Title: Lecture - Causal Inference, PHYS 777

Speakers: Robert Spekkens

Collection/Series: Causal Inference (Elective), PHYS 777, March 31 - May 2, 2025

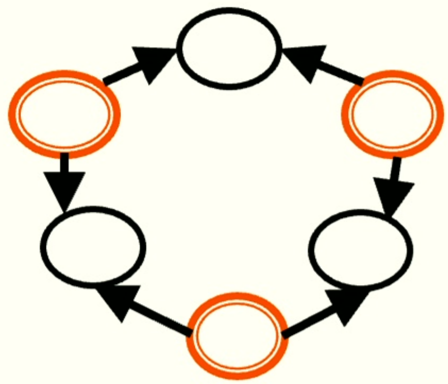
Subject: Quantum Foundations

Date: April 15, 2025 - 11:30 AM

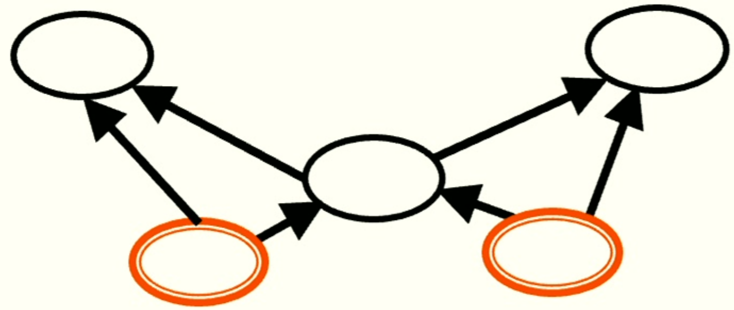
URL: https://pirsa.org/25040042

Inequality constraints for causal models

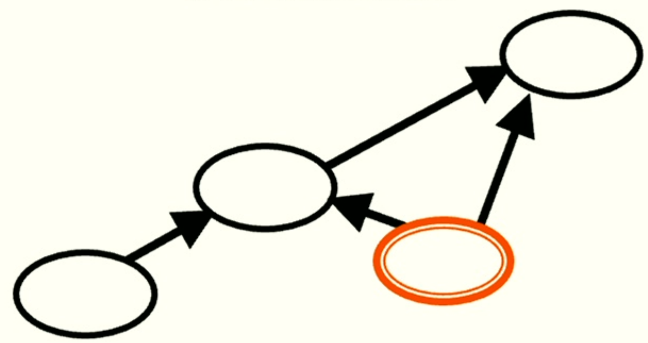
Triangle



Evans

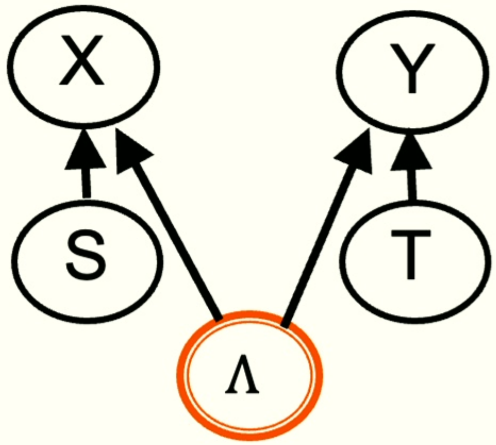


Instrumental



Inequality constraints for causal models by brute-force quantifier elimination

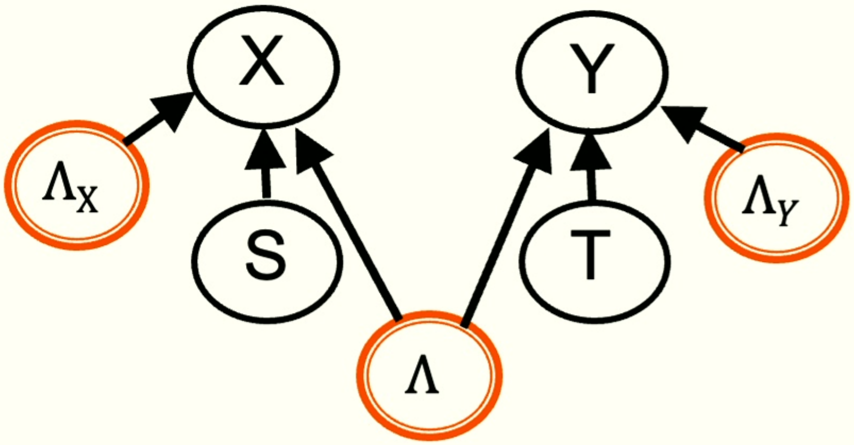
Bell scenario



$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|T\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

Bell scenario

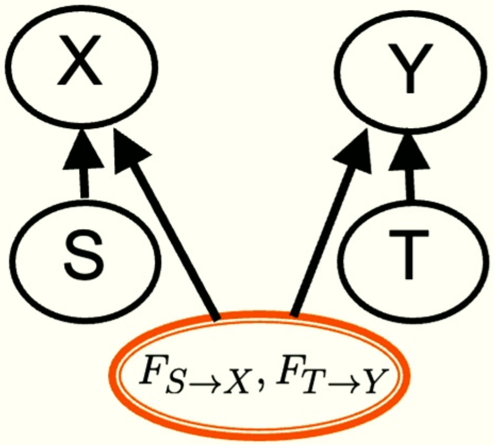


$$X = f_X(S, \Lambda, \Lambda_X)$$
$$Y = f_Y(T, \Lambda, \Lambda_Y)$$
$$P_\Lambda$$
$$P_{\Lambda_X}$$
$$P_{\Lambda_Y}$$

$$P_{XY|ST} = \sum_{\Lambda,\Lambda_X,\Lambda_Y} \delta_{X,f_X(S,\Lambda,\Lambda_X)} \delta_{Y,f_Y(T,\Lambda,\Lambda_Y)} P_\Lambda P_{\Lambda_X} P_{\Lambda_Y}$$

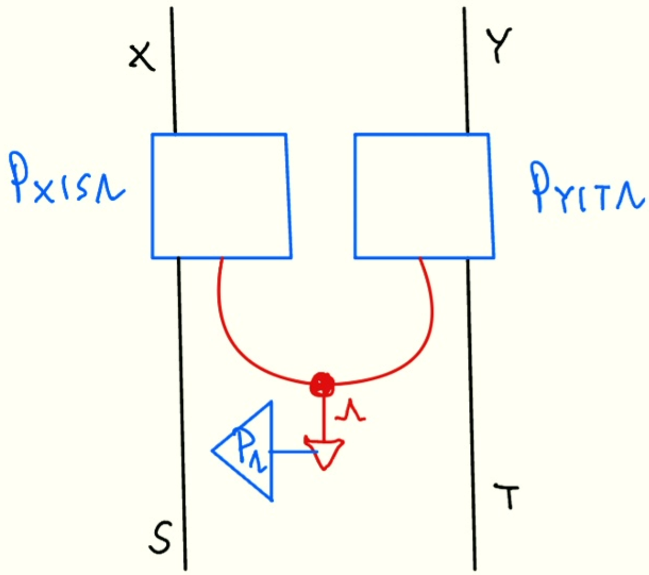
$$P_{X|S\Lambda} = \sum_{\Lambda_X} \delta_{X,f_X(S,\Lambda,\Lambda_X)} P_{\Lambda_X}$$
$$P_{Y|T\Lambda} = \sum_{\Lambda_Y} \delta_{Y,f_Y(T,\Lambda,\Lambda_Y)} P_{\Lambda_Y}$$

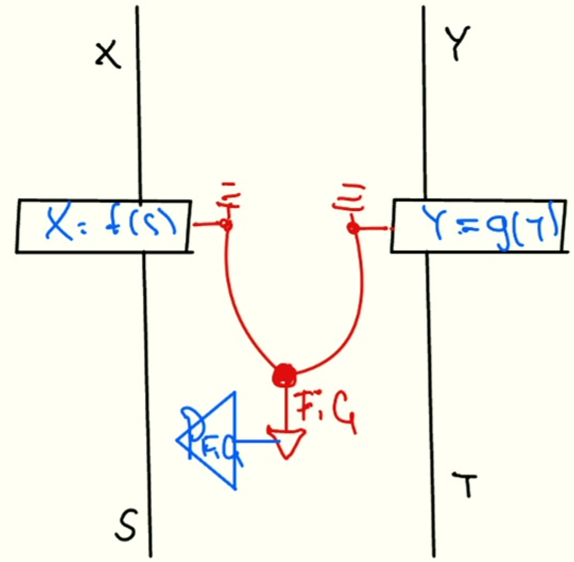
Bell scenario



$P_{F_{S\to X}, F_{T\to Y}}$

$$P_{XY|ST} = \sum_{f,f'} \delta_{X,f(S)} \delta_{Y,f'(T)} P_{F_{S\to X}, F_{T\to Y}}(f, f')$$


X
Y
P_{X|SΛ}
P_{Y|TΛ}
Λ
P_Λ
S
T


X
Y
X = f(S)
Y = g(T)
F, G
P_{F,G}
S
T

$$P_{XY|ST} = \sum_{f,g} \delta_{X,f(S)} \delta_{Y,g(T)} P_{F,G}(f,g)$$

If X,Y,S,T are binary, Λ can have cardinality 16

$$p_{xy|st} := P_{XY|ST}(xy|st) \qquad q_{fg} := P_{F,G}(f,g)$$

$$x,y,s,t \in \{0,1\} \qquad f,g \in \{\mathbb{I}, \mathbb{F}, \mathbb{R}_0, \mathbb{R}_1\}$$

$$p_{00|00} = q_{\mathbb{R}_0,\mathbb{R}_0} + q_{\mathbb{R}_0,\mathbb{I}} + q_{\mathbb{I},\mathbb{R}_0} + q_{\mathbb{I},\mathbb{I}}$$

$$p_{00|01} = q_{\mathbb{R}_0,\mathbb{R}_0} + q_{\mathbb{R}_0,\mathbb{F}} + q_{\mathbb{I},\mathbb{R}_0} + q_{\mathbb{I},\mathbb{F}}$$

$$\vdots$$

$$0 \leq q_{fg} \leq 1 \ \forall f,g$$

16 linear equalities + inequalities

Do linear quantifier elimination on the 16 q's.

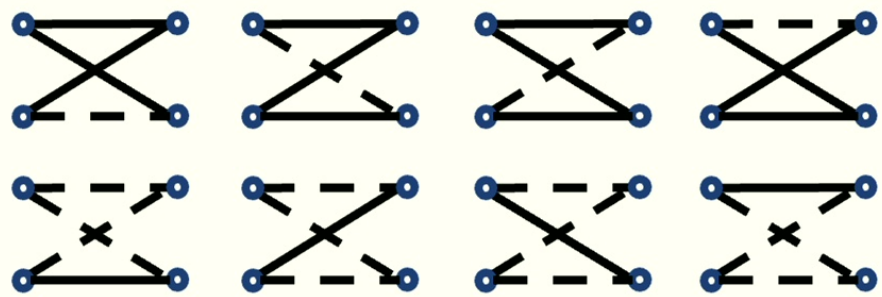
This yields the conditional independence relations and the 8 CHSH inequalities

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

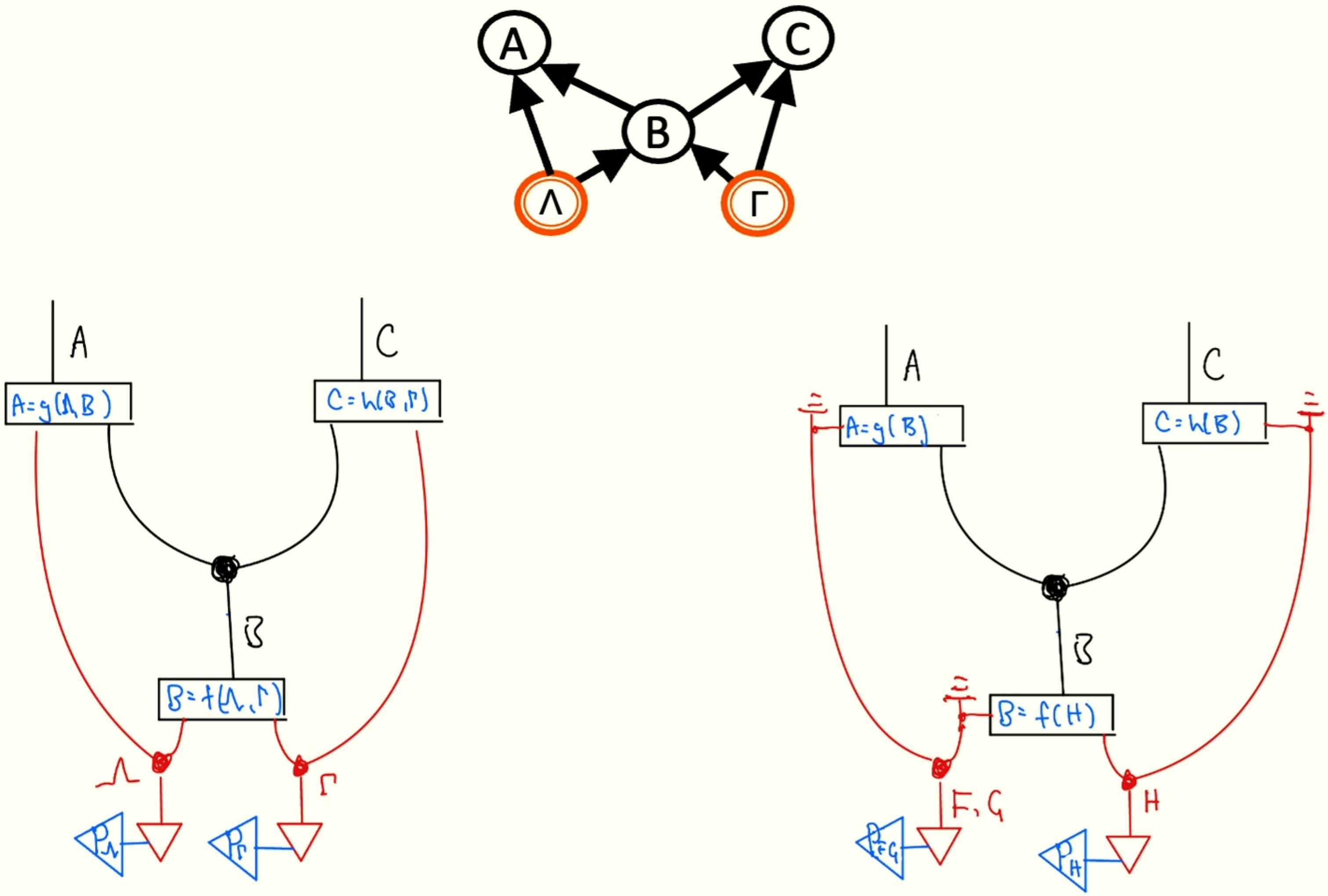
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

+ 7 others

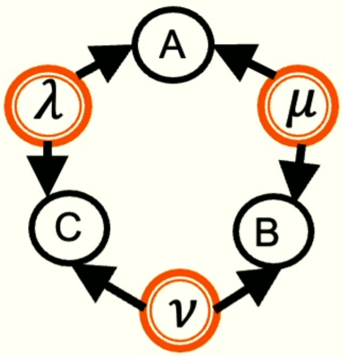
Corresponding to the 8 frustrated four-node networks



Clauser, Horne, Shimony and Holte, Phys. Rev. Lett.23, 880 (1967)


A
C
B
Λ
Γ
A
C
A=g(Λ,B)
C=h(B,Γ)
B
B=f(Λ,Γ)
Λ
Γ
A
C
A=g(B)
C=h(B)
B
B=f(H)
F, G
H

DAGs for which one can deduce the cardinalities of latent variables in this manner are termed **gearable**
(R. Evans, Annals of Statistics, **46**, 2623 (2018))



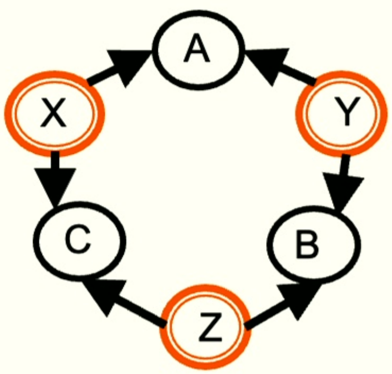
Triangle

The triangle scenario is an example of a DAG that is not gearable

Techniques for determining upper bounds on cardinalities of the latent variables in more general causal structures

R. Evans, Annals of Statistics, 46, 2623 (2018)

D. Rosset, N. Gisin, and E. Wolfe. Quantum Inf. & Comp. **18,** 0910 (2018)



A, B, C binary → Sufficient for X, Y, Z to be 6-valued

With more than one latent variable,
we require **nonlinear** quantifier elimination
which scales badly

Entropic inequalities by quantifier elimination

Shannon entropy

$$H(X) := -\sum_x P_X(x) \log P_X(x)$$

Conditional entropy

$$H(X|Y) := H(XY) - H(Y)$$

Mutual information

$$I(X:Y) = H(X) + H(Y) - H(XY)$$

Conditional mutual information

$$I(X:Y|Z) = H(XZ) + H(YZ) - H(XYZ) - H(Z)$$

Def'n: A and B are conditionally independent given C

$$P_{AB|C} = P_{A|C}P_{B|C}$$
$$P_{B|AC} = P_{B|C}$$
$$P_{A|BC} = P_{A|C}$$

$$P_{ABC} = P_{A|C}P_{B|C}P_C$$

$$I(A:B|C) = 0$$

Denote this
$(A \perp B|C)$

Entropy vector

For the joint distribution of the random variables X_1, ...X_n , the components of the entropy vector are the entropies of the marginals for all possible subsets of variables:

$$(H(X_1), H(X_2), \dots, H(X_n), H(X_1X_2), H(X_1X_3), \dots, H(X_1, X_2, \cdots, X_n))$$

Entropy cone

The closure of this set of vectors is called the
entropy cone

It is a convex cone
therefore characterized by **linear** inequalities

An outer approximation to the entropy cone: the Shannon cone

Monotonicity

$H(XA) \geq H(X)$

for every variable A and sets of variables X

Submodularity

$H(X) + H(XAB) \leq H(XA) + H(XB)$

where A and B are variables not in the set X

Inequalities describing the Shannon cone are termed **Shannon-type**

Valid inequalities for the entropy cone that are not Shannon-type are termed **non-Shannon-type** (R. Yeung, IEEE Trans. Inf. Th., 43, 1997)

Example: for distributions on X, Y, Z, the linear equalities defining the Shannon cone are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} H(X) \\ H(Y) \\ H(Z) \\ H(XY) \\ H(XZ) \\ H(YZ) \\ H(XYZ) \end{pmatrix} \geq 0$$

So far, these are statements about a joint distribution, with no causal content

What are the constraints on the entropies of observed variables for a given causal structure?

The entropic technique for deriving inequality constraints:

The set of all constraints on observed and latent variables are the conditional independence relations among these
These imply linear equalities on the components of the entropy vector for observed and latent variables

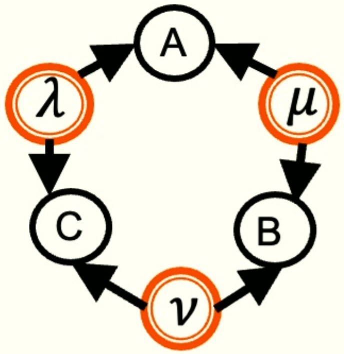
Add the linear inequalities of Shannon-type

Implicitize all entropic quantities that refer to latent variables

The result is the **marginal Shannon cone,** described by linear inequalities on entropies over observed variables only

Entropic constraint for the triangle scenario

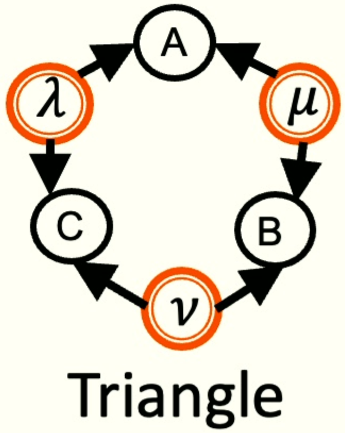
T. Fritz, 2012



Triangle

$$I(A:B) + I(A:C) \leq H(A)$$

Entropic constraint for the triangle scenario



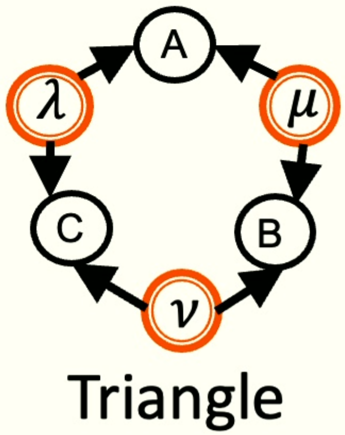
T. Fritz, 2012

$$A \perp B|\mu \quad \lambda \perp \mu\nu$$
$$A \perp C|\lambda \quad \mu \perp \lambda\nu$$
$$B \perp C|\nu \quad \nu \perp \lambda\mu$$

$$A \perp B|\mu \implies I(A:B|\mu) = 0 \implies I(A:B) \leq I(A:\mu)$$
$$A \perp C|\lambda \implies I(A:C|\lambda) = 0 \implies I(A:C) \leq I(A:\lambda)$$

By Shannon-type inequalities

$$I(A:B) + I(A:C) \leq I(A:\mu) + I(A:\lambda)$$



Entropic constraint for the triangle scenario

T. Fritz, 2012

$$A \perp B|\mu \quad \lambda \perp \mu\nu$$
$$A \perp C|\lambda \quad \mu \perp \lambda\nu$$
$$B \perp C|\nu \quad \nu \perp \lambda\mu$$

$$A \perp B|\mu \implies I(A:B|\mu) = 0 \implies I(A:B) \leq I(A:\mu)$$
$$A \perp C|\lambda \implies I(A:C|\lambda) = 0 \implies I(A:C) \leq I(A:\lambda)$$

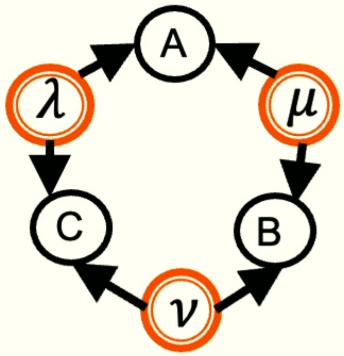
By Shannon-type inequalities

$$I(A:B) + I(A:C) \leq I(A:\mu) + I(A:\lambda)$$
$$\leq H(A) + I(\mu:\lambda)$$

By Shannon-type inequalities

$$\mu \perp \lambda \implies I(\mu:\lambda) = 0$$

$$I(A:B) + I(A:C) \leq H(A)$$

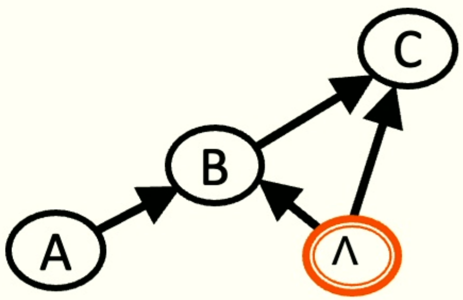


Triangle

$$\Longrightarrow \quad I(A:B) + I(A:C) \leq H(A)$$

Note that this inequality detects the incompatibility of

$$P_{ABC} = \tfrac{1}{2}[000] + \tfrac{1}{2}[111]$$



Instrumental

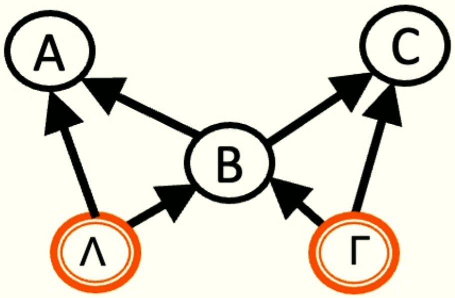
$$\Rightarrow \quad I(A:BC) \leq H(B)$$

Note that this inequality **also** detects the incompatibility of

$$P^{\text{pinch}}_{ABC} = \left(\tfrac{1}{2}[00]_{AC} + \tfrac{1}{2}[11]_{AC}\right)[0]_B$$

But not our example separating Evans and instrumental

$$P^{\text{pinch2}}_{ABC} = \tfrac{1}{2}\left(\tfrac{1}{2}[00]_{AC} + \tfrac{1}{2}[11]_{AC}\right)[0]_B + \tfrac{1}{2}[00]_{AC}[1]_B$$

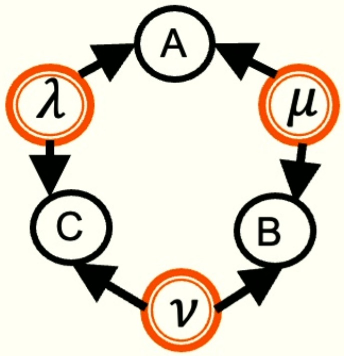


Evans

$$I(A:C|B) \leq H(B)$$

Note that this inequality detects the incompatibility of

$$P_{ABC}^{\text{pinch}} = \left(\tfrac{1}{2}[00]_{AC} + \tfrac{1}{2}[11]_{AC}\right)[0]_B$$



Triangle

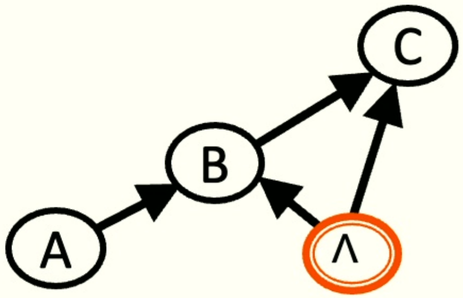
$$I(A:B) + I(A:C) \leq H(A)$$

Note that this inequality detects the incompatibility of

$$P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$$

But it fails to detect the incompatibility of

$$P_{ABC} = \frac{1}{3}[001] + \frac{1}{3}[010] + \frac{1}{3}[100]$$



Instrumental

$$\Longrightarrow \quad I(A:BC) \leq H(B)$$

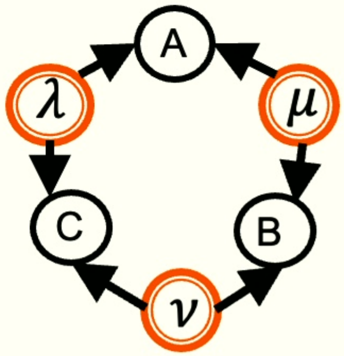
Note that this inequality **also** detects the incompatibility of

$$P^{\text{pinch}}_{ABC} = \left(\tfrac{1}{2}[00]_{AC} + \tfrac{1}{2}[11]_{AC}\right)[0]_B$$

But not our example separating Evans and instrumental

$$P^{\text{pinch2}}_{ABC} = \tfrac{1}{2}\left(\tfrac{1}{2}[00]_{AC} + \tfrac{1}{2}[11]_{AC}\right)[0]_B + \tfrac{1}{2}[00]_{AC}[1]_B$$

(Other examples of the separation **are** detected by the inequality)



Triangle

$$\Rightarrow \quad I(A:B) + I(A:C) \leq H(A)$$

Note that this inequality detects the incompatibility of

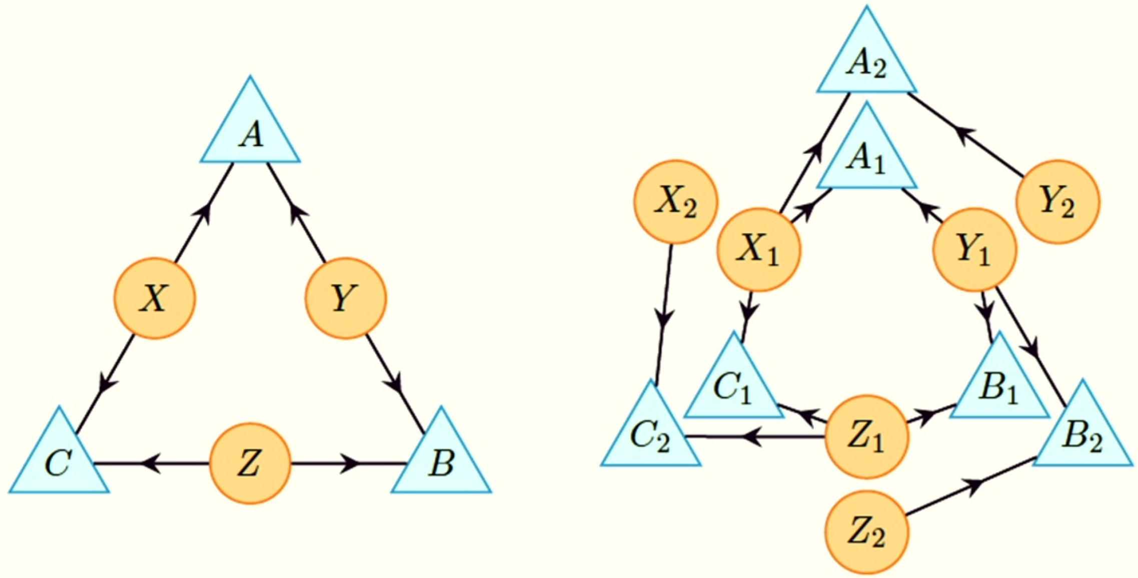
$$P_{ABC} = \tfrac{1}{2}[000] + \tfrac{1}{2}[111]$$

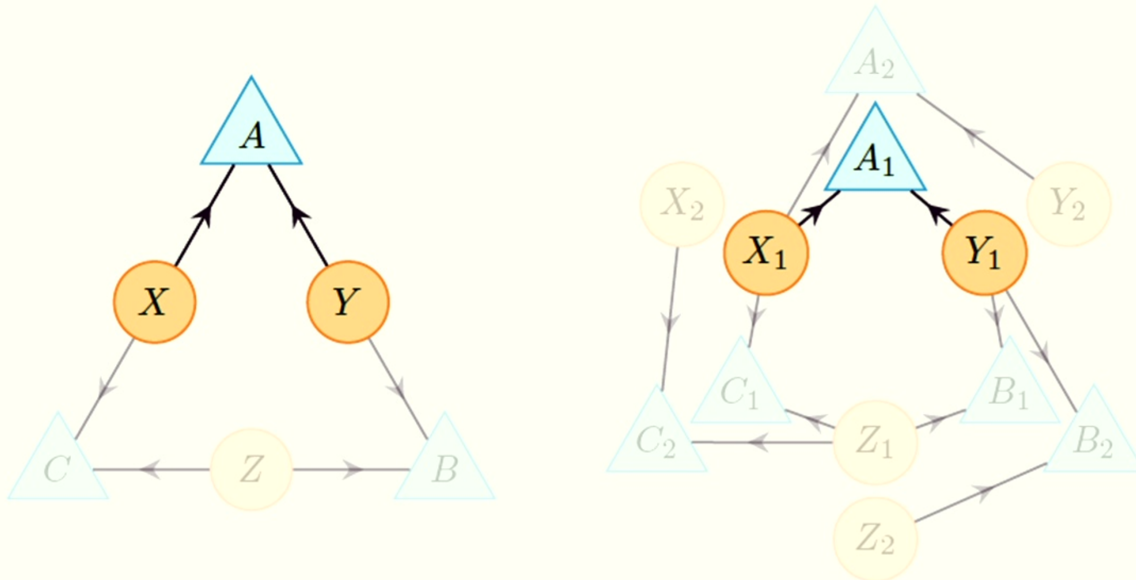
But it fails to detect the incompatibility of

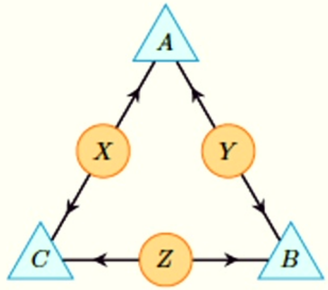
$$P_{ABC} = \tfrac{1}{3}[001] + \tfrac{1}{3}[010] + \tfrac{1}{3}[100]$$

The move to entropies has thrown away too much information to witness certain incompatibilities

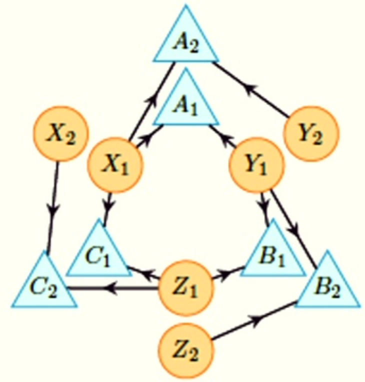
The inflation technique


A
X
Y
C
Z
B
A2
A1
X2
X1
Y1
Y2
C1
B1
C2
Z1
B2
Z2

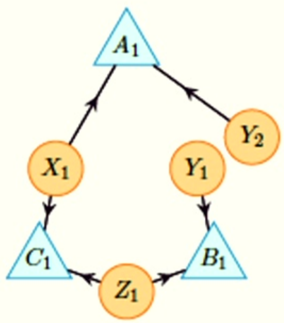

A
X
Y
C
Z
B
A2
A1
X2
X1
Y1
Y2
C1
B1
C2
Z1
B2
Z2



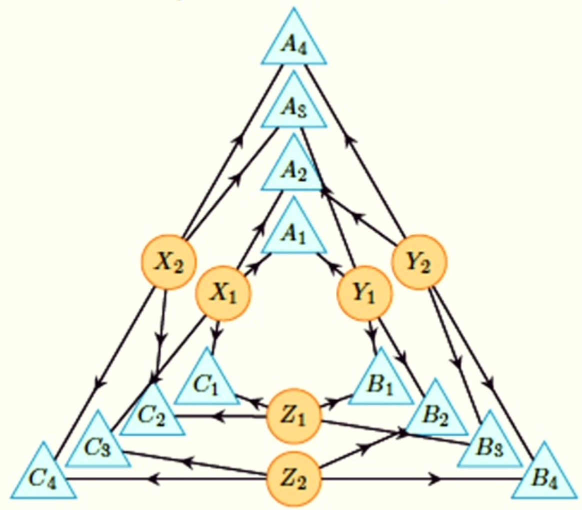
The Triangle Scenario



Spiral Inflation

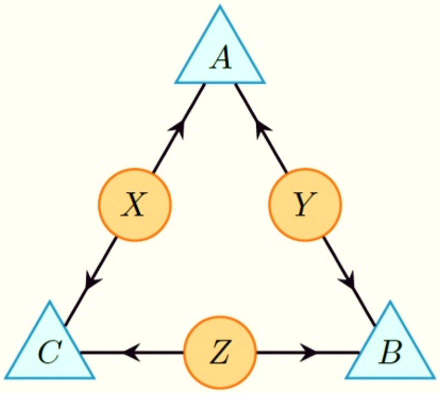


Cut Inflation



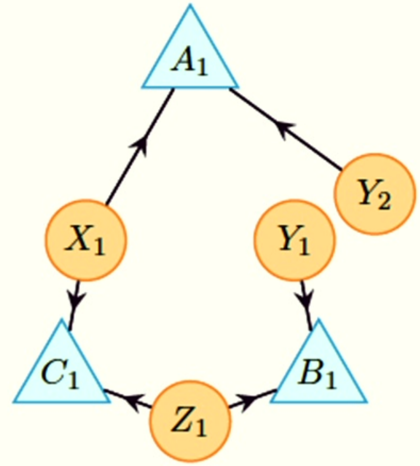
Large Inflation

model M on DAG G



$P_{A|XY}$
$P_{B|YZ}$
$P_{C|XZ}$
P_X
P_Y
P_Z

M' = G → G' Inflation of M

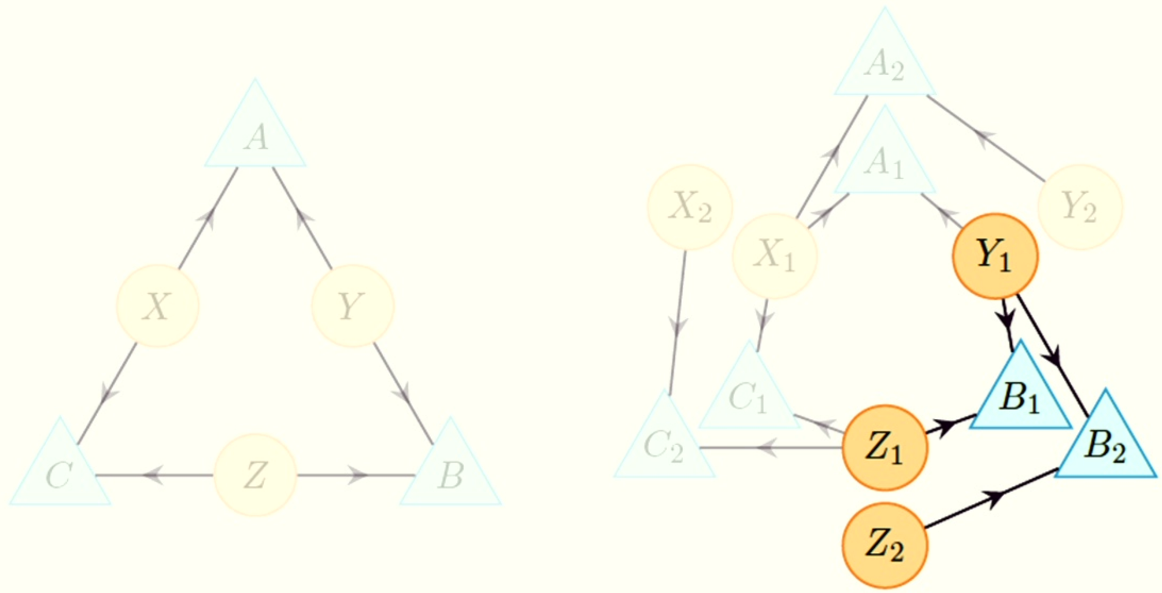


$P_{A_1|X_1Y_2}$
$P_{B_1|Y_1Z_1}$
$P_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

with symmetry constraint:

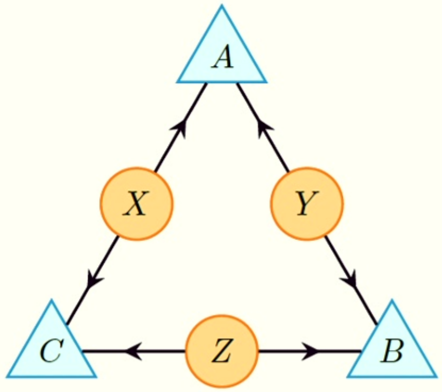
$$P_{Y_1} = P_{Y_2}$$

Injectable sets of observed variables
in the inflation DAG



$\{B_1 B_2\}$ is *not* an injectable set

model M on DAG G



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|YZ}$

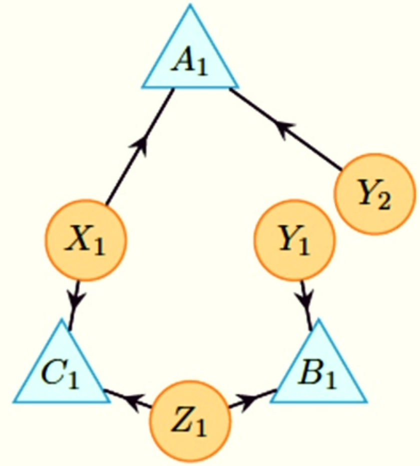
$\mathrm{P}_{C|XZ}$

P_{X}

P_{Y}

P_{Z}

M’ = G $\rightarrow$ G’ Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|Y_1Z_1}$

$\mathrm{P}_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

P_{Y_2}

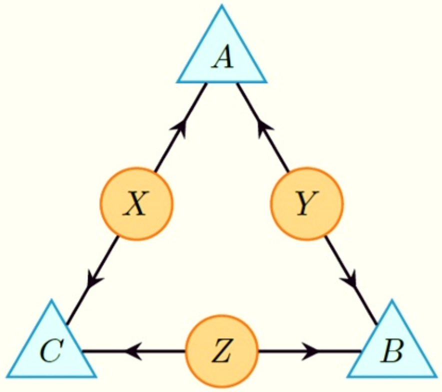
P_{Z_1}

$\{A_1C_1\}$ is an injectable set

$$\mathrm{P}_{A_1C_1} = \sum_{X_1Y_2Z_1} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{C_1|X_1Z_1}\mathrm{P}_{X_1}\mathrm{P}_{Y_2}\mathrm{P}_{Z_1}$$

$$\mathrm{P}_{AC} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{C|XZ}\mathrm{P}_{X}\mathrm{P}_{Y}\mathrm{P}_{Z}$$

model M on DAG G



$P_{A|XY}$

$P_{B|YZ}$

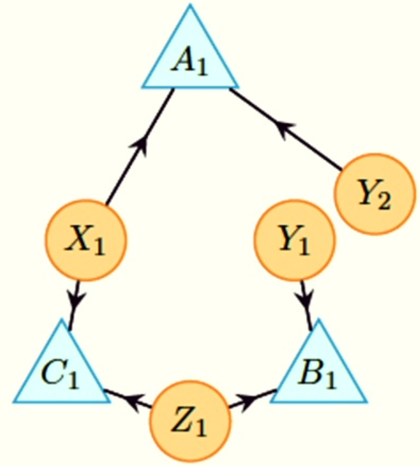
$P_{C|XZ}$

P_X

P_Y

P_Z

M' = G → G' Inflation of M



$P_{A_1|X_1Y_2}$

$P_{B_1|Y_1Z_1}$

$P_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

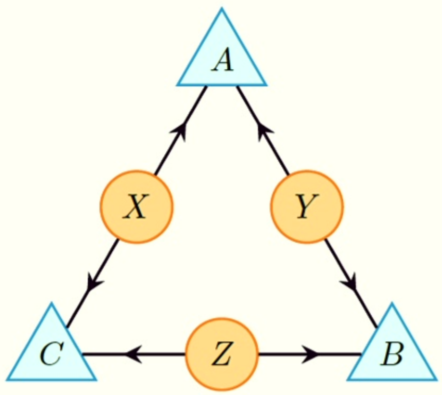
P_{Y_2}

P_{Z_1}

$\{A_1B_1\}$ is *not* an injectable set

$$P_{A_1B_1} = \left(\sum_{X_1Y_2} P_{A_1|X_1Y_2} P_{Y_2} P_{X_1}\right)\left(\sum_{Z_1Y_1} P_{B_1|Y_1Z_1} P_{Y_1} P_{Z_1}\right)$$

model M on DAG G



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|YZ}$

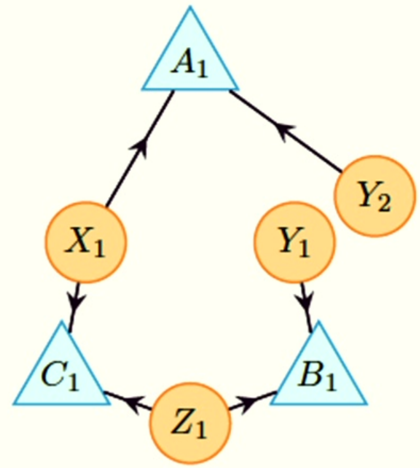
$\mathrm{P}_{C|XZ}$

P_{X}

P_{Y}

P_{Z}

M' = G $\rightarrow$ G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|Y_1Z_1}$

$\mathrm{P}_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

P_{Y_2}

P_{Z_1}

Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_1C_1\}, \{B_1C_1\}$

$$(P_A, P_B, P_C, P_{AC}, P_{BC}) \text{ compatible with } M \implies (P_{A_1}, P_{B_1}, P_{C_1}, P_{A_1C_1}, P_{B_1C_1}) \text{ compatible with } M'$$

where

$\mathrm{P}_{A_1} = \mathrm{P}_A$ $\quad$ $\mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$

$\mathrm{P}_{B_1} = \mathrm{P}_B$ $\quad$ $\mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$

$\mathrm{P}_{C_1} = \mathrm{P}_C$

M′ = G → G′ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$$
is compatible with M

$$\Longrightarrow$$

$$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

Deriving causal compatibility inequalities by the inflation technique

M′ = G → G′ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\qquad$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$
is compatible with M

$\Longrightarrow$

$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

$\Downarrow$

$I_{\mathcal{S}'}$ is **satisfied** for
$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$

M′ = G → G′ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

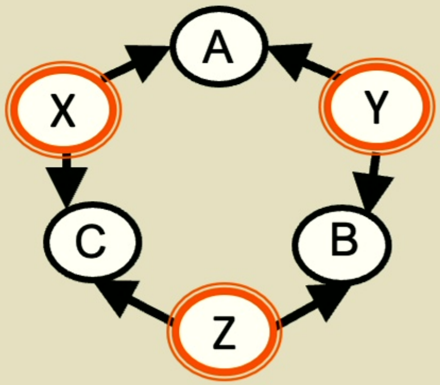
$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

is compatible with M

$$\Downarrow$$

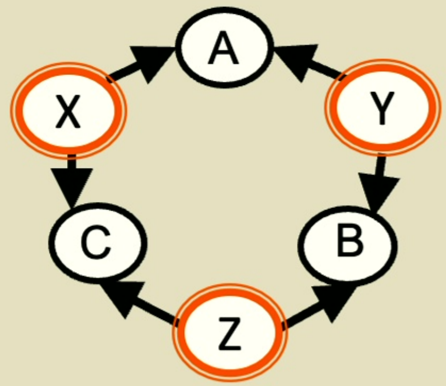
$I_{\mathcal{S}}$ is **satisfied** for

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$



$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

causal compatibility inequality



$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

causal compatibility inequality

Consider binary X, Y and Z

$\exists P_{XYZ}$ with $P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ}$ as marginals

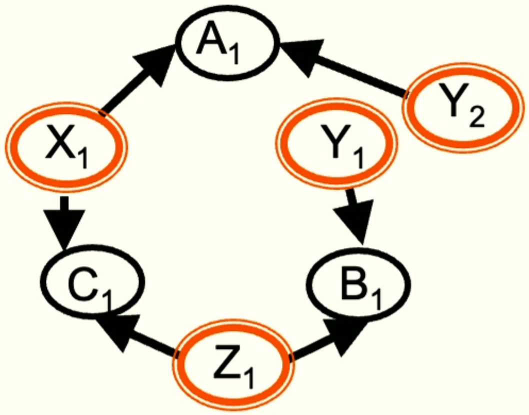
⇩

Marginal compatibility constraints

$$P_X = \sum_Y P_{XY} = \sum_Z P_{XZ}$$
$$P_Y = \sum_X P_{XY} = \sum_Z P_{YZ}$$
$$P_Z = \sum_X P_{XZ} = \sum_Y P_{YZ}$$
$$P_X + P_Y + P_Z - P_{XY} - P_{YZ} - P_{XZ} \leq 1$$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

Linear quantifier elimination

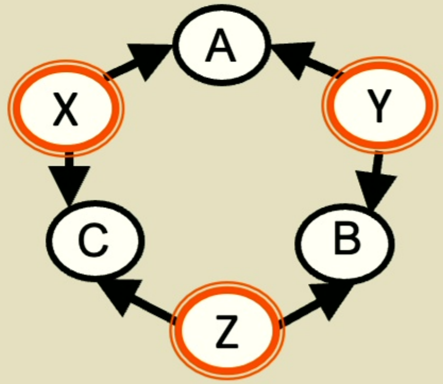


$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

$$A_1 \perp_d B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$$

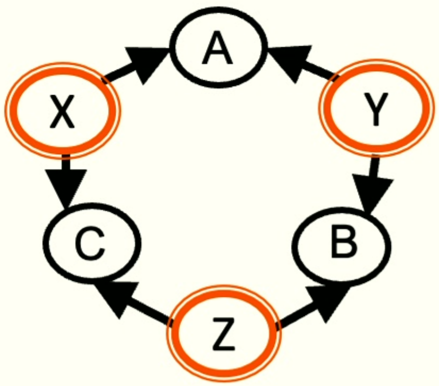
$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M' $\implies$ $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

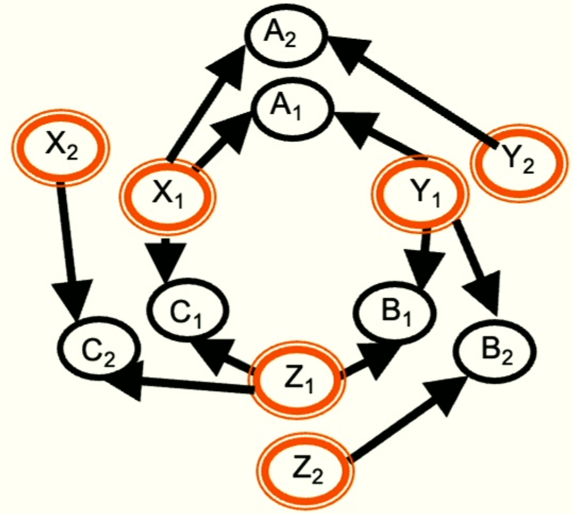
This is a causal compatibility inequality for M'



$$P_A(1)P_B(1)P_C(1)$$
$$\leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1)$$
$$+ P_{AC}(11)P_B(1) + P_{ABC}(000)$$

causal compatibility inequality





$$
\begin{aligned}
&P_A(1)P_B(1)P_C(1)\\
&\leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1)\\
&+ P_{AC}(11)P_B(1) + P_{ABC}(000)
\end{aligned}
$$

$$
\begin{aligned}
&P_{A_2}(1)P_{B_2}(1)P_{C_2}(1)\\
&\leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1)\\
&+ P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)
\end{aligned}
$$

Polynomial inequality constraints for causal compatibility with the original DAG

⇐

Linear inequality constraints from marginal compatibility
(from linear quantifier elimination)

+

Polynomial equality constraints from causal compatibility with the inflated DAG
(e.g., from d-separation relations)

The technique defines an algorithm for deriving causal compatibility inequalities and for testing compatibility

Proof that this provides a convergent heirarchy of tests:
Navascués & Wolfe, J. Causal Inf. 8(1) 70 (2020)

Approaches to Bell arguments that follow essentially
the logic of the inflation technique:

Fine's proof of CHSH inequalities
Hardy's proof of Bell's theorem
The Greenberger-Horne-Zeilinger proof of Bell's theorem
Bell's theorem by way of Kochen-Specker noncontextuality
Braunstein-Caves entropic inequalities
Symmetric extensions / shareability of local correlations
...

Linear inequality constraints from marginal compatibility (from linear quantifier elimination)

Polynomial inequality constraints for causal compatibility with the original DAG

⇐

+

Polynomial equality constraints from causal compatibility with the inflated DAG (e.g., from d-separation relations)

The technique defines an algorithm for deriving causal compatibility inequalities and for testing compatibility

Proof that this provides a convergent heirarchy of tests:
Navascués & Wolfe, J. Causal Inf. 8(1) 70 (2020)

Approaches to Bell arguments that follow essentially the logic of the inflation technique:

Fine's proof of CHSH inequalities
Hardy's proof of Bell's theorem
The Greenberger-Horne-Zeilinger proof of Bell's theorem
Bell's theorem by way of Kochen-Specker noncontextuality
Braunstein-Caves entropic inequalities
Symmetric extensions / shareability of local correlations
...

Approaches to Bell arguments that follow essentially the logic of the inflation technique:

Fine's proof of CHSH inequalities
Hardy's proof of Bell's theorem
The Greenberger-Horne-Zeilinger proof of Bell's theorem
Bell's theorem by way of Kochen-Specker noncontextuality
Braunstein-Caves entropic inequalities
Symmetric extensions / shareability of local correlations
...