Title: Lecture - Causal Inference, PHYS 777

Speakers: Robert Spekkens

Collection/Series: Causal Inference (Elective), PHYS 777, March 31 - May 2, 2025

Subject: Quantum Foundations

Date: April 11, 2025 - 11:30 AM

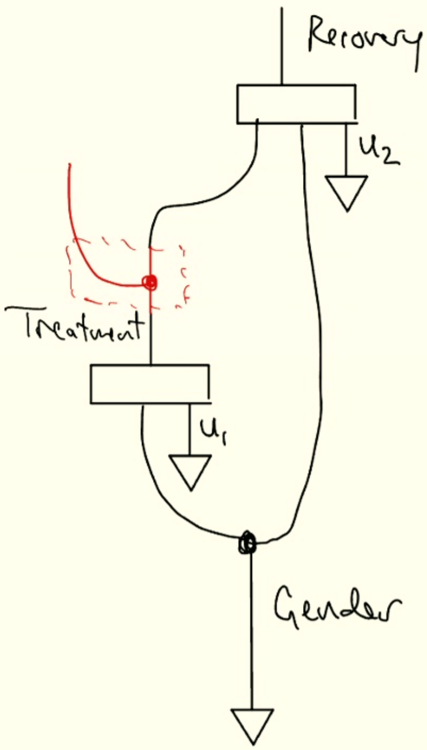
URL: https://pirsa.org/25040040

The observational and interventional dominance orders of causal structures

arXiv:2502.07891
arxiv:2407.01686

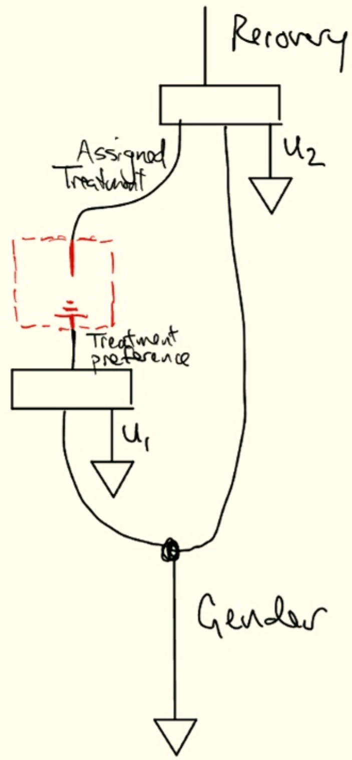
Probing schemes

Observe
probing scheme on X



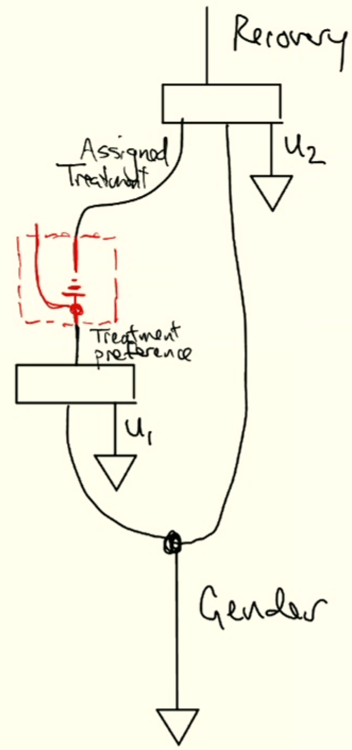
Record value of X
without disturbing

Do
Probing scheme on X



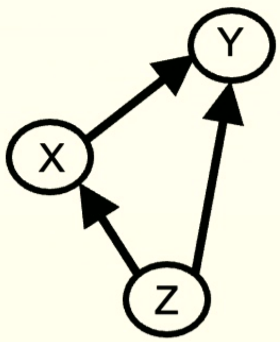
Ignore value of X
+ input new version of X

Observe-and-do
probing scheme on X



Record value of X
+ input new version of X

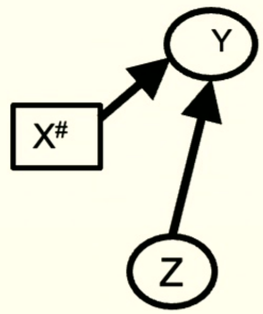
Observe
probing scheme on X



$P_{Y|XZ}$
$P_{X|Z}$
P_Z

$$P_{XYZ} = P_{Y|XZ} P_{X|Z} P_Z$$

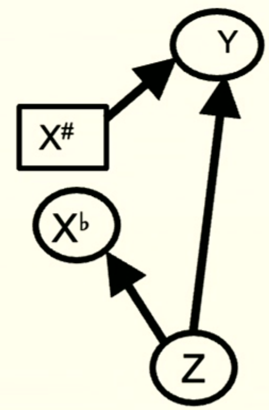
Do
Probing scheme on X



$P_{Y|X^\sharp Z}$
P_Z

$$P_{YZ|X^\sharp} = P_{Y|X^\sharp Z} P_Z$$

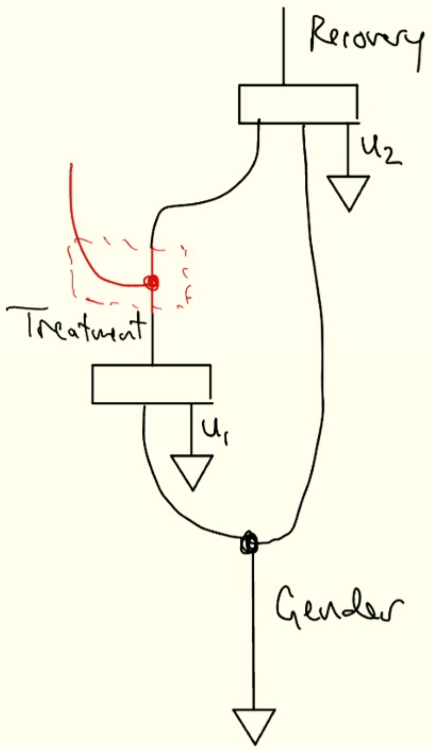
Observe-and-do
probing scheme on X



$P_{Y|X^\sharp Z}$
$P_{X^\flat|Z}$
P_Z

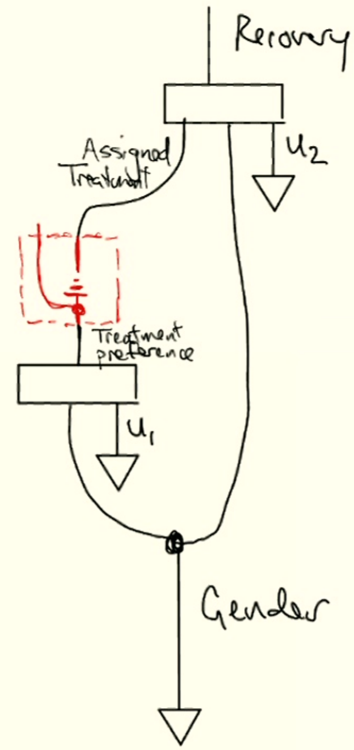
$$P_{YX^\flat Z|X^\sharp} = P_{Y|X^\sharp Z} P_{X^\flat|Z} P_Z$$

Observe
probing scheme on X



Record value of X
without disturbing

Observe-and-do
probing scheme on X



Record value of X
+ input new version of X

Definition: A probing scheme is said to be **informationally complete** if it can learn everything that can be learned by any probing scheme restricted to the visible nodes

A probing scheme that implements observe-and-do on every visible node X is informationally complete

Restriction of scope to sets of causal structures with a fixed ordering of visible nodes

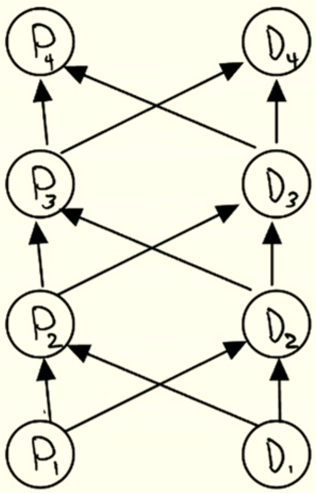
If causal relations among variables are described by a DAG, then these variables must be **temporally localized**

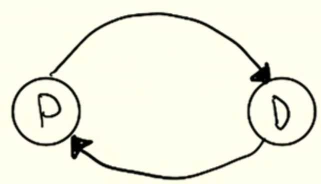
What about cases where variables are not temporally localized?

Example: price and demand

But if we coarse-grain a number of temporally localized variables into a single temporally delocalized variable

We lose the property of acyclicity, i.e., we leave the DAG framework





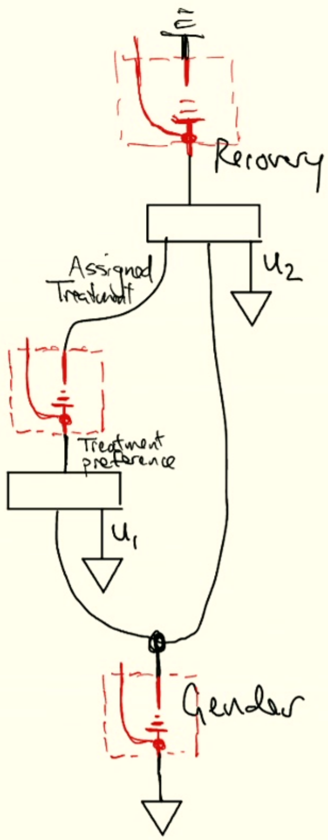
variables are temporally localized → variables are temporally ordered

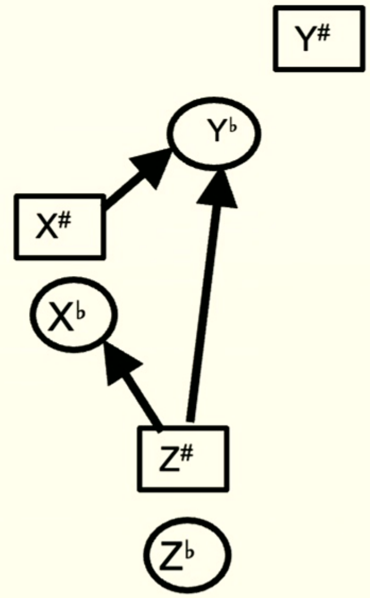
* assuming no space-like separation

Note also: knowledge regarding temporal ordering is a prerequisite for implementing fully general probing schemes in practice

Therefore, we consider the set of causal structures consistent with a **fixed temporal ordering** of the visible nodes

Observe&Do realizability of a distribution





$P_{Y^\flat|X^\sharp Z^\sharp}$
$P_{X^\flat|Z^\sharp}$
$P_{Z^\flat}$

$$P_{Y^\flat X^\flat Z^\flat|X^\sharp Y^\sharp Z^\sharp} = P_{Y^\flat|X^\sharp Z^\sharp} P_{X^\flat|Z^\sharp} P_{Z^\flat}$$

pDAG (partitioned DAG) = DAG where the nodes are partitioned into visible and latent
Vnodes(G) = visible nodes of G
Lnodes(G) = latent nodes of G

M_{obs} (G,$\mathbf{c}_{\text{Vnodes(G)}}$) = The set of probability distributions over visible variables of cardinalities $\mathbf{c}_{\text{Vnodes(G)}}$ that are observationally realizable by pDAG G.

$M_{O\&D}$ (G,$\mathbf{c}_{\text{Vnodes(G)}}$) = The set of probability distributions over visible variables of cardinalities $\mathbf{c}_{\text{Vnodes(G)}}$ that are observe&do realizable by pDAG G.

Observational dominance

Definition: Let G and G′ be two pDAGs such that Vnodes(G)= Vnodes(G′). We say that G ***observationally dominates*** G′ (denoted G⪰G′) when the set of observationally realizable distributions of G includes the set of observationally realizable distributions of G′, regardless of the assignment of cardinalities of the visible variables.

That is,

$$\mathcal{G} \succeq_{\text{obs}} \mathcal{G}' \quad \text{iff} \quad \forall \vec{c}_{\texttt{Vnodes}(\mathcal{G})} \in \mathbb{N}^{|\texttt{Vnodes}(\mathcal{G})|} : \mathcal{M}_{obs}(\mathcal{G}', \vec{c}_{\texttt{Vnodes}(\mathcal{G}')}) \subseteq \mathcal{M}_{obs}(\mathcal{G}, \vec{c}_{\texttt{Vnodes}(\mathcal{G})})$$

where M_{obs} (G,$\mathbf{c}_{\text{Vnodes(G)}}$) = The set of probability distributions over visible variables of cardinalities $\mathbf{c}_{\text{Vnodes(G)}}$ that are observationally realizable by pDAG G.

Dominance ($G \succeq G'$)

Nondominance ($G \not\succeq G'$)

Strict dominance ($G \succ G'$)

Equivalence ($G \simeq G'$)
= mutual dominance ($G \succeq G'$ and $G' \succeq G$)

Incomparability
= mutual nondominance ($G \not\succeq G'$ and $G' \not\preceq G$).

Observe&Do dominance

Definition: Let G and G′ be two pDAGs such that Vnodes(G)= Vnodes(G′). We say that G ***Observe&Do dominates*** G′ (denoted G$\succeq$G′) when the set of O&D-realizable distributions of G includes the set of O&D-realizable distributions of G′, regardless of the assignment of cardinalities of the visible variables.

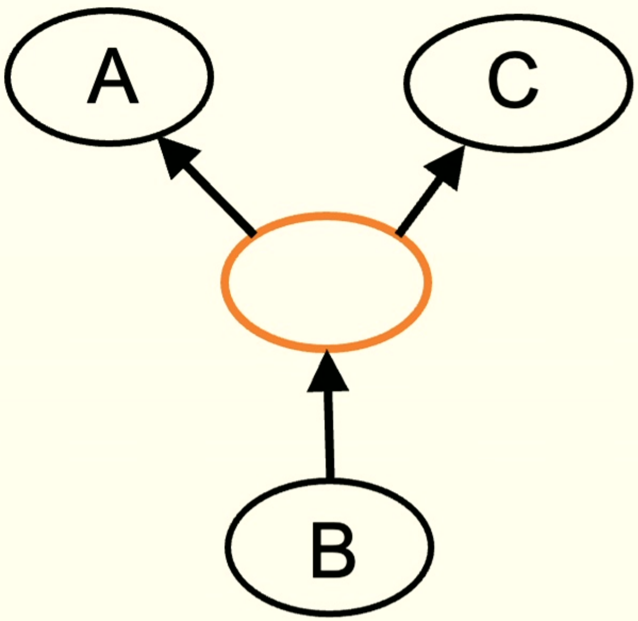
That is,

$$\mathcal{G} \succeq_{\mathrm{O\&D}} \mathcal{G}' \quad \text{iff} \quad \begin{array}{c} \forall \vec{c}_{\mathtt{Vnodes}(\mathcal{G})} \in \mathbb{N}^{|\mathtt{Vnodes}(\mathcal{G})|}: \\ \mathcal{M}_{O\&D}(\mathcal{G}', \vec{c}_{\mathtt{Vnodes}(\mathcal{G}')}) \subseteq \mathcal{M}_{O\&D}(\mathcal{G}, \vec{c}_{\mathtt{Vnodes}(\mathcal{G})}). \end{array}$$

where $M_{\mathrm{O\&D}}$ (G,$\mathbf{c}_{\mathrm{Vnodes(G)}}$) = The set of probability distributions over visible variables of cardinalities $\mathbf{c}_{\mathrm{Vnodes(G)}}$ that are Observe&Do realizable by pDAG G.

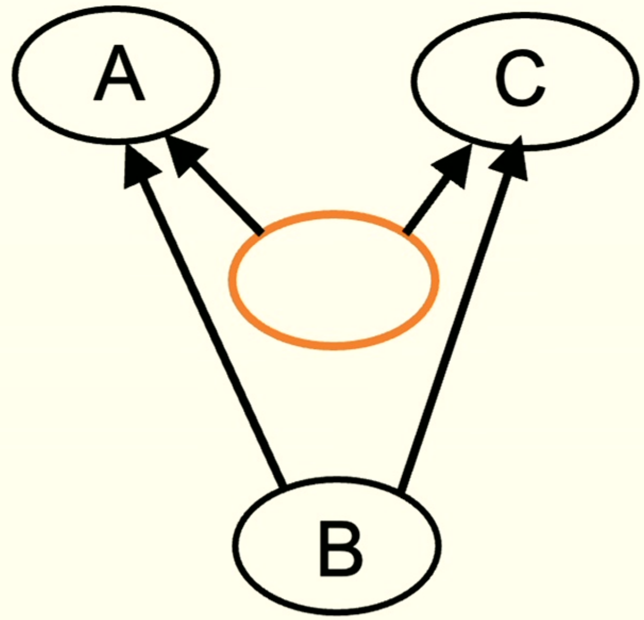
When is it impossible to distinguish two causal structures even when there is access to informationally complete probing schemes?

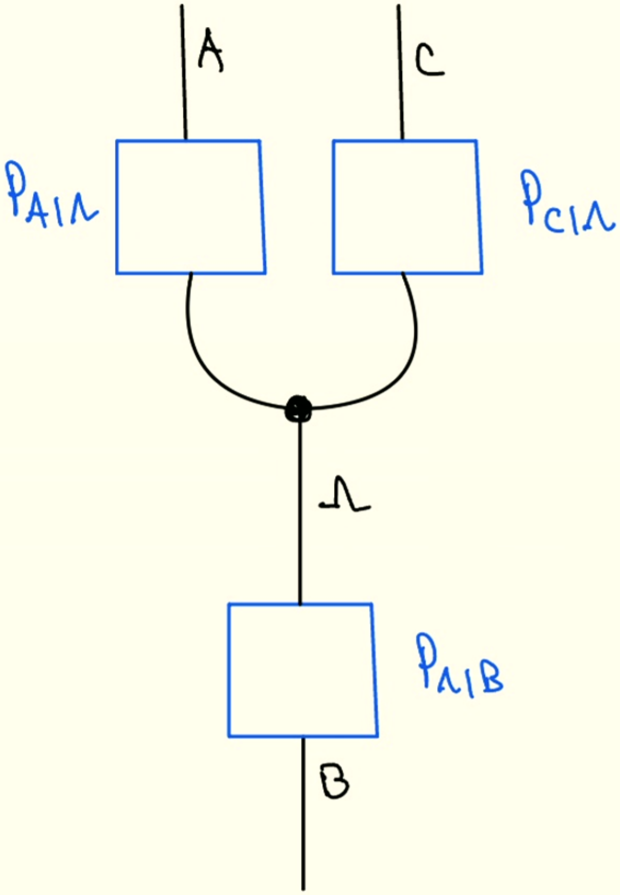
Exogenization rule



$\simeq$

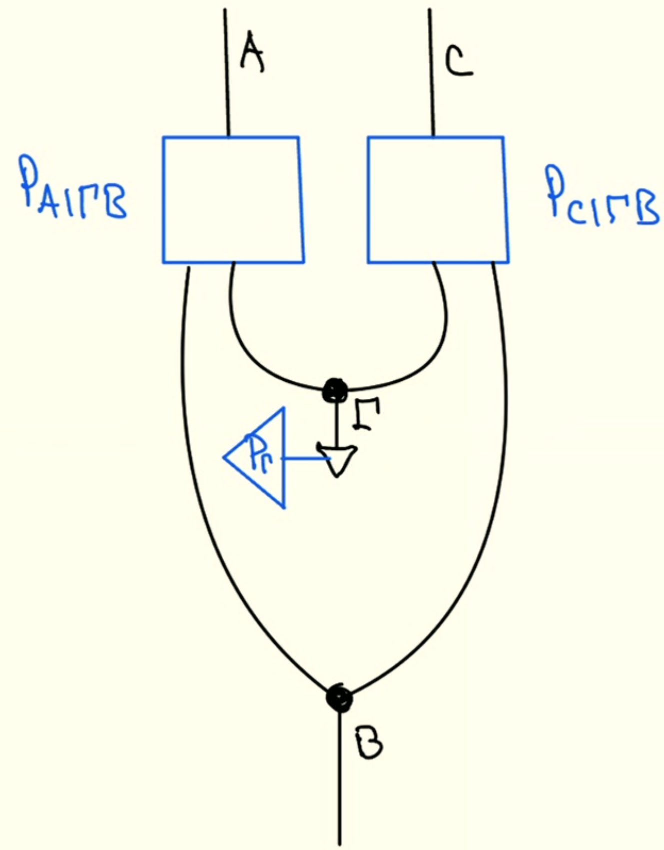
interventionally
equivalent

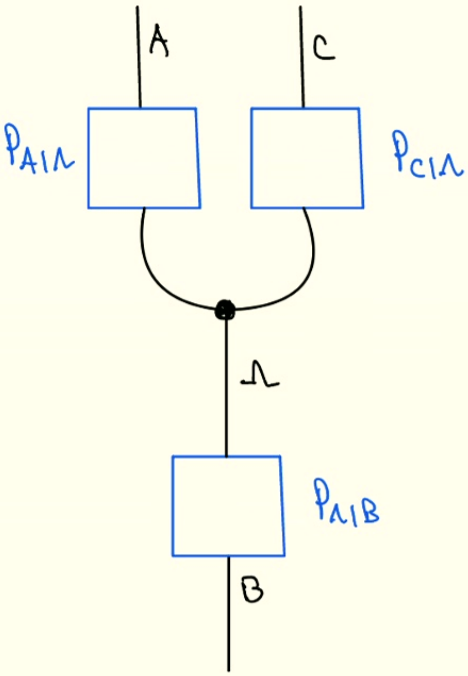


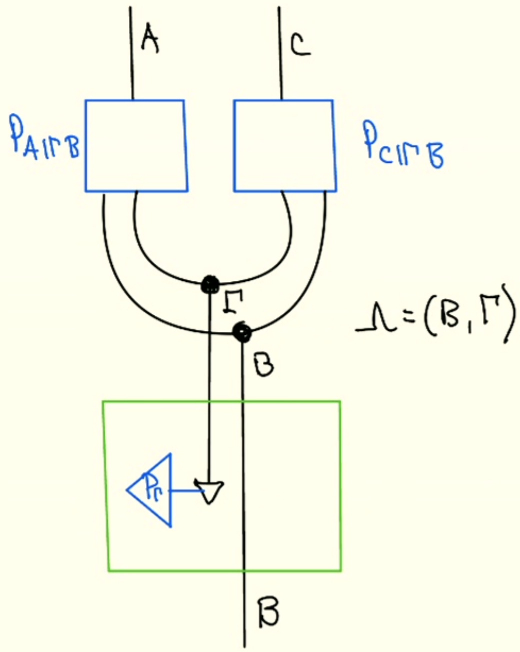


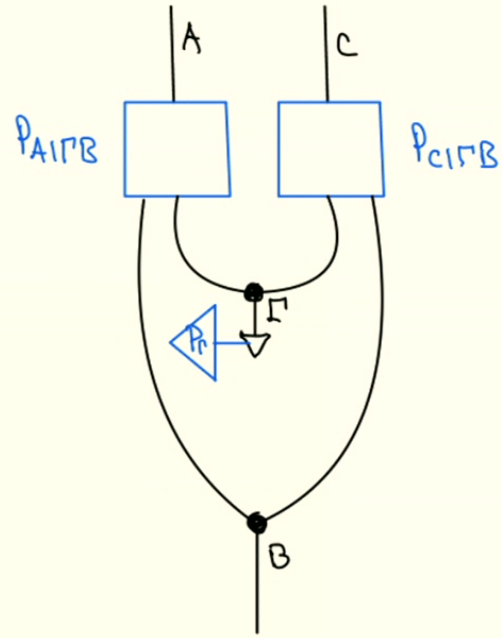
≃

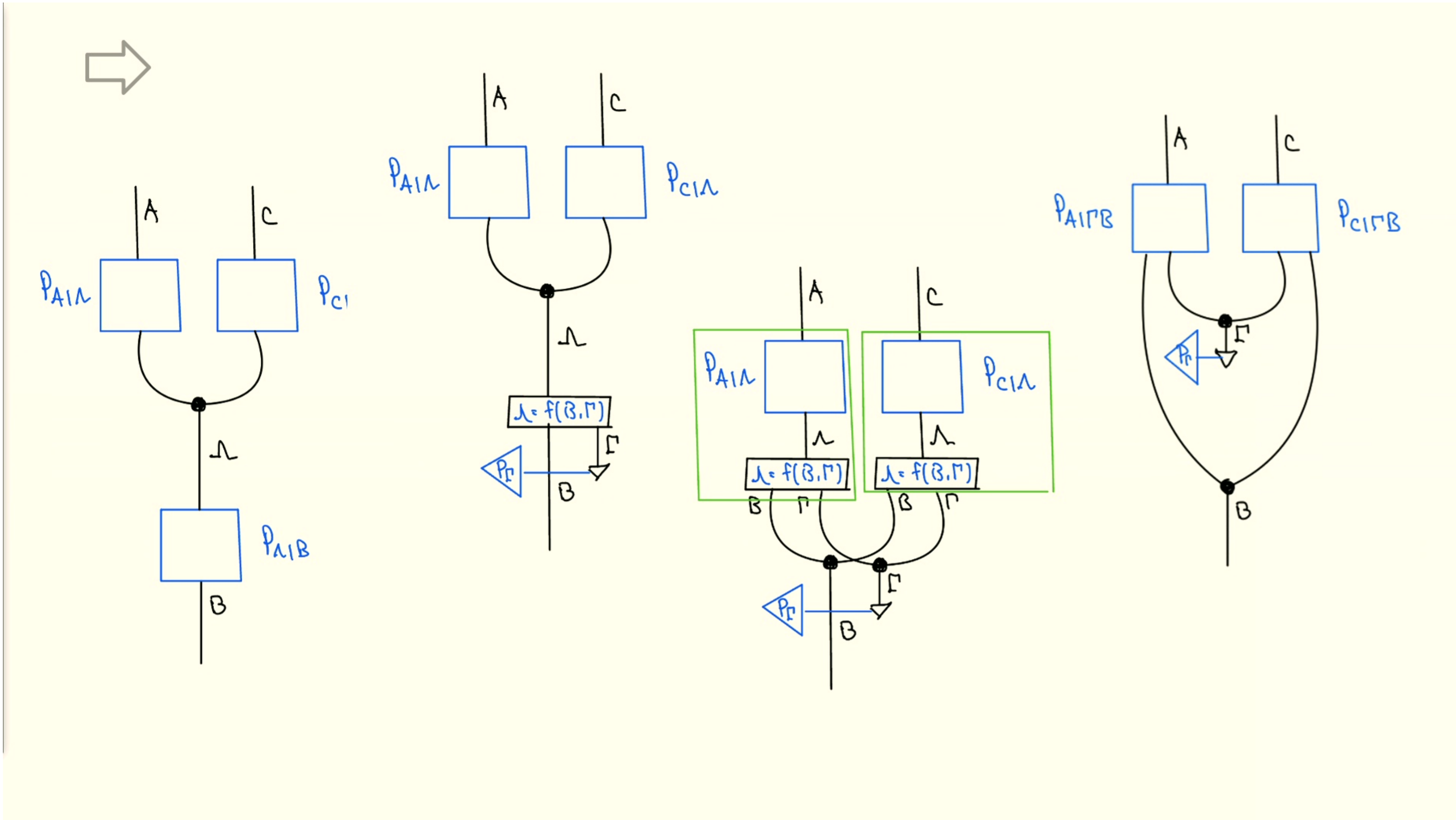
interventionally equivalent

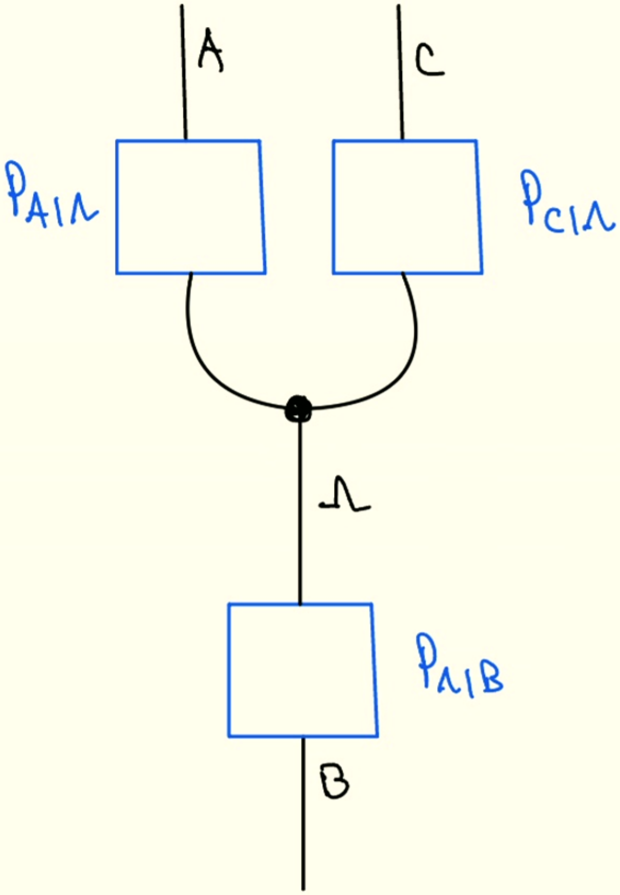



A
C
$P_{A|\Lambda}$
$P_{C|\Lambda}$
Λ
$P_{\Lambda|B}$
B


A
C
$P_{A|\Gamma B}$
$P_{C|\Gamma B}$
Γ
B
$\Lambda = (B, \Gamma)$
P_{Γ}
B

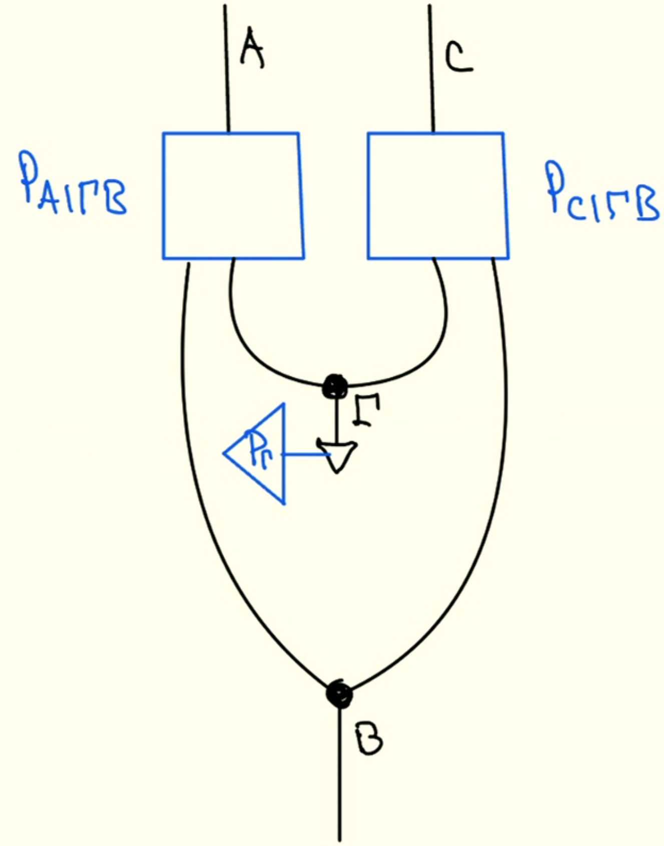

A
C
$P_{A|\Gamma B}$
$P_{C|\Gamma B}$
Γ
P_{Γ}
B


A
C
P_{A|Λ}
P_{C|
Λ
P_{Λ|B}
B
P_{C|Λ}
Λ = f(B, Γ)
Γ
P_Γ
P_{A|ΓB}
P_{C|ΓB}

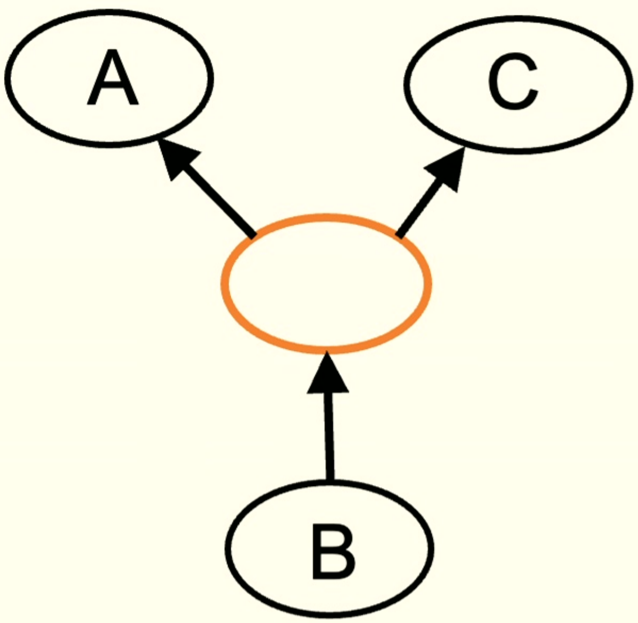


$\simeq$

interventionally equivalent

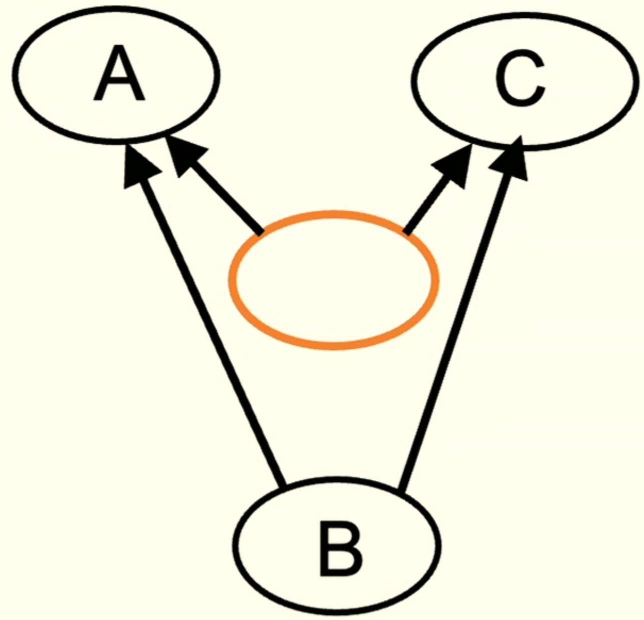


Exogenization rule

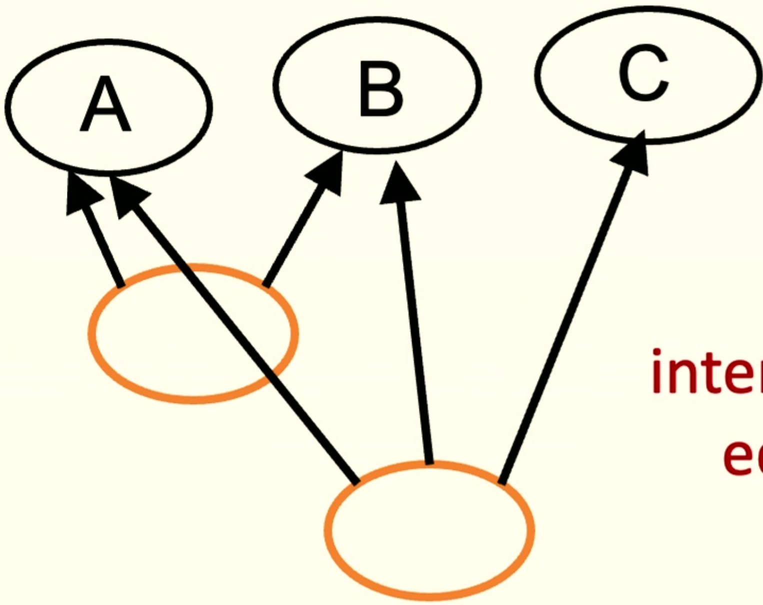


≃

interventionally
equivalent

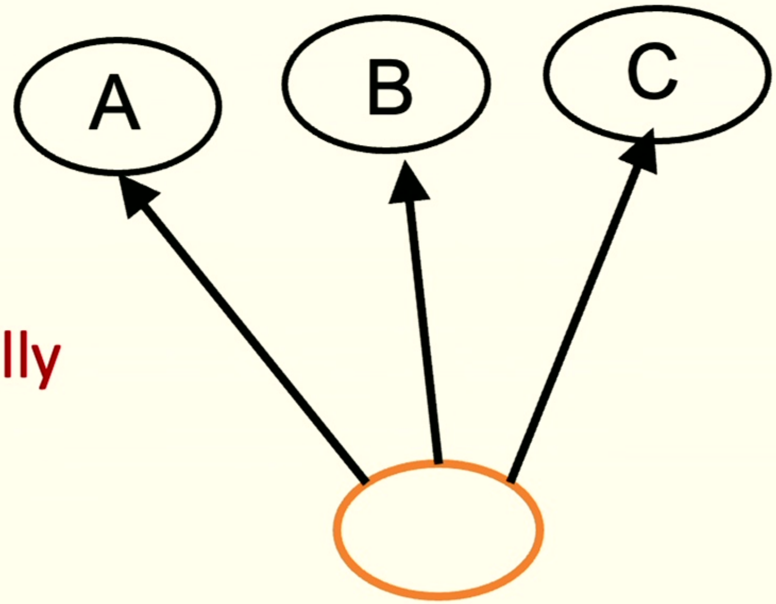


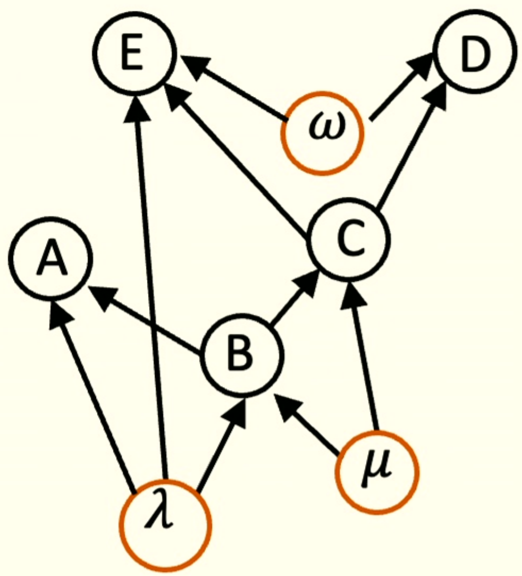
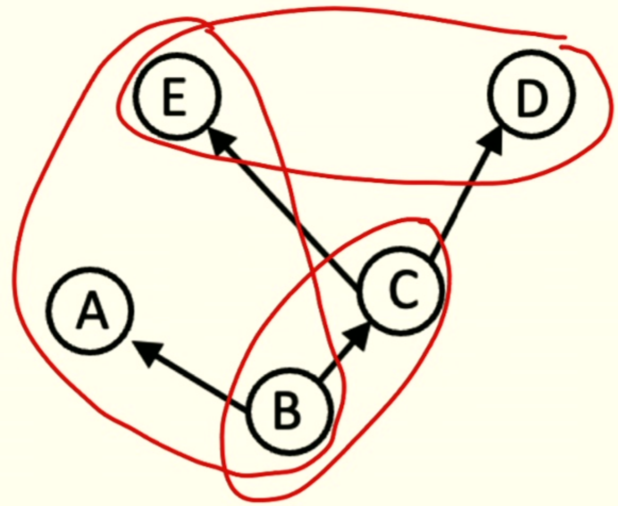
Eliminating redundant latents rule



$\simeq$

interventionally equivalent




E
D
ω
A
C
B
μ
λ

E
D
A
C
B

Definition: an mDAG with node set V is a pair (D, *B*), where D is a DAG over V (which we refer to as the directed structure) and *B* is a simplicial complex over V (the elements of which describe the nodes that share a latent common cause)

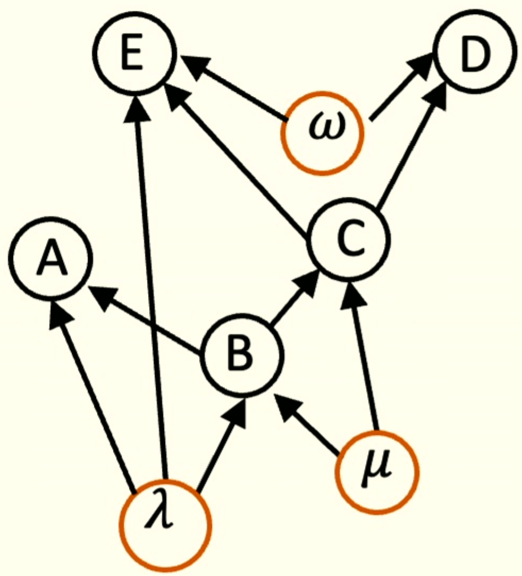
Definition: An abstract Simplicial Complex over a finite set V is a set *B* of subsets of V such that

- B includes all singleton sets

$\{v\} \in B$ for all $v \in V$

- If a subset of V is in B, then so are all of its subsets

If $S \subseteq T \subseteq V$ and $T \in B$, then $S \in B$


E
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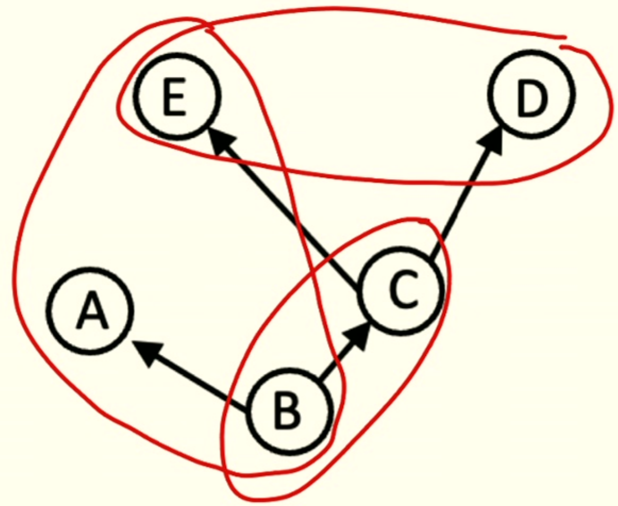
E
D
C
A
B

Definition: an mDAG with node set V is a pair (D, *B*), where D is a DAG over V (which we refer to as the directed structure) and *B* is a simplicial complex over V (the elements of which describe the nodes that share a latent common cause)

Definition: An abstract Simplicial Complex over a finite set V is a set *B* of subsets of V such that

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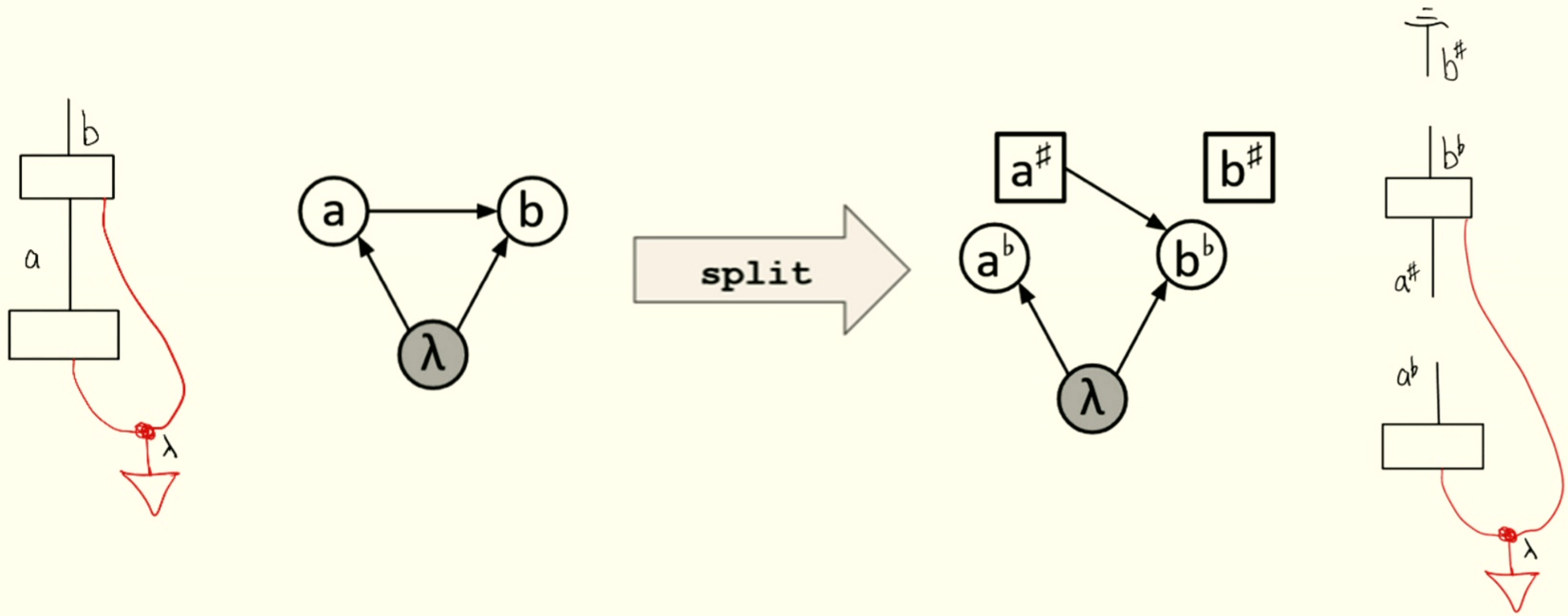
If $S \subseteq T \subseteq V$ and $T \in B$, then $S \in B$

Informationally complete probing schemes (such as Observe&Do) cannot discriminate pDAGs that are associated to the same mDAG.

Therefore, having different mDAGs is necessary for Observe&Do inequivalence. We will see that it is also sufficient.



b
a
λ
a
b
λ
split
a#
b#
a♭
b♭
λ
b#
b♭
a#
a♭
λ

Let G and G’ be two mDAGs with the same sets of nodes.

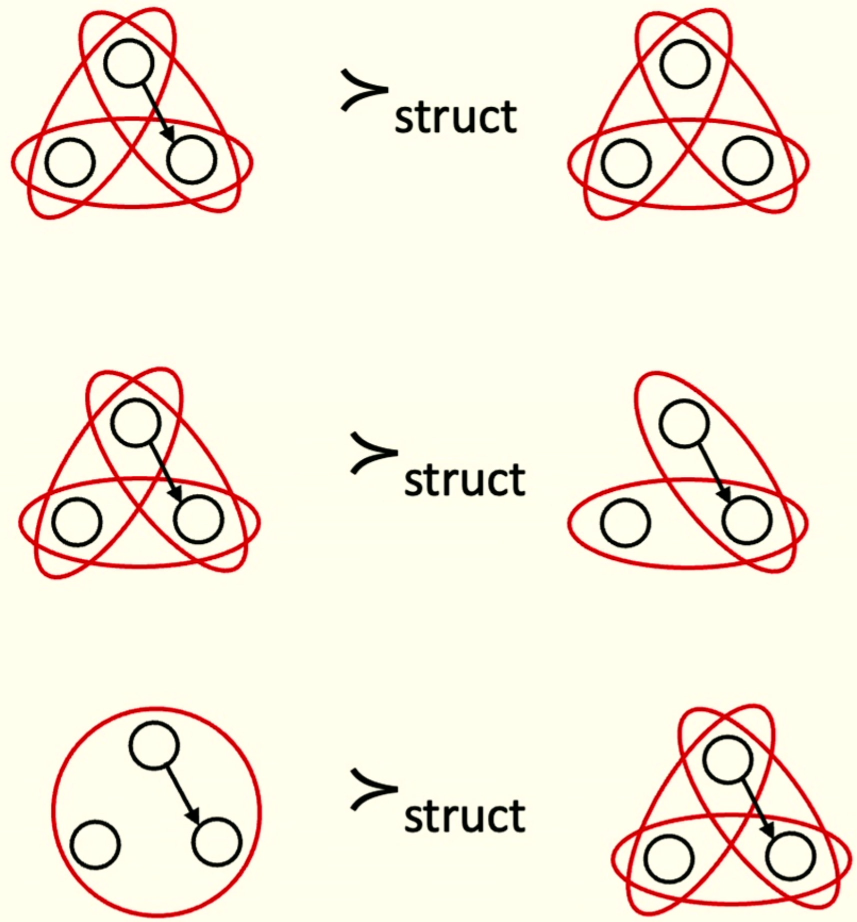
G **structurally dominates** G’ if:

(i) the directed structure of G’ can be obtained from the directed structure of G by dropping edges

DirectedEdges(G’) ⊆ DirectedEdges(G)

(i) the simplicial complex of G’ can be obtained from the simplicial complex of G by dropping faces

Faces(G’) ⊆ Faces(G)


≻struct
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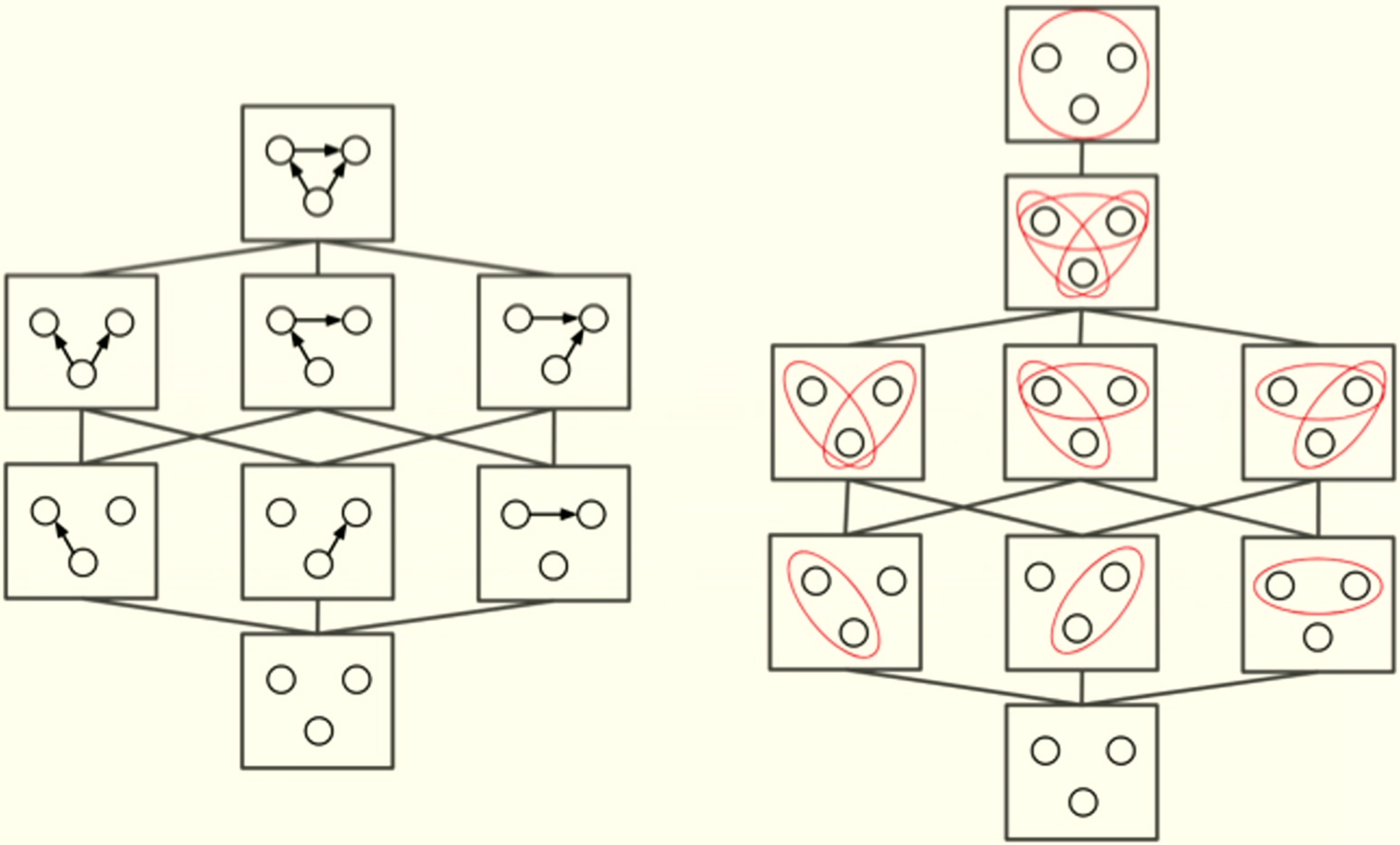
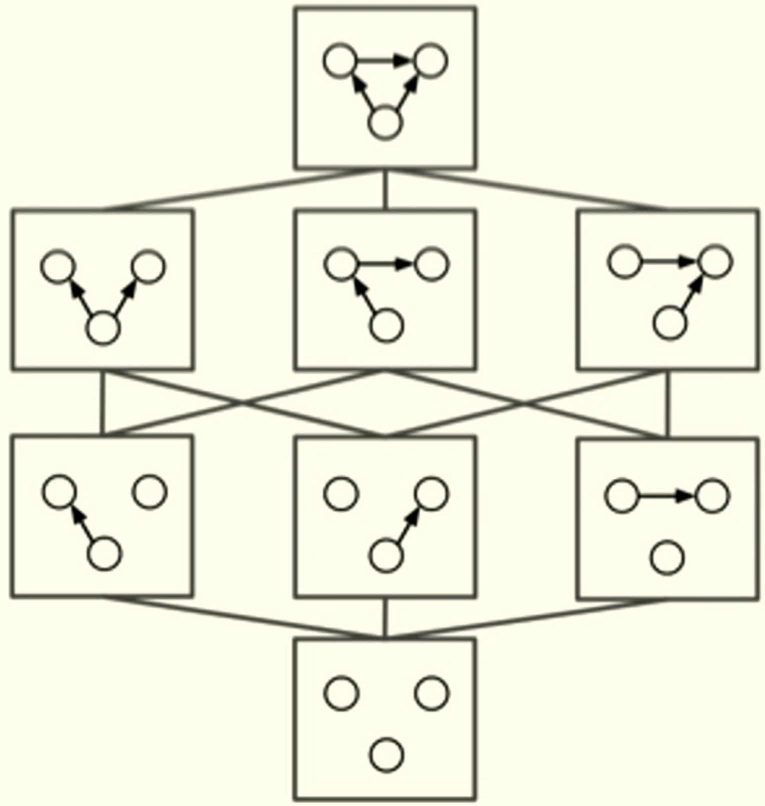


Special cases of observational dominance order



For confounder-free mDAGs

Observational dominance of mDAGs

$\Longleftrightarrow$

Structural dominance of mDAGs

$G \succeq_{obs} G'$

$G \succeq_{struct} G'$

For confounder-free mDAGs,
the conditional independence relations implied by the d-separation relations are **all** the constraints on the distribution

Recall the Markov condition characterizing all compatible distributions for a confounder-free mDAG

We saw that this set can also be characterized by the set of conditional independence relations described by the local Markov condition together with semi-graphoid axioms

For confounder-free mDAGs

Observational dominance of mDAGs

$\Leftrightarrow$

Structural dominance of mDAGs

$G \succeq_{obs} G'$

$G \succeq_{struct} G'$

If half: The case of structural equivalence is trivial. Strict structural dominance means that there is a directed edge in G that is not in G'. This will lead to G' having a set of d-separation relations (and hence conditional independence relations) that is a strict superset of that of G. As CI relations exhaust the constraints for confounder-free mDAGs, we conclude strict obs'l dominance of G over G'.

For confounder-free mDAGs

Observational dominance of mDAGs $\Longleftrightarrow$ Structural dominance of mDAGs

$G \succeq_{obs} G'$ $\quad$ $G \succeq_{struct} G'$

Only if half: Consider the contrapositive. Lack of structural dominance means that there is a directed edge in G' that is not in G. This will lead to a CI relation in G that is violated in G', hence no obs'l dominance of G over G'.

IC* algorithm and PC algorithm

Set of conditional independence relations in observational distribution ⇨ Feasible and infeasible confounder-free mDAGs

For directed-edge-free mDAGs

Observational dominance of mDAGs

$\Leftrightarrow$

Structural dominance of mDAGs

$G \geq_{obs} G'$

$G \geq_{struct} G'$

If half: If the simplicial complex of G has a face that G' lacks, then G can realize any distribution that G' can, and others besides. Specifically, perfect correlation among a set of nodes is possible if and only if they are part of the same face.

For directed-edge-free mDAGs

Observational dominance of mDAGs ⟺ Structural dominance of mDAGs

$G \succeq_{obs} G'$ $\quad$ $G \succeq_{struct} G'$

Only if half: Consider the contrapositive. Lack of structural dominance means that there is a face in G' that is not in G. In this case there there is a set of nodes that have a common ancestor in G' but not in G. Thus perfect correlation among these is achievable in G' but not in G. Thus, there is no obs'l dominance of G over G'.

Observe&Do dominance of mDAGs

$G \succeq_{O\&D} G'$

⇔

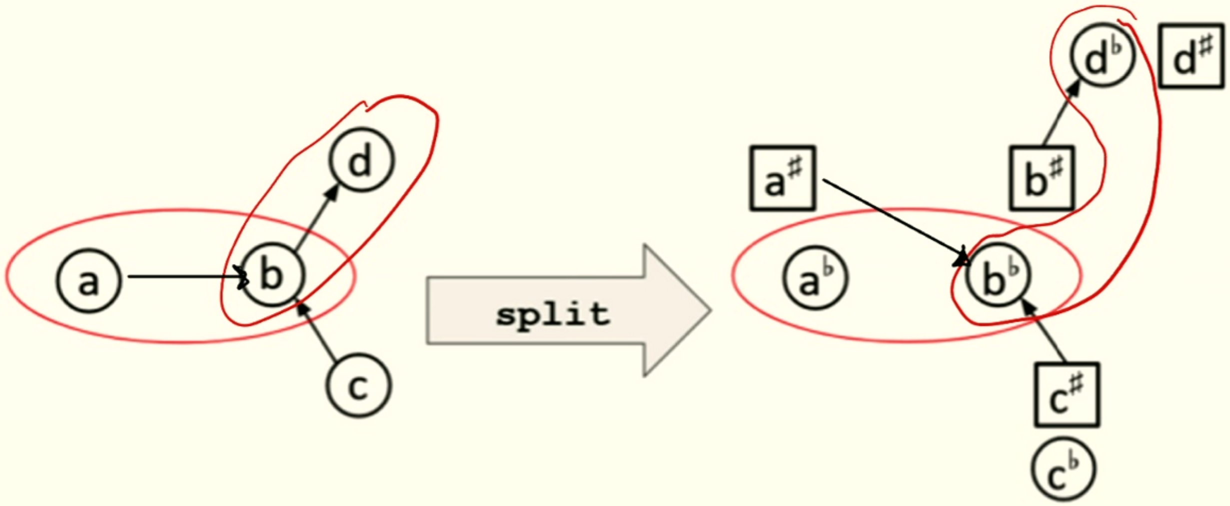
Observational dominance of node-split mDAGs

$\mathrm{split}(G) \succeq_{obs} \mathrm{split}(G')$

⇔

Structural dominance of node-split mDAGs

$\mathrm{split}(G) \succeq_{struc} \mathrm{split}(G')$



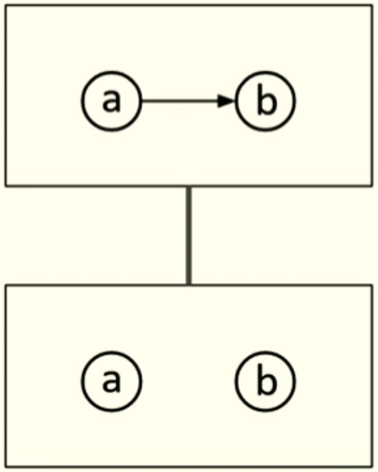
In G, a dropped directed edge need not generate a new CI relation, but in split(G) it does because flat nodes have no parents

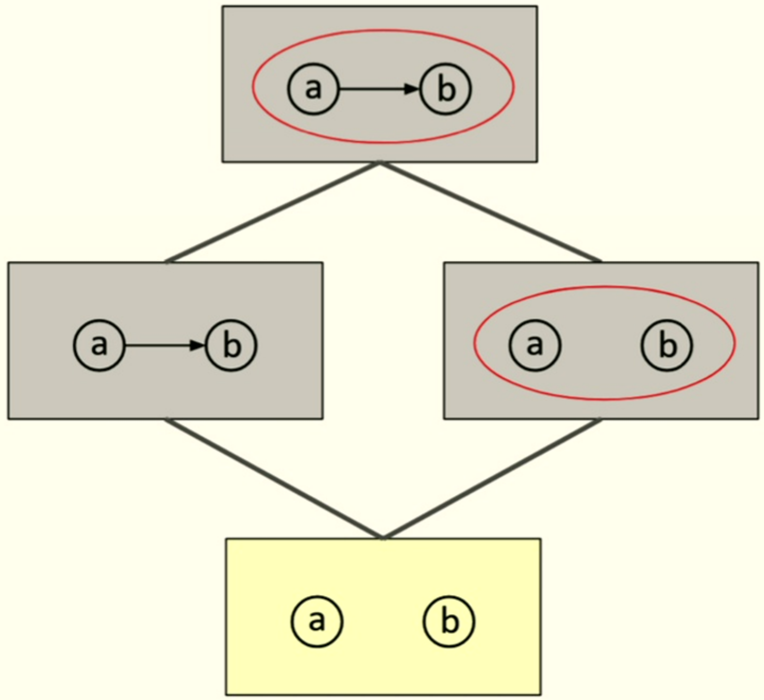
In G, dropping a face of the simplicial complex need not generate a new CI relation, but in split(G) it does because flat nodes do not have any directed edges between them.

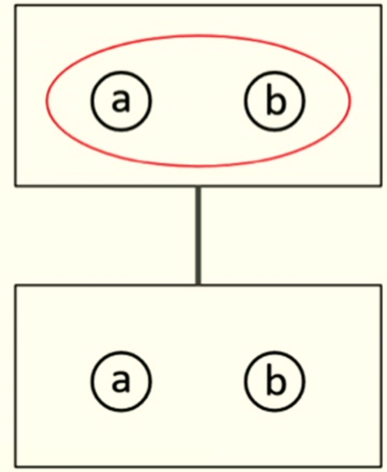
Observe&Do dominance of mDAGs $\Leftrightarrow$ Observational dominance of node-split mDAGs $\Leftrightarrow$ Structural dominance of node-split mDAGs

$G \succeq_{O\&D} G'$ $\quad$ $\mathrm{split}(G) \succeq_{obs} \mathrm{split}(G')$ $\quad$ $\mathrm{split}(G) \succeq_{struc} \mathrm{split}(G')$

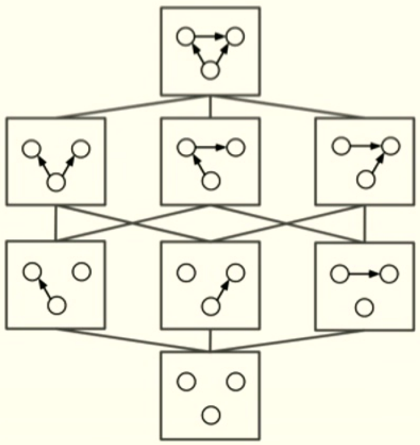
O&D dominance order of 2-node mDAGs

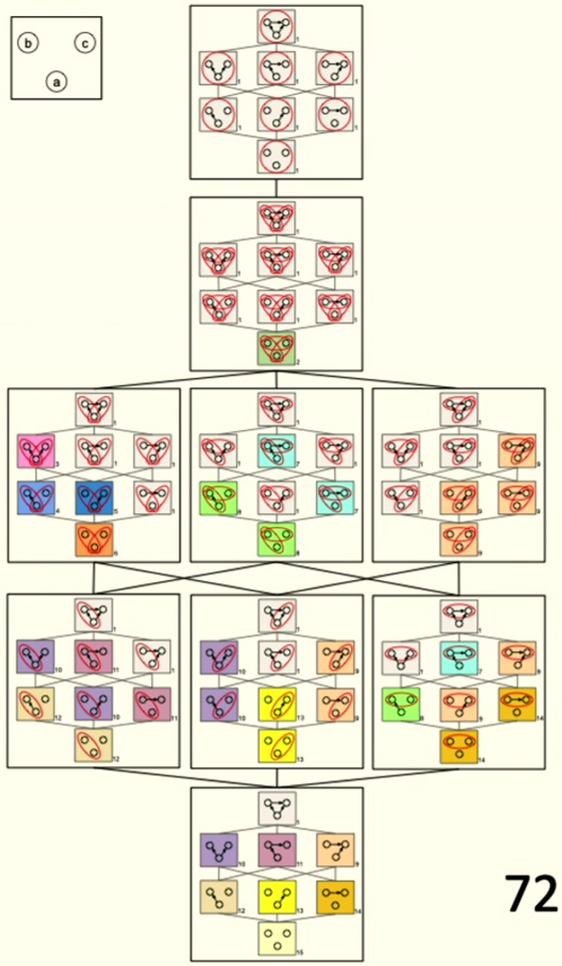


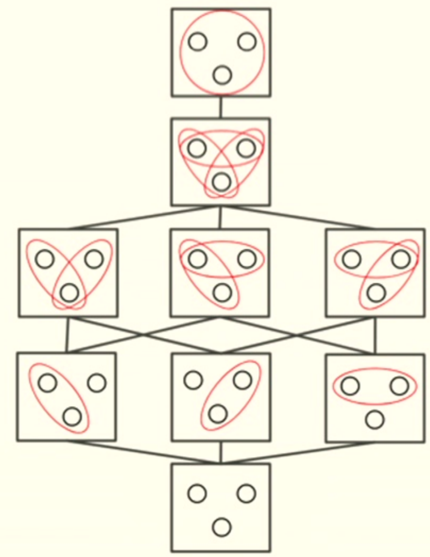




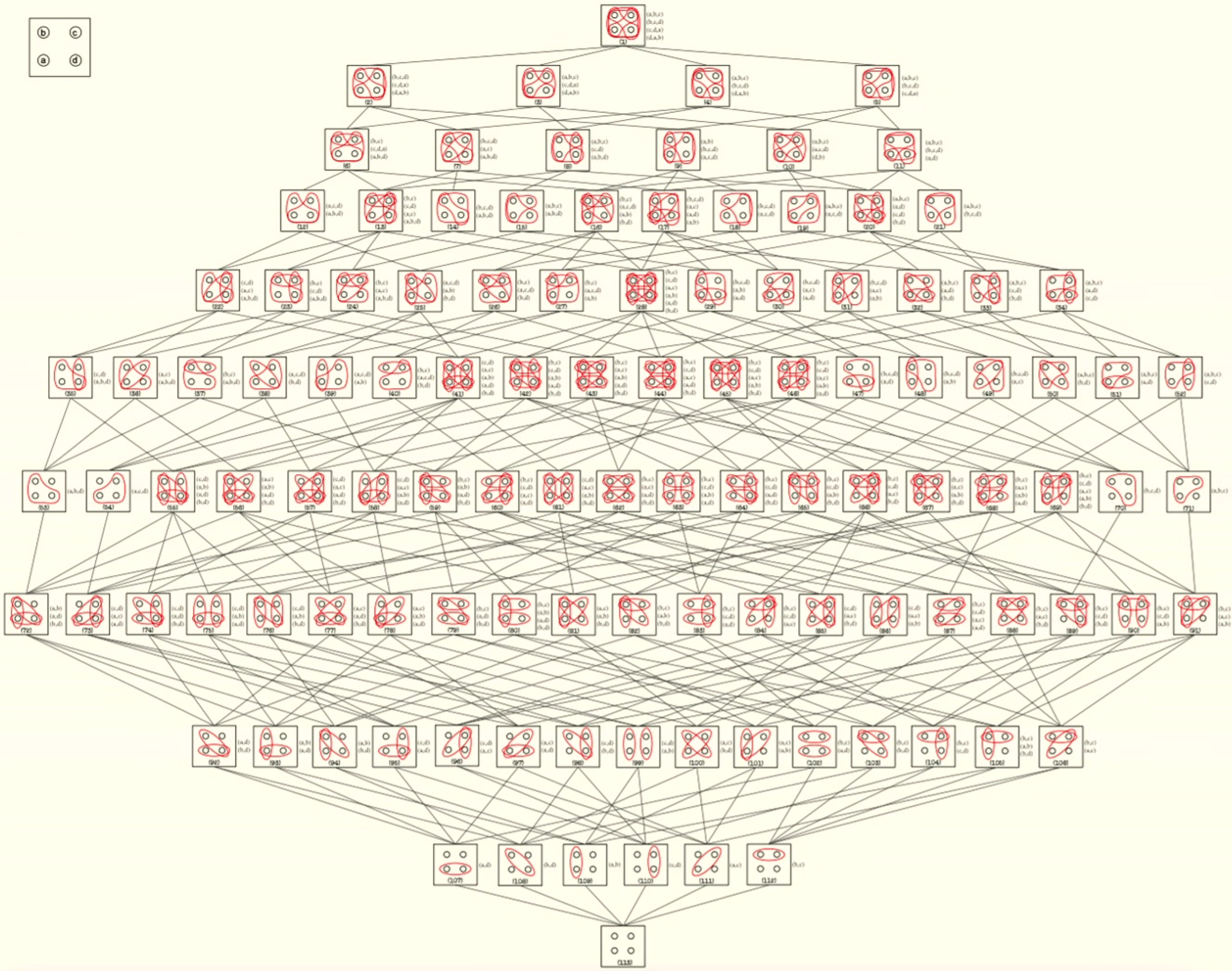
O&D dominance order of 3-node mDAGs

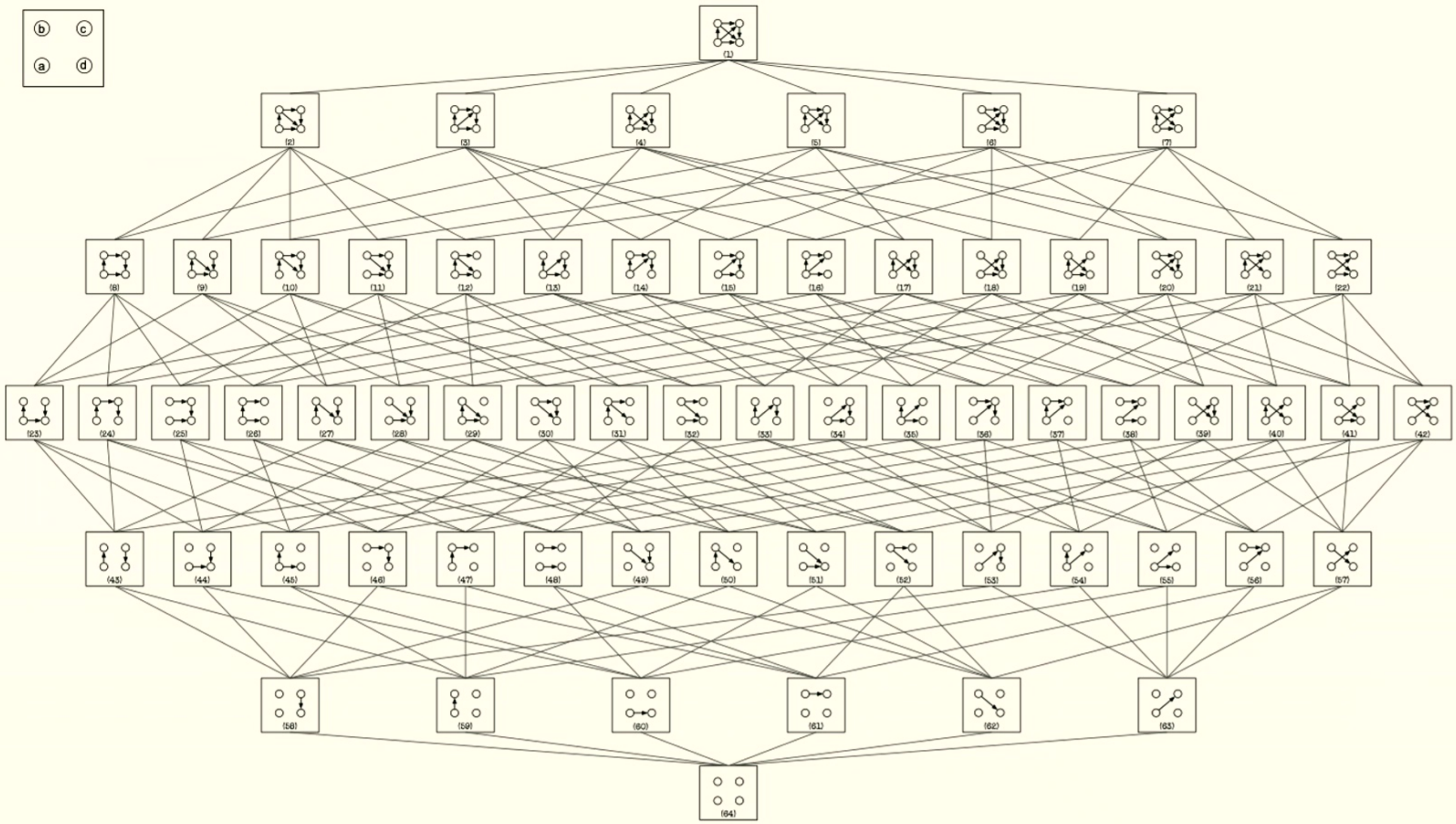




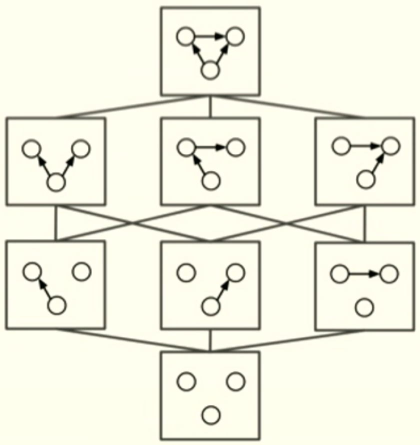


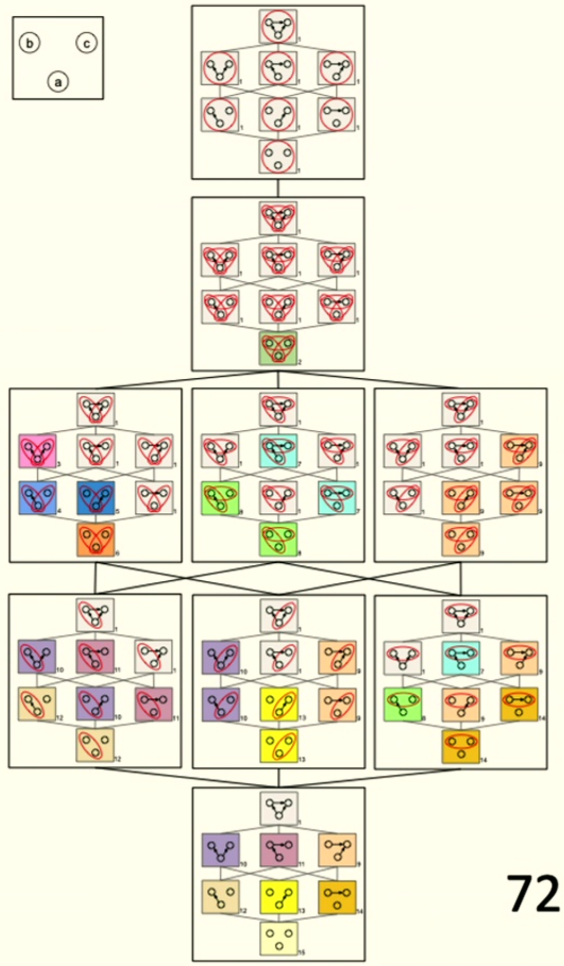
72 mDAGs

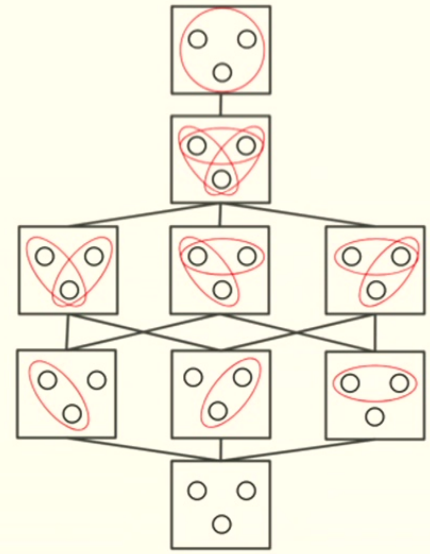




O&D dominance order of 3-node mDAGs

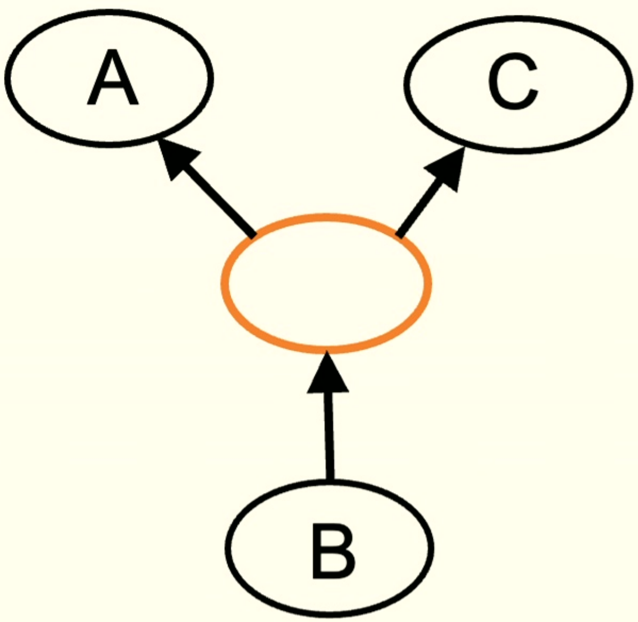






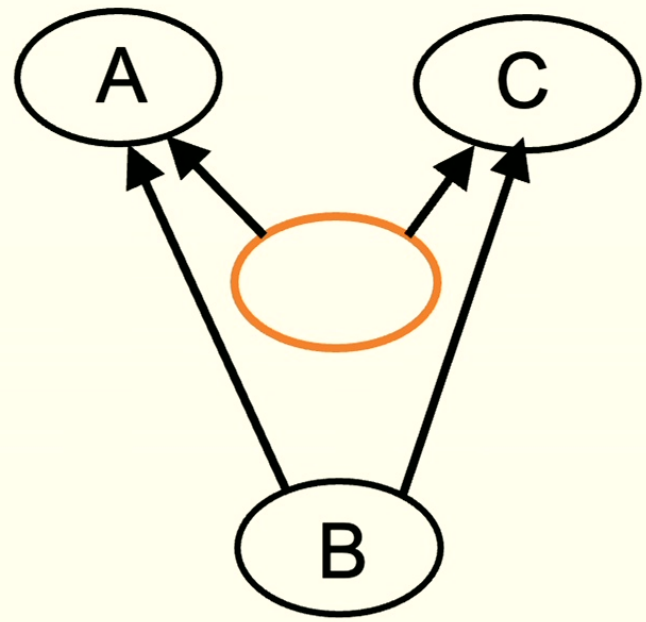
72 mDAGs

Exogenization rule

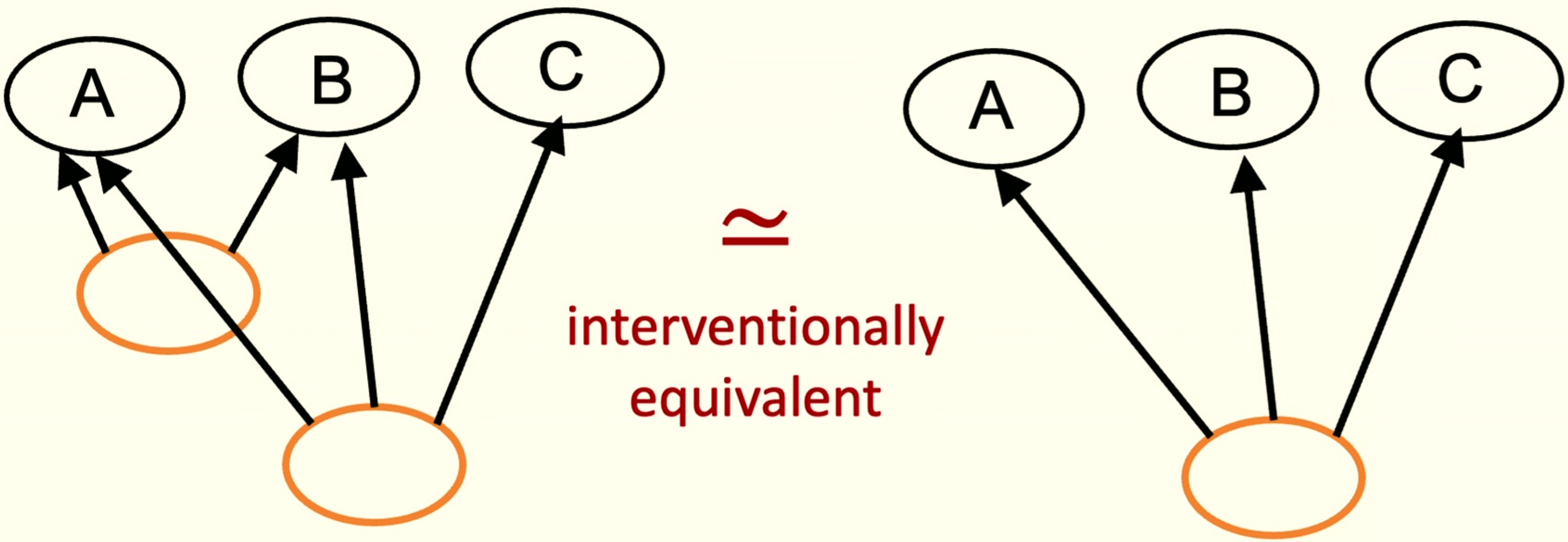


$\simeq$

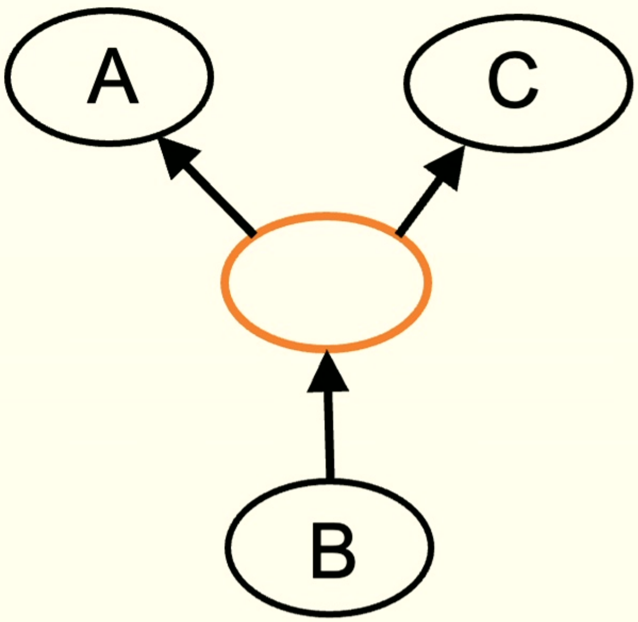
interventionally equivalent



Eliminating redundant latents rule

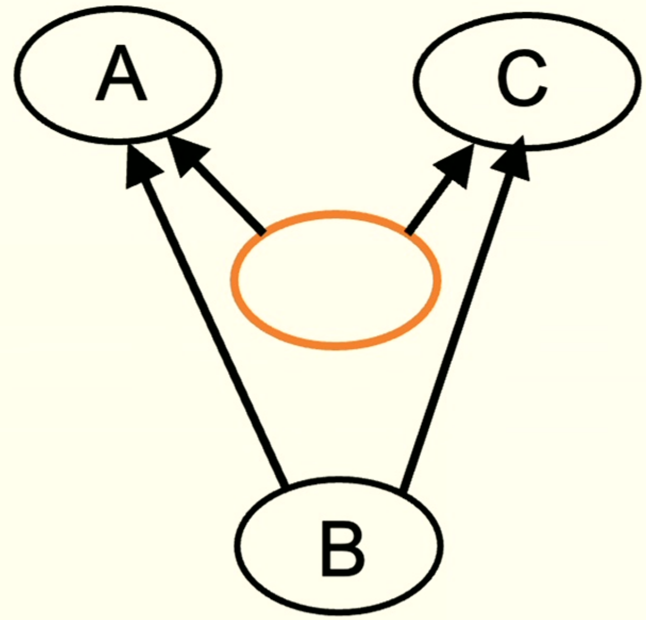


Exogenization rule

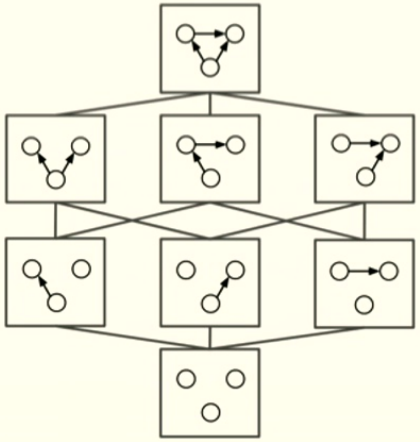


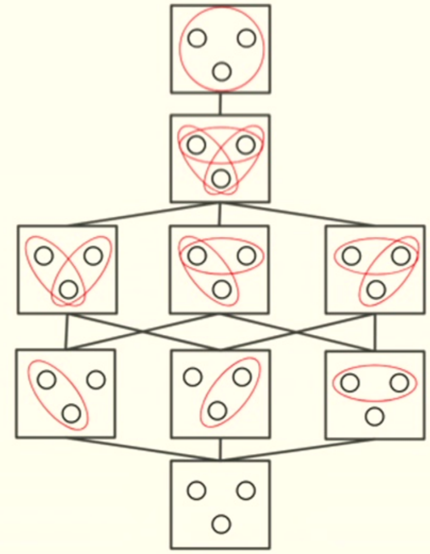
$\simeq$

interventionally equivalent



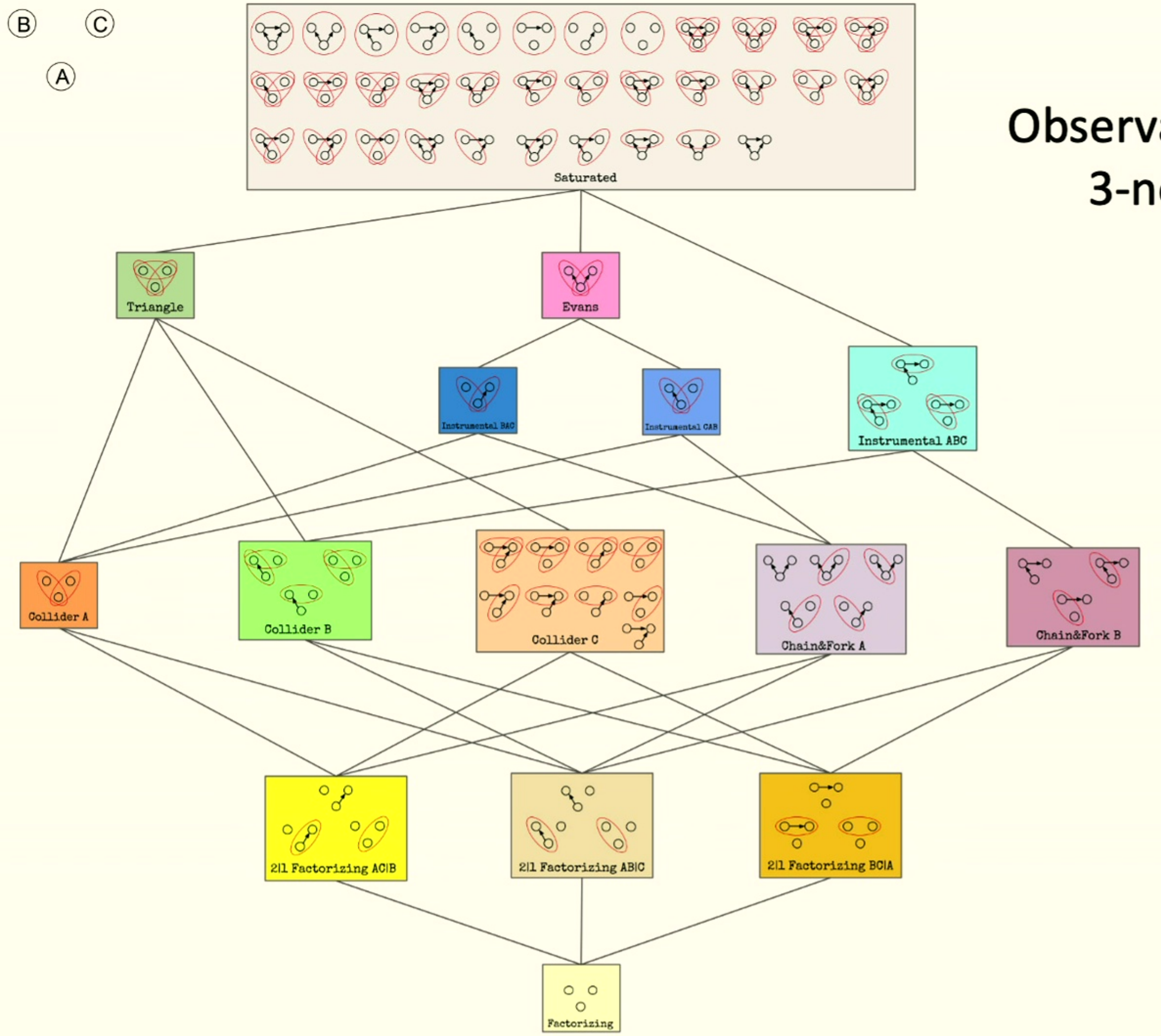
O&D dominance order of 3-node mDAGs



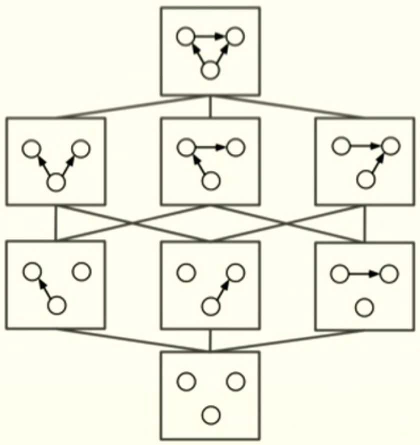


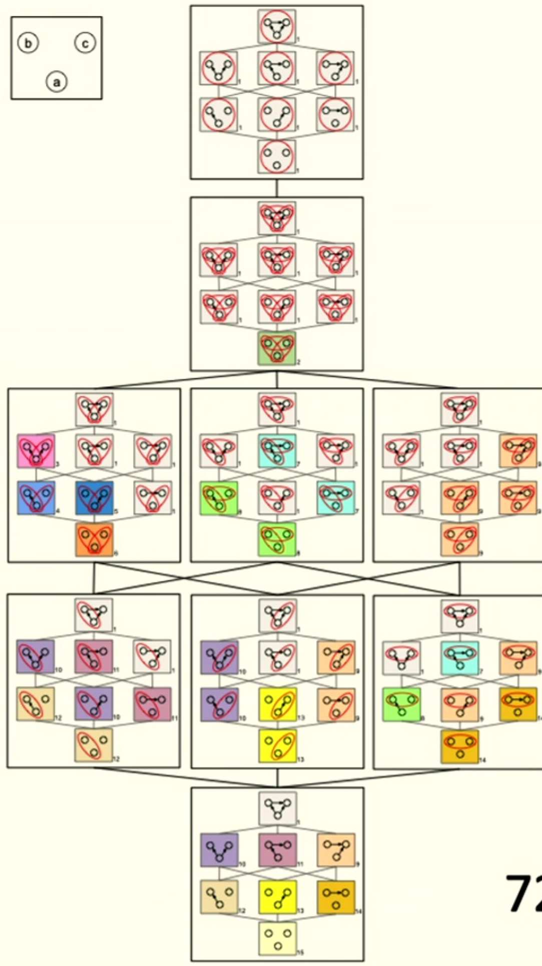
72 mDAGs

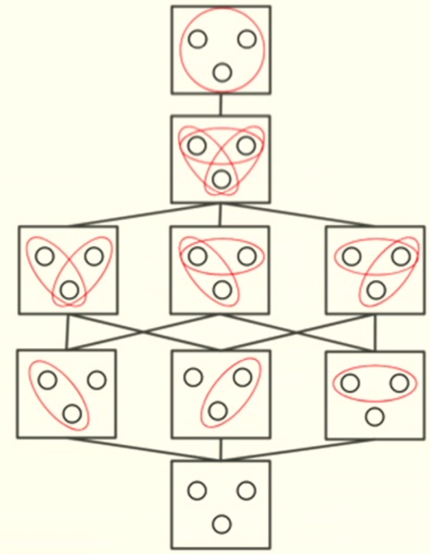
Observational order of 3-node mDAGs



O&D dominance order of 3-node mDAGs







72 mDAGs