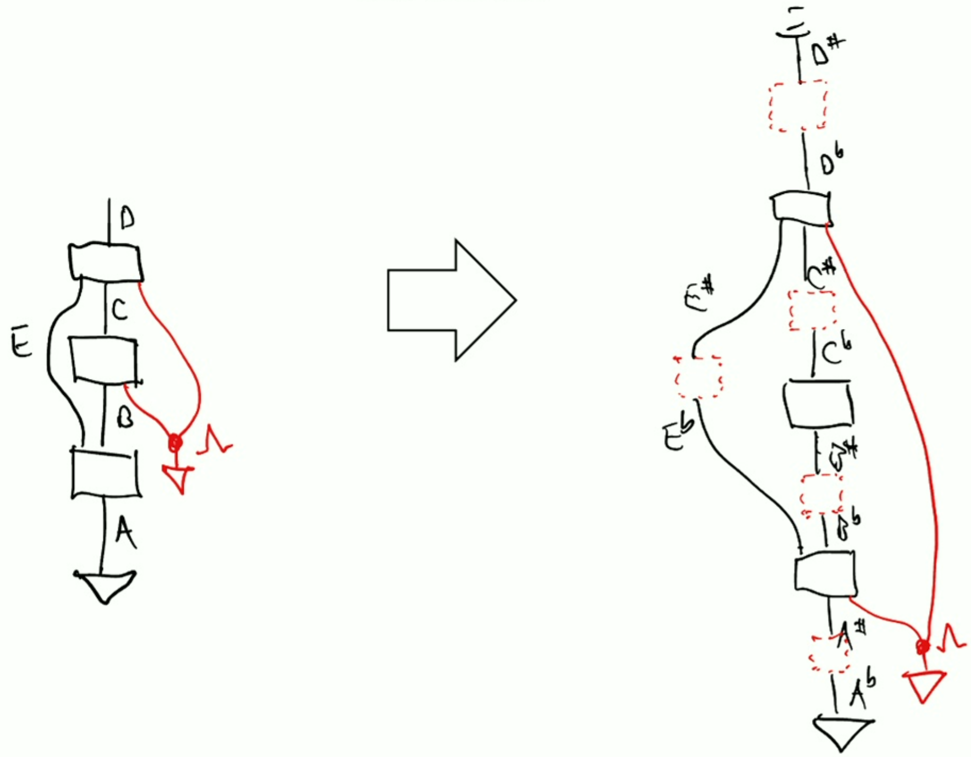


Lorenz and Barrett, Quantum 5, 511 (2021)

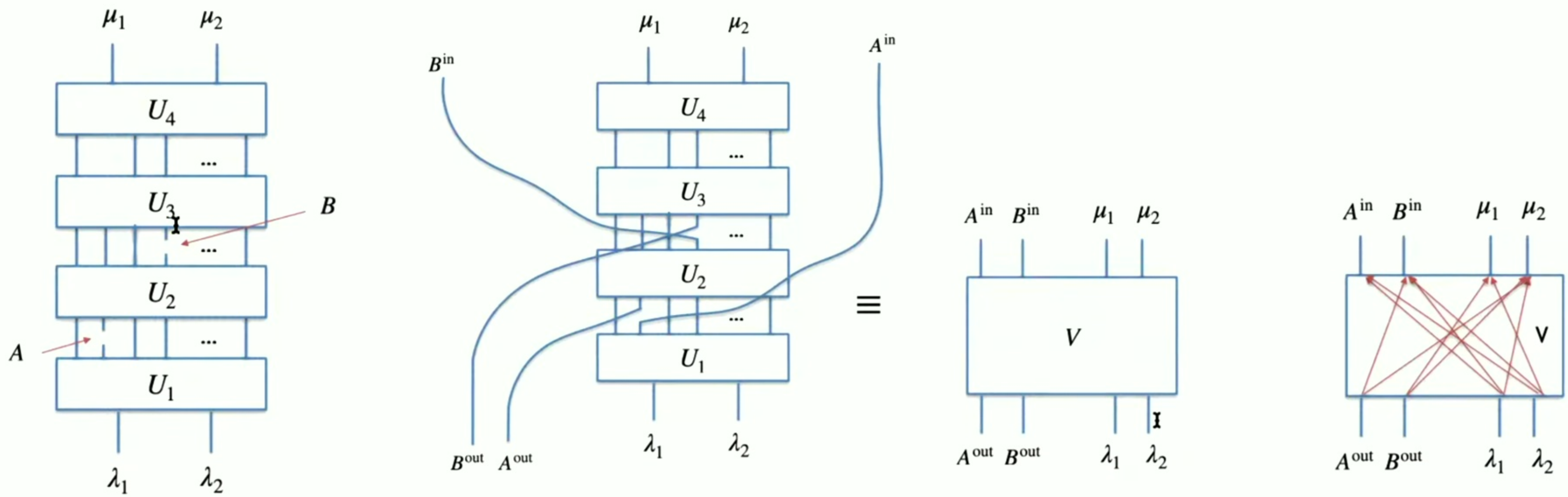
The most fundamental representation of a latent-permitting causal structure is one that allows *any probing scheme* among the visible variables



The most fundamental representation of a latent-permitting causal structure is one that allows *any probing scheme* among the visible variables

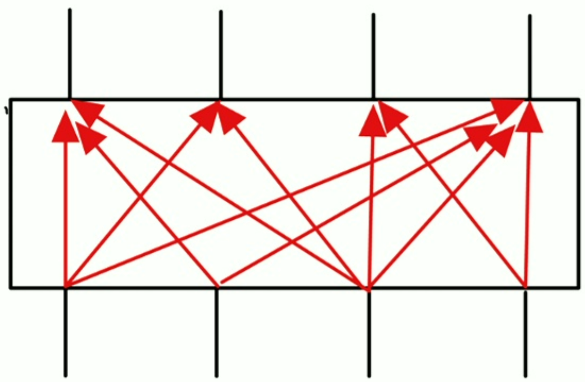


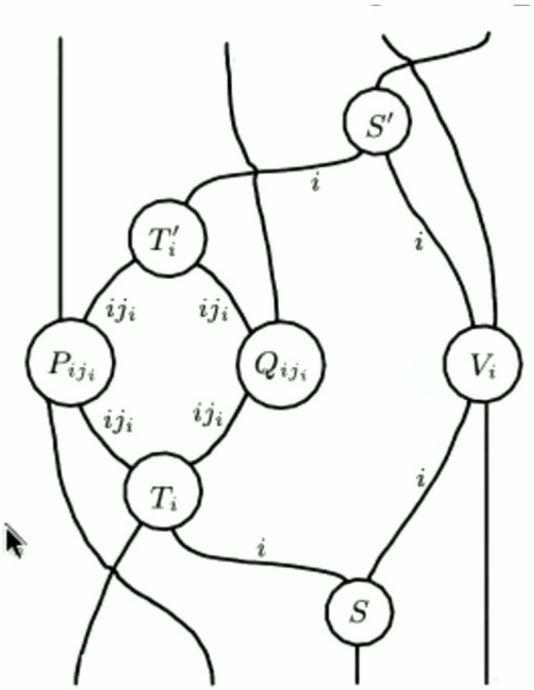
Therefore, it suffices to know how to model 2-layer causal structures


μ_1
μ_2
U_4
...
U_3
B
...
U_2
A
...
U_1
λ_1
λ_2
B^{in}
A^{in}
B^{out}
A^{out}
≡
V
A^{in}
B^{in}
A^{out}
B^{out}

A circuit decomposition is **causally faithful** if influences are given by structure of diagram

unitary





A circuit decomposition is **causally faithful** if influences are given by structure of diagram

It is an open question whether every multipartite unitary channel admits a causally faithful decomposition

Known to be true for:
3 inputs and N outputs
N inputs and 3 outputs
Some other special cases

Joint states for systems that are common-cause connected

	Classical	Quantum
State of knowledge	P_A	ρ_A
Normalization	$\sum_A P_A = 1$	$\mathrm{Tr}_A(\rho_A) = 1$
Joint state	P_{AB}	ρ_{AB}
Marginalization	$P_B = \sum_A P_{AB}$	$\rho_B = \mathrm{Tr}_A(\rho_{AB})$

Causally disconnected A and B

$$\rho_{AB} = \rho_A \otimes \rho_B$$

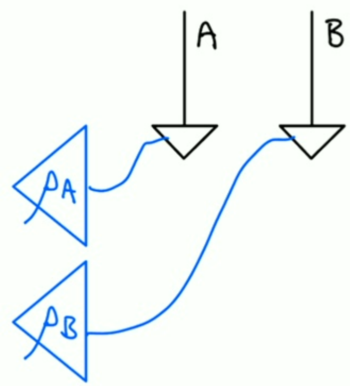
Denote this

$$(A \perp B)$$

$$I(A:B) = 0$$

where

$$I(A:B) = S(A) + S(B) - S(AB)$$
$$S(X) = -\mathrm{Tr}(\rho_X \log \rho_X)$$



Conditional states for systems that are cause-effect connected

Classical belief propagation

$$P_B = \Gamma_{B|A}[P_A]$$

Stochastic map preserving positivity and normalization

In terms of a conditional

$$P_B = \sum_A P_{B|A} P_A$$

$$\sum_B P_{B|A} = 1$$

$$P_{B|A} \geq 0$$

Quantum belief propagation

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

Completely positive trace-preserving map

In terms of a conditional

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$$\mathrm{Tr}_B(\rho_{B|A}) = I_A$$

$$\rho_{B|A}^{T_A} \geq 0$$

The Choi-Jamiolkowski isomorphism

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A}\cdot)$$

Proof: $$\mathrm{Tr}_A(\rho_{B|A}\rho_A) = \sum_{j,k} \Phi_{B|A'}(|j\rangle\langle k|_{A'})\langle j|\rho_A|k\rangle$$

$$= \Phi_{B|A}(\sum_{j,k} |j\rangle\langle j|_A \rho_A |k\rangle\langle k|_A)$$

$$= \Phi_{B|A}(\rho_A)$$ QED

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A} \cdot)$$

$$\Phi_{B|A'} \text{ is trace-preserving} \quad \leftrightarrow \quad \mathrm{Tr}_B(\rho_{B|A}) = I_A$$

Proof:
$$\mathrm{Tr}_B(\rho_{B|A}) = \mathrm{Tr}_B\left[(\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)\right]$$
$$= \sum_{j,k} \mathrm{Tr}_B \circ \Phi_{B|A'}\left(|j\rangle\langle k|_{A'}\right) |k\rangle\langle j|_A$$

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

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Proof:

$$\begin{aligned}
\mathrm{Tr}_B(\rho_{B|A}) &= \mathrm{Tr}_B\left[(\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)\right] \\
&= \sum_{j,k} \mathrm{Tr}_B \circ \Phi_{B|A'}(|j\rangle\langle k|_{A'})\, |k\rangle\langle j|_A \\
&= \sum_{j,k} \mathrm{Tr}_{A'}(|j\rangle\langle k|_{A'})\, |k\rangle\langle j|_A \\
&= \sum_{j,k} \delta_{j,k} |k\rangle\langle j|_A \\
&= I_A
\end{aligned}$$

QED

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A} \cdot)$$

$$\Phi_{B|A} \text{ is completely positive} \quad \leftrightarrow \quad \rho_{B|A}^{T_A} \geq 0$$

$$\rho_{B|A} \text{ is PPT}$$

Proof: $$\rho_{B|A} = (\Phi_{B|A'} \otimes \mathrm{id}_A)\left[(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |j\rangle\langle k|_A)^{T_A}\right]$$

$$= \left[(\Phi_{B|A'} \otimes \mathrm{id}_A)(d_A |\Psi^+\rangle_{A'A}\langle\Psi^+|)\right]^{T_A}$$

QED

Classical belief propagation

$$P_B = \Gamma_{B|A}[P_A]$$

Stochastic map preserving positivity and normalization

In terms of a conditional

$$P_B = \sum_A P_{B|A} P_A$$

$$\sum_B P_{B|A} = 1$$

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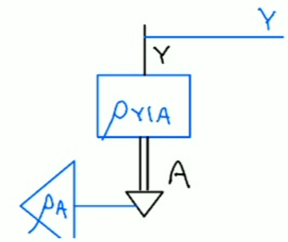
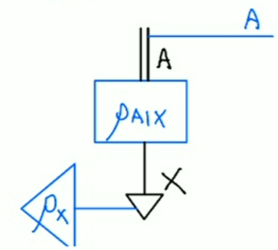
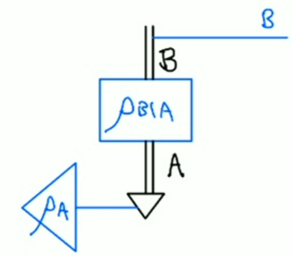
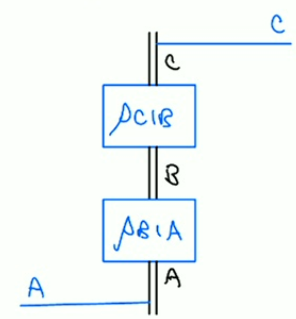
Completely positive trace-preserving map

In terms of a conditional

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

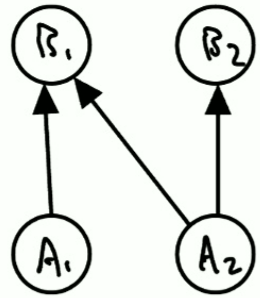
$$\mathrm{Tr}_B(\rho_{B|A}) = I_A$$

$$\rho_{B|A}^{T_A} \geq 0$$

	Conventional expression	In terms of conditional states	
Born's rule	$\forall y : P_Y(y) = \mathrm{Tr}_A(E_y^A \rho_A)$	$\rho_Y = \mathrm{Tr}_A(\rho_{Y\|A}\rho_A)$	
Ensemble averaging	$\rho_A = \sum_x P_X(x)\rho_x^A$	$\rho_A = \mathrm{Tr}_X(\rho_{A\|X}\rho_X)$	
Action of quantum channel	$\rho_B = \mathcal{E}_{B\|A}(\rho_A)$	$\rho_B = \mathrm{Tr}_A(\varrho_{B\|A}\rho_A)$	
Composition of channels	$\mathcal{E}_{C\|A} = \mathcal{E}_{C\|B} \circ \mathcal{E}_{B\|A}$	$\varrho_{C\|A} = \mathrm{Tr}_B(\varrho_{C\|B}\varrho_{B\|A})$	

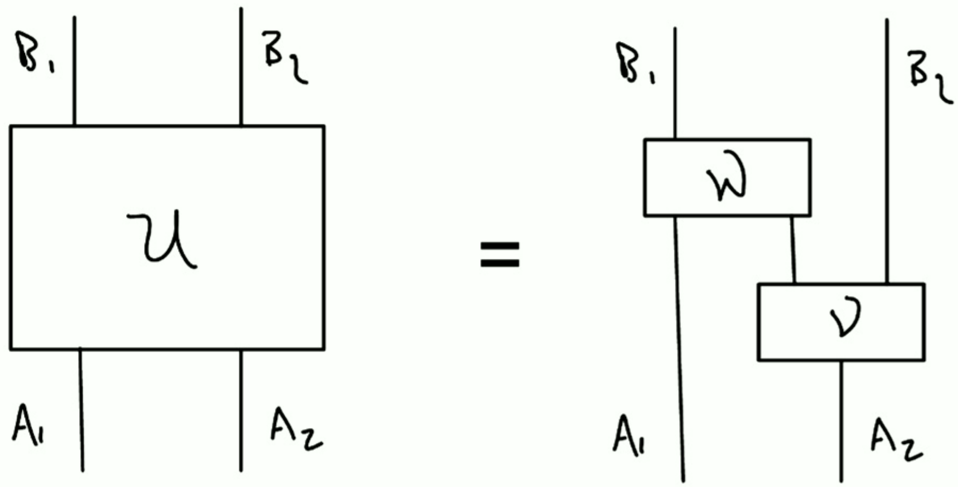
No-influence conditions imply independences

A_1 only influences B_1
A_2 influences B_1 and B_2



Quantumly:

$\exists\, \mathcal{V}, \mathcal{W}:$



Causally disconnected A and B

$$\rho_{B|A} = \rho_B$$

$$I(A:B) = 0$$

where

$$I(A:B) = S(A) + S(B) - S(AB)$$

for $\rho_{AB} = (\rho_{B|A})^{T_A} \frac{1}{d_A}$